

Boundary-induced heterogeneous absorbing states

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Abstract. We study two different types of systems with many absorbing states (with and without a conservation law) and scrutinize the effect of walls/boundaries (either absorbing or reflecting) into them. In some cases, non-trivial structured absorbing configurations (characterized by a background field) develop around the wall. We study such structures using a mean-field approach as well as computer simulations. The main results are: i) for systems in the directed percolation class, a very fast (exponential) convergence of the background to its bulk value is observed; ii) for systems with a conservation law, power-law decaying landscapes are induced by both types of walls: while for absorbing walls this effect is already present in the mean-field approximation, for reflecting walls the structured background is a noise-induced effect. The landscapes are shown to converge to their asymptotic bulk values with an exponent equal to the inverse of the bulk correlation length exponent. Finally, the implications of these results in the context of self-organizing systems are discussed.

Keywords: Non-equilibrium phase transition, absorbing states, self-organized criticality

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INTRODUCTION: BOUNDARIES IN SYSTEMS WITH ABSORBING STATES

Systems with absorbing states played a dominant role in the development of non-equilibrium statistical physics [1]. Directed percolation, the contact process, or the Domany-Kinzel automaton, among many other similar models, have been studied profusely. They all exhibit a phase transition from an active phase, with indefinitely sustained non-trivial dynamics, to an absorbing phase in which the system falls with certainty into a frozen state in which all dynamics ceases. Applications run from epidemics, to flow in porous media, auto-catalytic reactions, self-organization, damage spreading, population dynamics, turbulence, etc.

The critical behavior of most of these systems yields into the very robust directed percolation (DP) universality class [2], described at a coarse-grained level by the Reggeon field theory or Gribov process [1]. An experimental realization of DP has been recently obtained, in a breakthrough work, by Takeuchi et al. [3]. It is only in the presence of extra symmetries, long-range interactions, or conservation laws, that critical behavior different from DP can be observed [1, 11].

There are important physical situations in which the number of absorbing configurations grows exponentially with system size, being infinite in the thermodynamic limit. Two main universality classes of such systems are:

1. The *directed percolation class with many AS* [4]. Defined by models as the pair

contact process, the threshold transfer process, and models of catalytic surface reactions [5], this class has no extra symmetry/conservation-law with respect to DP. Its corresponding Langevin equation is:

$$\begin{aligned}\partial_t \rho(\mathbf{x}, t) &= a\rho - b\rho^2 + \gamma\rho\Psi(\mathbf{x}, t) + \nabla^2 \rho + \sigma\sqrt{\rho}\eta(\mathbf{x}, t) \\ \partial_t \Psi(\mathbf{x}, t) &= \alpha\rho - \beta\Psi\rho + D\nabla^2 \rho\end{aligned}\quad (1)$$

where a, b, γ, α and β are constants and η is a Gaussian white noise. Despite the non-trivial absorbing phase, characterized by the non-diffusive background-field $\Psi(\mathbf{x}, t)$, it exhibits DP-like (bulk) criticality (see [4] for more details).

2. The *C-DP* class, introduced to describe the criticality of stochastic sandpiles, as the *Manna* or the *Oslo* one [6, 7, 8], is represented by a Langevin equation with an extra conservation law, rendering it different from DP [9, 10, 11]:

$$\begin{aligned}\partial_t \rho(\mathbf{x}, t) &= a\rho - b\rho^2 + \omega\rho E(\mathbf{x}, t) + \nabla^2 \rho + \sigma\sqrt{\rho}\eta(\mathbf{x}, t) \\ \partial_t E(\mathbf{x}, t) &= D\nabla^2 \rho.\end{aligned}\quad (2)$$

a, b, D and ω are parameters and $E(\mathbf{x}, t)$ is the (conserved and non-diffusive) background, usually called *energy field*. Higher order, irrelevant, terms have been omitted.

It has been claimed, and confirmed in various ways, that the prototypical models of self-organized criticality (SOC), i.e. *sandpiles* [6], fluctuate around a critical point owing to the combined effect of slow driving (addition of energy) and open boundaries (energy dissipation), and that such a critical point is in the C-DP class, owing to the conserved nature of the (bulk) redistribution dynamics [9, 12].

A key issue to fully clarify this connection is to elucidate whether the heterogeneity introduced by open walls in self-organizing systems plays any relevant role far away from the wall, i.e. whether the boundaries induce long-range effects that might eventually alter the bulk dynamics, affecting (bulk) universal critical properties. Actually, if this was the case, then the understanding of SOC in terms of standard non-equilibrium (bulk) phase transitions into absorbing states [9] would be in jeopardy.

It is well known that, owing to the presence of diverging correlations, walls induce non-trivial effects in critical phenomena, i.e. *surface critical phenomena* [13]. For instance, in systems with a single absorbing state in the DP class, spreading exponents [1] are known to differ from their bulk counterparts if initial seeds of activity are localized in the neighborhood of a wall [14]. However, the universality class of the bulk transition remains unaltered.

As we will show, in systems with a non-trivial background field (i.e. systems with many absorbing states) walls can induce long-range modifications of this field deep into the bulk (similar situations have been addressed in the context of directed/anisotropic sandpiles [15]), opening the possibility for relevant changes in the bulk dynamics to occur.

In what follows, we analyze the effect of walls (both absorbing and reflecting) in both Eq.(1) and Eq.(2) in *one dimensional* systems. First, we perform mean-field analyses and second we study the full problem employing computer simulations. Finally, we discuss the previous issues using the new insight.

MEAN FIELD RESULTS

In mean field approximation the noise can be neglected (i.e. $\sigma = 0$), but in order to explore spatial structures we keep the Laplacian terms. Absorbing boundaries are implemented as

$$\rho(0,t) = 0, \quad \Phi(0,t) = 0 \quad (3)$$

while reflecting ones correspond to

$$\nabla(\rho(0,t)) = 0, \quad \nabla(\Phi(0,t)) = 0 \quad (4)$$

(i.e. Dirichlet and Neumann conditions, respectively), where Φ stands for the background field: Ψ or E . All the forthcoming discussions assume implicitly that a well-defined stationary state exists. For this reason, we approach the critical point from the active phase (to avoid getting trapped into absorbing configurations, which prevents the system from relaxing to a true self-averaging steady state).

DP class with many AS: The analysis of Eq.(1) with $\sigma = 0$ becomes trivial if one subtracts the second equation from the first one (multiplied by D) and assumes stationarity. In this way, the discretized Laplacian terms (either at the wall or away from it) cancel out and one obtains a site-independent equation, leading to spatially homogeneous solutions for either absorbing and reflecting boundaries.

Note that we have analyzed Eq.(1), which includes explicitly a $\nabla^2 \rho(x,t)$ term in the equation for the background field. Such a term, present at the coarse-grained Langevin theory [4], is neglected in many studies interested only in critical properties. Actually, it can be argued to be irrelevant in the renormalization group sense; still, as we have seen, it is important to study spatial properties.

C-DP: Imposing (Neumann) reflecting conditions, and proceeding as above (i.e. subtracting one equation from the other to get rid of the Laplacians) it is straightforward to see that the only possible steady state is a flat one, as in the DP case.

On the other hand, considering absorbing boundaries, the problem becomes more subtle. Integrating the background equation in space, the total amount of energy decreases as $-\nabla \rho(0,t) < 0$, which implies that there is a “leakage of energy” at the open boundary and, hence, the only true steady state is the trivial one $\rho(x) = E(x) = 0$.

However, for long but finite times and starting from a flat initial state, there is a non-trivial profile matching the two boundaries: $E(0,t) = 0$ and $E(\infty,t) = 1 = E_{bulk}$, where E_{bulk} corresponds to the initial condition. A simple calculation allows to derive the shape of this landscape. We look for a solution using the following ansatz: $E(x,t) = E_{bulk} - x^{-\alpha} \mathcal{F}(x^2/t)$, where α is some exponent and $\mathcal{F}(x^2/t)$ an unknown function of the scaling variable x^2/t . As the linear coefficient for the activity equation is linear in E , we can try a solution of the form $\rho(x,t) = \rho_{bulk} - x^{-\alpha} \mathcal{G}(x^2/t)$. Plugging this into the equation for the activity at the bulk critical point, $a = -w E_{bulk}$ (for which $\rho_{bulk} = 0$) and equating the lowest orders, it follows that $\alpha = 2$ and that $\mathcal{G}(x^2/t)$ needs to be of the form $\frac{x^2}{t} \mathcal{F}(x^2/t)$ for a solution to exist. For asymptotically large times the leading order

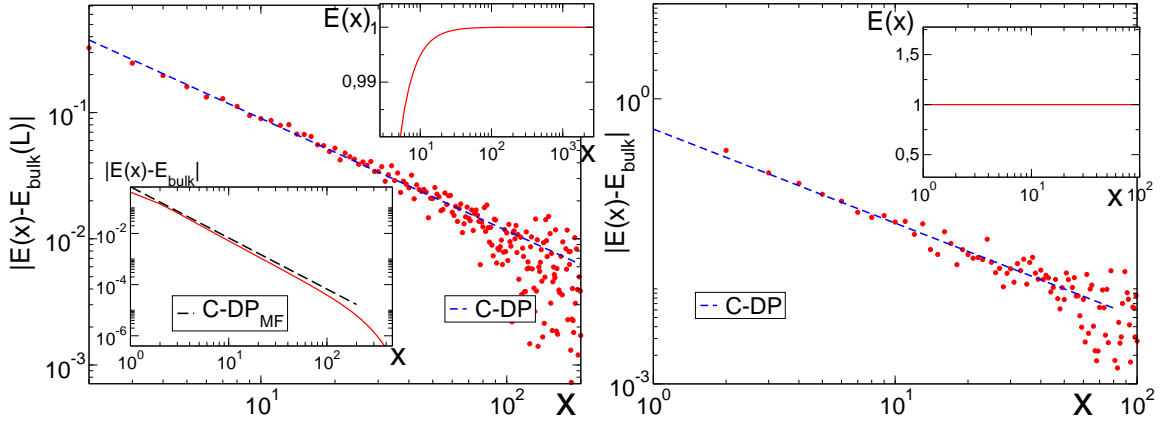


FIGURE 1. Background field for the C-DP equations, Eq.(2), with an absorbing (left) and a reflecting boundary (right), respectively. The main plots correspond to the numerical integration of the full set of equations; in both cases a power-law decay is observed for $|E(x,t) - E_{bulk}|$. The two smaller insets show mean field results: a structured background in the absorbing case (left) and a flat one for reflecting walls (right). The large inset in the left figure, illustrates the power-law decay of $|E(x,t) - E_{bulk}|$ in the mean field solution for absorbing walls at a finite time.

of such a calculation gives rather straightforwardly

$$\rho(x,t) = t^{-1} \mathcal{F}(x^2/t) \quad (5)$$

$$E(x,t) - E_{bulk} = x^{-2} \mathcal{F}(x^2/t) \quad (6)$$

where \mathcal{F} is a degenerate hypergeometric function. In conclusion: there is a non-stationary structured background for any finite time. It consists of a power-law with exponent $\alpha = 2$ converging asymptotically in the bulk to the closed-boundaries initial value E_{bulk} , multiplied by a scaling function of x^2/t (see Eq.(6)).

The exponent $\alpha = 2$ can also be derived using naïve power-counting arguments. As said above, the background field contributes linearly to the coefficient of the linear term in the activity equation, it behaves as the distance to the critical point and, therefore, it scales with the inverse of the correlation length critical exponent, ν , which in mean-field approximation is $\nu = 1/2$ [1], entailing $\alpha = 2$.

Numerical integration of Eq.(2) with $\sigma = 0$ (i.e. in its noiseless version) confirms these conclusions (see the insets of Fig.1a and Fig. 1b, for absorbing and reflecting conditions, respectively).

BEYOND MEAN FIELD

In order to go beyond mean field approximation, we switch-on back the noise term both in Eq.(1) and Eq.(2). Analytical solutions do not exist anymore and we need to resort to numerical simulations in one-dimensional discretized lattices. Absorbing and reflecting boundaries correspond, respectively to: $f(-1) = f(0) = 0$ and $f(-1) = f(1)$ (where f stand for either ρ , Ψ , or E). Using this, the discretized Laplacian operator, $\nabla^2 f(x) = f(x+1) + f(x-1) - 2f(x)$ can be replaced, at the wall, by $\nabla^2 = \nabla - I$ (where

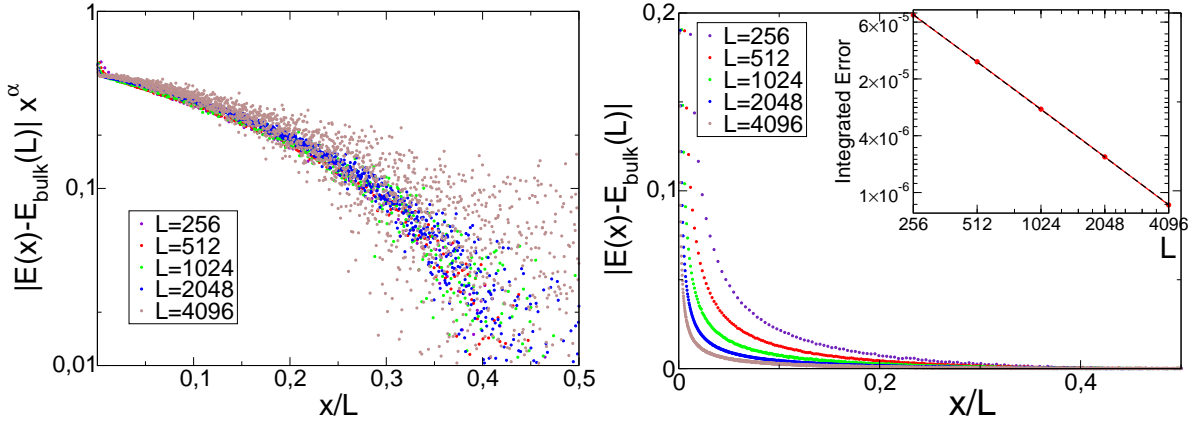


FIGURE 2. Left: Collapsed background fields, $|E(x) - E_{bulk}(L)| x^\alpha$, as a function of x/L for a system into the C-DP universality class, with an absorbing boundary, at criticality, and for different sizes. Using $\alpha = 0.80(5)$ all curves collapse into a unique one. Right: Stationary background field for the same C-DP system at criticality, for different values of L . The inset shows a plot of the area below the different curves, normalized by system size, as a function of L .

I is the identity) for absorbing walls, and $\nabla^2 = 2\nabla$ for reflecting ones. Integration is performed by using the scheme proposed recently by Dornic et al [16], which allows to integrate square-root noise stochastic equations in an efficient way. We consider systems of different sizes L (from 2^8 to 2^{12}).

For each of them we determine numerically the critical point (its location is slightly affected by finite size effects). Contrarily to the mean field case, now, the activity can reach the absorbing state $\rho(x) = 0$ in finite time and, hence, a steady state for the background field exists for both absorbing or reflecting boundaries, as we will illustrate. In numerics, a steady state is achieved by perturbing with some small amount of activity the background field, leaving the system relax to a new absorbing configuration, and iterating this procedure as much as needed.

DP class with many AS: Numerical integration of Eq.(1) shows results qualitatively similar to those obtained in the mean-field approach. For both absorbing and reflecting walls there is an extremely fast convergence (it involves only a few sites) from $E(0) = 0$ to $E = E_{bulk}$ (results not shown).

C-DP: Figure 1(main plots) shows results for the stationary backgrounds coming out of a numerical integration of Eq.(2) for absorbing and reflecting walls respectively ($L = 4096$). Both log-log plots show a power-law convergence to the bulk value. In the absorbing case there is an under-density nearby the wall, while in the reflecting case there is an over-density.

One might wonder why arguments as those presented in the previous section to discard the possibility of structured backgrounds in the reflecting case fail here. More specifically: if the background field is to be stationary on average, then $\nabla^2 \rho$ cannot have any structure on average (see Eq.(2)). From this simple relation, in mean-field

approximation we concluded that there cannot be a non-trivial structure neither for ρ nor for $E(x)$. Instead, in the presence of fluctuations, even if $\rho(x)$ does not have any non-trivial spatial structure (i.e. even if $\nabla^2 \rho$ vanishes on average), $\rho^2(x)$, $\sqrt{\rho(x)}$ and $E(x)$ can have one and actually, we have verified in numerical simulations that they do have one. In other words; even if the first moment of ρ is structureless, the second one and the average of its square-root take different values at different sites, having therefore a non-trivial structure. This can happen only for fluctuating variables implying that the non-trivial structure in this case is a *noise-induced one*.

Fig.2a illustrates that, in the absorbing case, the backgrounds for different sizes collapse into a unique scaling curve by assuming $|E_{bulk}(L) - E(x)| \sim x^{-\alpha} \mathcal{F}(x/L)$ with $\alpha = 0.80(5)$. Using scaling arguments as the mean-field ones above, α should coincide with $1/\nu$. ν is known from previous work (and our own direct measurements) to be $\nu \approx 1.33(5)$ [11], implying $\alpha \approx 0.75(5)$, compatible with our previous estimation. Note that this solution coincides qualitatively with Eq.(6), but the exponent α takes the value $1/\nu = 2$ in mean-field and its renormalized value $1/\nu \approx 0.75$ here.

In Fig.2b we plot the deviation of the background field from its asymptotic bulk value, $|E(x) - E_{bulk}|$, as a function of x/L . For all values of L , the decay exhibits a fat tail. However, as shown in the inset of Fig.2b, the global deviation from the bulk value, defined as the spatial integral of $|E_{bulk} - E(x)|$ divided by L , decays exponentially fast to 0 as L is increased, i.e. $\int dx |E_{bulk} - E(x)|$ is sub-extensive. In other words, *the global effect of an absorbing boundary on the bulk dynamics becomes arbitrarily small for sufficiently large system sizes; this guarantees the existence of a well defined bulk in the thermodynamic limit*. In other words, the bulk behavior in the thermodynamic limit can be systematically approached by considering a sequence of finite-size open-boundaries systems with larger and larger sizes. Identical results are obtained in the reflecting case (not shown).

DISCUSSION AND CONCLUSIONS

The summary of the previous findings is as follows. Models in the DP class have a mostly non-structured background field at criticality both for absorbing and reflecting walls. This result is obtained in a mean field calculation as well as in numerical simulations. Walls in the DP class affect boundary/surface properties, but not bulk ones.

On the other hand, systems with a conservation-law (in the C-DP class) exhibit, for both absorbing and reflecting walls, a power-law convergence to their corresponding bulk value of the background field. This stems from the purely diffusive and conserved nature of the corresponding background-field equation.

The absorbing case is qualitatively well described by our mean-field (noiseless) approach (which leads to a local under-density nearby the wall, converging as a power law to the bulk value), even if with an exponent different from the mean-field one.

Instead, in the reflecting case, the mean-field calculation does not lead to any non-trivial structure. This is so because the derivative of the background field at the boundary is fixed to 0 and this leads ineluctably to a flat background. Instead, in the presence of fluctuations, locally non-vanishing derivatives of the field appear at the boundary; these are then amplified, a local over-density of the background field is generated, and

(in the long time limit) a power-law decaying stationary background structure sets in. Therefore, once fluctuations are switched on, a *noise-induced non-trivial background structure emerges*.

For both absorbing and reflecting backgrounds the spatial convergence to the corresponding bulk value is described by a power law with an exponent α equal to the inverse of the (bulk) correlation length exponent.

Finally, let us emphasize that a well defined bulk exists in all cases owing to the fact that the global deviation of the averaged background field with respect to its bulk value converges exponentially fast to zero in the large system size limit (as illustrated in Fig.2b). It is important to underline that this does not imply that *boundary critical exponents* cannot be affected by the presence of walls, and actually they typically are [17]. For instance, it is well known that walls change the surviving probability for the propagation of activity from a localized seed nearby the boundary, affecting spreading exponents. Instead, bulk properties are not affected in any case, despite of the power-law convergence of the background reported above. The global effect of the wall on bulk properties can be made as small as wanted by enlarging the system size, implying the existence of a well defined bulk in the thermodynamic limit. This provides further conceptual support for the understanding of self-organizing sandpiles (with open boundaries) from the perspective of standard phase bulk transitions in systems with many absorbing states and a conservation law [9].

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