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Three-point boundary value problems for nonlinear second-order impulsive q-difference equations

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Abstract

The quantum calculus on finite intervals was studied recently by the authors in Adv. Differ. Equ. 2013:282, 2013, where the concepts of q_k-derivative and q_k-integral of a function $f : J_k := [t_k, t_{k+1}] \to \mathbb{R}$ have been introduced. In this paper, we prove existence and uniqueness results for nonlinear second-order impulsive q_k-difference three-point boundary value problems, by using Banach's contraction mapping principle and Krasnoselskii's fixed-point theorem.

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Keywords: q_k-derivative; q_k-integral; impulsive q_k-difference equation; existence; uniqueness; three-point boundary conditions; fixed-point theorems

1 Introduction

In this article, we investigate the nonlinear second-order impulsive q_k-difference equation with three-point boundary conditions

$$\begin{cases} D^2_{q_k} x(t) = f(t, x(t)), & t \in J := [0, T], t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k)), & k = 1, 2, \dots, m, \\ D_{q_k} x(t_k^+) - D_{q_{k-1}} x(t_k) = I_k^*(x(t_k)), & k = 1, 2, \dots, m, \\ x(0) = 0, \quad x(T) = x(\eta), \end{cases} \tag{1.1}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $I_k, I_k^* \in C(\mathbb{R}, \mathbb{R})$, $\Delta x(t_k) = x(t_k^+) - x(t_k)$ for $k = 1, 2, \dots, m$, $x(t_k^+) = \lim_{h\to 0} x(t_k + h)$, $\eta \in (t_j, t_{j+1})$ a constant for some $j \in \{0, 1, 2, \dots, m\}$ and $0 < q_k < 1$ for $k = 0, 1, 2, \dots, m$.

The theory of quantum calculus on finite intervals was developed recently by the authors in [1]. In [1] the concepts of q_k-derivative and q_k-integral of a function $f : J_k := [t_k, t_{k+1}] \to \mathbb{R}$, are defined and their basic properties proved. As applications, existence and uniqueness results for initial value problems for first- and second-order impulsive q_k-difference equations are proved.

The book by Kac and Cheung [2] covers many of the fundamental aspects of the quantum calculus. In recent years, the topic of q-calculus has attracted the attention of several researchers and a variety of new results can be found in the papers [3–15] and the references cited therein.

Impulsive differential equations, that is, differential equations involving an impulse effect, appear as a natural description of observed evolution phenomena of several real-world problems. For some monographs on impulsive differential equations we refer to [16–18].

In the present paper we prove existence and uniqueness results for the impulsive boundary value problem (1.1) by using Banach's contraction mapping principle and Krasnoselskii's fixed-point theorem. The rest of this paper is organized as follows: In Section 2 we present the notions of q_k-derivative and q_k-integral on finite intervals and collect their properties. The main results are proved in Section 3, while examples illustrating the results are presented in Section 4.

2 Preliminaries

In this section we present the notions of q_k-derivative and q_k-integral on finite intervals. For a fixed $k \in \mathbb{N} \cup \{0\}$ let $J_k := [t_k, t_{k+1}] \subset \mathbb{R}$ be an interval and $0 < q_k < 1$ be a constant. We define q_k-derivative of a function $f : J_k \to \mathbb{R}$ at a point $t \in J_k$ as follows.

Definition 2.1 Assume $f : J_k \to \mathbb{R}$ is a continuous function and let $t \in J_k$. Then the expression

$$D_{q_k}f(t) = \frac{f(t) - f(q_k t + (1 - q_k)t_k)}{(1 - q_k)(t - t_k)}, \quad t \neq t_k, \qquad D_{q_k}f(t_k) = \lim_{t \to t_k} D_{q_k}f(t), \tag{2.1}$$

is called the q_k-derivative of function f at t.

We say that f is q_k-differentiable on J_k provided $D_{q_k}f(t)$ exists for all $t \in J_k$. Note that if $t_k = 0$ and $q_k = q$ in (2.1), then $D_{q_k}f = D_q f$, where D_q is the well-known q-derivative of the function $f(t)$ defined by

$$D_q f(t) = \frac{f(t) - f(qt)}{(1 - q)t}. \tag{2.2}$$

In addition, we should define the higher q_k-derivative of functions.

Definition 2.2 Let $f : J_k \to \mathbb{R}$ is a continuous function, we call the second-order q_k-derivative $D^2_{q_k}f$ provided $D_{q_k}f$ is q_k-differentiable on J_k with $D^2_{q_k}f = D_{q_k}(D_{q_k}f) : J_k \to \mathbb{R}$. Similarly, we define the higher-order q_k-derivative $D^n_{q_k} : J_k \to \mathbb{R}$.

The properties of the q_k-derivative are summarized in the following theorem.

Theorem 2.3 *Assume $f, g : J_k \to \mathbb{R}$ are q_k-differentiable on J_k. Then:*

(i) *The sum $f + g : J_k \to \mathbb{R}$ is q_k-differentiable on J_k with*

$$D_{q_k}\big(f(t) + g(t)\big) = D_{q_k}f(t) + D_{q_k}g(t).$$

(ii) *For any constant α, $\alpha f : J_k \to \mathbb{R}$ is q_k-differentiable on J_k with*

$$D_{q_k}(\alpha f)(t) = \alpha D_{q_k}f(t).$$

(iii) *The product* $fg : J_k \to \mathbb{R}$ *is* q_k*-differentiable on* J_k *with*

$$\begin{aligned} D_{q_k}(fg)(t) &= f(t)D_{q_k}g(t) + g\big(q_k t + (1-q_k)t_k\big)D_{q_k}f(t) \\ &= g(t)D_{q_k}f(t) + f\big(q_k t + (1-q_k)t_k\big)D_{q_k}g(t). \end{aligned}$$

(iv) *If* $g(t)g(q_k t + (1-q_k)t_k) \neq 0$, *then* $\frac{f}{g}$ *is* q_k*-differentiable on* J_k *with*

$$D_{q_k}\left(\frac{f}{g}\right)(t) = \frac{g(t)D_{q_k}f(t) - f(t)D_{q_k}g(t)}{g(t)g(q_k t + (1-q_k)t_k)}.$$

Definition 2.4 Assume $f : J_k \to \mathbb{R}$ is a continuous function. Then the q_k-integral is defined by

$$\int_{t_k}^t f(s)\, d_{q_k}s = (1-q_k)(t-t_k)\sum_{n=0}^{\infty} q_k^n f\big(q_k^n t + \big(1-q_k^n\big)t_k\big), \tag{2.3}$$

for $t \in J_k$. Moreover, if $a \in (t_k, t)$ then the definite q_k-integral is defined by

$$\begin{aligned} \int_a^t f(s)\, d_{q_k}s &= \int_{t_k}^t f(s)\, d_{q_k}s - \int_{t_k}^a f(s)\, d_{q_k}s \\ &= (1-q_k)(t-t_k)\sum_{n=0}^{\infty} q_k^n f\big(q_k^n t + \big(1-q_k^n\big)t_k\big) \\ &\quad - (1-q_k)(a-t_k)\sum_{n=0}^{\infty} q_k^n f\big(q_k^n a + \big(1-q_k^n\big)t_k\big). \end{aligned}$$

Note that if $t_k = 0$ and $q_k = q$, then (2.3) reduces to the q-integral of a function $f(t)$, defined by $\int_0^t f(s)\, d_q s = (1-q)t\sum_{n=0}^{\infty} q^n f(q^n t)$ for $t \in [0, \infty)$.

Theorem 2.5 *For* $t \in J_k$, *the following formulas hold*:

(i) $D_{q_k}\int_{t_k}^t f(s)\, d_{q_k}s = f(t)$;

(ii) $\int_{t_k}^t D_{q_k}f(s)\, d_{q_k}s = f(t)$;

(iii) $\int_a^t D_{q_k}f(s)\, d_{q_k}s = f(t) - f(a)$ *for* $a \in (t_k, t)$.

3 Main results

Let $J = [0, T]$, $J_0 = [t_0, t_1]$, $J_k = (t_k, t_{k+1}]$ for $k = 1, 2, \ldots, m$. Let $PC(J, \mathbb{R}) = \{x : J \to \mathbb{R} :$ $x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k), k = 1, 2, \ldots, m\}$. $PC(J, \mathbb{R})$ is a Banach space with the norm $\|x\|_{PC} = \sup\{|x(t)|; t \in J\}$.

Lemma 3.1 *The unique solution of problem* (1.1) *is given by*

$$\begin{aligned} x(t) &= -t\sum_{k=1}^{j}\left(\int_{t_{k-1}}^{t_k} f\big(s, x(s)\big)\, d_{q_{k-1}}s + I_k^*\big(x(t_k)\big)\right) \\ &\quad - \frac{t}{T-\eta}\sum_{k=j+1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f\big(\sigma, x(\sigma)\big)\, d_{q_{k-1}}\sigma\, d_{q_{k-1}}s + I_k\big(x(t_k)\big)\right) \end{aligned}$$

$$\begin{aligned}
&- \frac{t}{T-\eta} \sum_{k=j+1}^{m} \left(\int_{t_{k-1}}^{t_k} f\big(s, x(s)\big)\, d_{q_{k-1}} s + I_k^*\big(x(t_k)\big) \right) (T - t_k) \\
&+ \frac{t}{T-\eta} \int_{t_j}^{\eta} \int_{t_j}^{s} f\big(\sigma, x(\sigma)\big)\, d_{q_j}\sigma\, d_{q_j} s \\
&- \frac{t}{T-\eta} \int_{t_m}^{T} \int_{t_m}^{s} f\big(\sigma, x(\sigma)\big)\, d_{q_m}\sigma\, d_{q_m} s \\
&+ \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} f\big(\sigma, x(\sigma)\big)\, d_{q_{k-1}}\sigma\, d_{q_{k-1}} s + I_k\big(x(t_k)\big) \right) \\
&+ \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} f\big(s, x(s)\big)\, d_{q_{k-1}} s + I_k^*\big(x(t_k)\big) \right) (t - t_k) \\
&+ \int_{t_k}^{t} \int_{t_k}^{s} f\big(\sigma, x(\sigma)\big)\, d_{q_k}\sigma\, d_{q_k} s, \qquad (3.1)
\end{aligned}$$

with $\sum_{0<0}(\cdot) = 0$.

Proof For $t \in J_0$, taking the q_0-integral for the first equation of (1.1), we get

$$D_{q_0} x(t) = D_{q_0} x(0) + \int_0^t f\big(s, x(s)\big)\, d_{q_0} s, \qquad (3.2)$$

which yields

$$D_{q_0} x(t_1) = D_{q_0} x(0) + \int_0^{t_1} f\big(s, x(s)\big)\, d_{q_0} s. \qquad (3.3)$$

For $t \in J_0$ we obtain by q_0-integrating (3.2),

$$\begin{aligned}
x(t) &= x(0) + D_{q_0} x(0) t + \int_0^t \int_0^s f\big(\sigma, x(\sigma)\big)\, d_{q_0}\sigma\, d_{q_0} s \\
&:= A + Bt + \int_0^t \int_0^s f\big(\sigma, x(\sigma)\big)\, d_{q_0}\sigma\, d_{q_0} s \quad \big(x(0) = A, D_{q_0} x(0) = B\big).
\end{aligned}$$

In particular, for $t = t_1$

$$x(t_1) = A + Bt_1 + \int_0^{t_1} \int_0^s f\big(\sigma, x(\sigma)\big)\, d_{q_0}\sigma\, d_{q_0} s. \qquad (3.4)$$

For $t \in J_1 = (t_1, t_2]$, q_1-integrating (1.1), we have

$$D_{q_1} x(t) = D_{q_1} x\big(t_1^+\big) + \int_{t_1}^t f\big(s, x(s)\big)\, d_{q_1} s.$$

Using the third condition of (1.1) with (3.3), it follows that

$$D_{q_1} x(t) = B + \int_0^{t_1} f\big(s, x(s)\big)\, d_{q_0} s + I_1^*\big(x(t_1)\big) + \int_{t_1}^t f\big(s, x(s)\big)\, d_{q_1} s. \qquad (3.5)$$

Taking the q_1-integral to (3.5) for $t \in J_1$, we obtain

$$x(t) = x(t_1^+) + \left[B + \int_0^{t_1} f(s, x(s))\, d_{q_0}s + I_1^*(x(t_1))\right](t - t_1) + \int_{t_1}^t \int_{t_1}^s f(\sigma, x(\sigma))\, d_{q_1}\sigma\, d_{q_1}s. \tag{3.6}$$

Applying the second equation of (1.1) with (3.4) and (3.6), we get

$$\begin{aligned} x(t) &= A + Bt_1 + \int_0^{t_1} \int_0^s f(\sigma, x(\sigma))\, d_{q_0}\sigma\, d_{q_0}s + I_1(x(t_1)) \\ &\quad + \left[B + \int_0^{t_1} f(s, x(s))\, d_{q_0}s + I_1^*(x(t_1))\right](t - t_1) \\ &\quad + \int_{t_1}^t \int_{t_1}^s f(\sigma, x(\sigma))\, d_{q_1}\sigma\, d_{q_1}s \\ &= A + Bt + \int_0^{t_1} \int_0^s f(\sigma, x(\sigma))\, d_{q_0}\sigma\, d_{q_0}s + I_1(x(t_1)) \\ &\quad + \left[\int_0^{t_1} f(s, x(s))\, d_{q_0}s + I_1^*(x(t_1))\right](t - t_1) \\ &\quad + \int_{t_1}^t \int_{t_1}^s f(\sigma, x(\sigma))\, d_{q_1}\sigma\, d_{q_1}s. \end{aligned}$$

Repeating the above process, for $t \in J$, we get

$$\begin{aligned} x(t) &= A + Bt \\ &\quad + \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^s f(\sigma, x(\sigma))\, d_{q_{k-1}}\sigma\, d_{q_{k-1}}s + I_k(x(t_k))\right) \\ &\quad + \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} f(s, x(s))\, d_{q_{k-1}}s + I_k^*(x(t_k))\right)(t - t_k) \\ &\quad + \int_{t_k}^t \int_{t_k}^s f(\sigma, x(\sigma))\, d_{q_k}\sigma\, d_{q_k}s. \end{aligned} \tag{3.7}$$

The first boundary condition of (1.1) implies $A = 0$. The second boundary condition of (1.1) yields

$$\begin{aligned} &\sum_{k=1}^m \left(\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^s f(\sigma, x(\sigma))\, d_{q_{k-1}}\sigma\, d_{q_{k-1}}s + I_k(x(t_k))\right) \\ &\quad + \sum_{k=1}^m \left(\int_{t_{k-1}}^{t_k} f(s, x(s))\, d_{q_{k-1}}s + I_k^*(x(t_k))\right)(T - t_k) \\ &\quad + \int_{t_m}^T \int_{t_m}^s f(\sigma, x(\sigma))\, d_{q_m}\sigma\, d_{q_m}s + BT \\ &= \sum_{k=1}^j \left(\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^s f(\sigma, x(\sigma))\, d_{q_{k-1}}\sigma\, d_{q_{k-1}}s + I_k(x(t_k))\right) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{j}\left(\int_{t_{k-1}}^{t_k} f\big(s,x(s)\big)\,d_{q_{k-1}}s + I_k^*\big(x(t_k)\big)\right)(\eta - t_k)\\
&+\int_{t_j}^{\eta}\int_{t_j}^{s} f\big(\sigma,x(\sigma)\big)\,d_{q_j}\sigma\,d_{q_j}s + B\eta,
\end{aligned}
$$

which implies

$$
\begin{aligned}
B ={}& -\sum_{k=1}^{j}\left(\int_{t_{k-1}}^{t_k} f\big(s,x(s)\big)\,d_{q_{k-1}}s + I_k^*\big(x(t_k)\big)\right)\\
&-\frac{1}{T-\eta}\sum_{k=j+1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f\big(\sigma,x(\sigma)\big)\,d_{q_{k-1}}\sigma\,d_{q_{k-1}}s + I_k\big(x(t_k)\big)\right)\\
&-\frac{1}{T-\eta}\sum_{k=j+1}^{m}\left(\int_{t_{k-1}}^{t_k} f\big(s,x(s)\big)\,d_{q_{k-1}}s + I_k^*\big(x(t_k)\big)\right)(T - t_k)\\
&+\frac{1}{T-\eta}\int_{t_j}^{\eta}\int_{t_j}^{s} f\big(\sigma,x(\sigma)\big)\,d_{q_j}\sigma\,d_{q_j}s - \frac{1}{T-\eta}\int_{t_m}^{T}\int_{t_m}^{s} f\big(\sigma,x(\sigma)\big)\,d_{q_m}\sigma\,d_{q_m}s.
\end{aligned}
$$

Substituting the constant B into (3.7), we obtain (3.1) as required. □

In view of Lemma 3.1, we define an operator $\mathcal{A}: PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ by

$$
\begin{aligned}
(\mathcal{A}x)(t) ={}& -t\sum_{k=1}^{j}\left(\int_{t_{k-1}}^{t_k} f\big(s,x(s)\big)\,d_{q_{k-1}}s + I_k^*\big(x(t_k)\big)\right)\\
&-\frac{t}{T-\eta}\sum_{k=j+1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f\big(\sigma,x(\sigma)\big)\,d_{q_{k-1}}\sigma\,d_{q_{k-1}}s + I_k\big(x(t_k)\big)\right)\\
&-\frac{t}{T-\eta}\sum_{k=j+1}^{m}\left(\int_{t_{k-1}}^{t_k} f\big(s,x(s)\big)\,d_{q_{k-1}}s + I_k^*\big(x(t_k)\big)\right)(T - t_k)\\
&+\frac{t}{T-\eta}\int_{t_j}^{\eta}\int_{t_j}^{s} f\big(\sigma,x(\sigma)\big)\,d_{q_j}\sigma\,d_{q_j}s\\
&-\frac{t}{T-\eta}\int_{t_m}^{T}\int_{t_m}^{s} f\big(\sigma,x(\sigma)\big)\,d_{q_m}\sigma\,d_{q_m}s\\
&+\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f\big(\sigma,x(\sigma)\big)\,d_{q_{k-1}}\sigma\,d_{q_{k-1}}s + I_k\big(x(t_k)\big)\right)\\
&+\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k} f\big(s,x(s)\big)\,d_{q_{k-1}}s + I_k^*\big(x(t_k)\big)\right)(t - t_k)\\
&+\int_{t_k}^{t}\int_{t_k}^{s} f\big(\sigma,x(\sigma)\big)d_{q_k}\sigma\,d_{q_k}s. \qquad (3.8)
\end{aligned}
$$

It should be noticed that problem (1.1) has solutions if and only if the operator $\mathcal{A}$ has fixed points.

For convenience, we set

$$\Phi_k = \left[(t_k - t_{k-1})(T - t_k) + \frac{(t_k - t_{k-1})^2}{1 + q_{k-1}}\right]M_1 + M_2 + (T - t_k)M_3, \tag{3.9}$$

$$\Psi_k = \left[(t_k - t_{k-1})(T - t_k) + \frac{(t_k - t_{k-1})^2}{1 + q_{k-1}}\right]L_1 + L_2 + (T - t_k)L_3, \tag{3.10}$$

for $k = 1, \dots, m$.

Theorem 3.2 *Assume that*:

(H_1) *The function* $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ *is a continuous and there exists a constant* $L_1 > 0$ *such that* $|f(t,x) - f(t,y)| \le L_1|x - y|$, *for each* $t \in J$ *and* $x, y \in \mathbb{R}$.

(H_2) *The functions* $I_k, I_k^* : \mathbb{R} \to \mathbb{R}$ *are continuous and there exist constants* $L_2, L_3 > 0$ *such that* $|I_k(x) - I_k(y)| \le L_2|x - y|$ *and* $|I_k^*(x) - I_k^*(y)| \le L_3|x - y|$ *for each* $x, y \in \mathbb{R}$, $k = 1, 2, \dots, m$.

If

$$\begin{aligned}\Lambda &:= T\sum_{k=1}^{j}\bigl[(t_k - t_{k-1})L_1 + L_3\bigr] + \frac{T}{T - \eta}\sum_{k=j+1}^{m}\Psi_k + \frac{TL_1}{T - \eta}\left(\frac{(\eta - t_j)^2}{1 + q_j} + \frac{(T - t_m)^2}{1 + q_m}\right) \\ &\quad + \sum_{k=1}^{m}\Psi_k + \frac{(T - t_m)^2}{1 + q_m}L_1 \le \delta < 1, \end{aligned} \tag{3.11}$$

then the impulsive q_k*-difference boundary value problem* (1.1) *has a unique solution on* J.

Proof First, we transform the problem (1.1) into a fixed-point problem, $x = \mathcal{A}x$, where the operator $\mathcal{A}$ is defined by (3.8). By using Banach's contraction principle, we shall show that $\mathcal{A}$ has a fixed point which is the unique solution of problem (1.1).

Set $\sup_{t\in J}|f(t,0)| = M_1 < \infty$, $\sup\{|I_k(0)| : k = 1, 2, \dots, m\} = M_2 < \infty$, $\sup\{|I_k^*(0)| : k = 1, 2, \dots, m\} = M_3 < \infty$ and a constant

$$\begin{aligned}\rho &= T\sum_{k=1}^{j}\bigl[(t_k - t_{k-1})M_1 + M_3\bigr] + \frac{T}{T - \eta}\sum_{k=j+1}^{m}\Phi_k \\ &\quad + \frac{TM_1}{T - \eta}\left(\frac{(\eta - t_j)^2}{1 + q_j} + \frac{(T - t_m)^2}{1 + q_m}\right) \\ &\quad + \sum_{k=1}^{m}\Phi_k + \frac{(T - t_m)^2}{1 + q_m}M_1. \end{aligned} \tag{3.12}$$

Choosing $r \ge \frac{\rho}{1-\varepsilon}$, where $\delta \le \varepsilon < 1$, we show that $\mathcal{A}B_r \subset B_r$, where $B_r = \{x \in PC(J, \mathbb{R}) : \|x\| \le r\}$. For $x \in B_r$, we have

$$\begin{aligned}&\|\mathcal{A}x\| \\ &\quad \le \sup_{t\in J}\left\{t\sum_{k=1}^{j}\left(\int_{t_{k-1}}^{t_k}\bigl|f\bigl(s, x(s)\bigr)\bigr|\,d_{q_{k-1}}s + \bigl|I_k^*\bigl(x(t_k)\bigr)\bigr|\right)\right.\end{aligned}$$

$$
\begin{aligned}
&+\frac{t}{T-\eta}\sum_{k=j+1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\left|f\left(\sigma,x(\sigma)\right)\right|d_{q_{k-1}}\sigma\, d_{q_{k-1}}s+\left|I_k\left(x(t_k)\right)\right|\right)\\
&+\frac{t}{T-\eta}\sum_{k=j+1}^{m}\left(\int_{t_{k-1}}^{t_k}\left|f\left(s,x(s)\right)\right|d_{q_{k-1}}s+\left|I_k^*\left(x(t_k)\right)\right|\right)(T-t_k)\\
&+\frac{t}{T-\eta}\int_{t_j}^{\eta}\int_{t_j}^{s}\left|f\left(\sigma,x(\sigma)\right)\right|d_{q_j}\sigma\, d_{q_j}s+\frac{t}{T-\eta}\int_{t_m}^{T}\int_{t_m}^{s}\left|f\left(\sigma,x(\sigma)\right)\right|d_{q_m}\sigma\, d_{q_m}s\\
&+\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\left|f\left(\sigma,x(\sigma)\right)\right|d_{q_{k-1}}\sigma\, d_{q_{k-1}}s+\left|I_k\left(x(t_k)\right)\right|\right)\\
&+\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\left|f\left(s,x(s)\right)\right|d_{q_{k-1}}s+\left|I_k^*\left(x(t_k)\right)\right|\right)(t-t_k)\\
&+\int_{t_k}^{t}\int_{t_k}^{s}\left|f\left(\sigma,x(\sigma)\right)\right|d_{q_k}\sigma\, d_{q_k}s\Bigg\}\\
\le{}& T\sum_{k=1}^{j}\left(\int_{t_{k-1}}^{t_k}\left(\left|f\left(s,x(s)\right)-f(s,0)\right|+\left|f(s,0)\right|\right)d_{q_{k-1}}s+\left|I_k^*\left(x(t_k)\right)-I_k^*(0)\right|+\left|I_k^*(0)\right|\right)\\
&+\frac{T}{T-\eta}\sum_{k=j+1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\left(\left|f\left(\sigma,x(\sigma)\right)-f(\sigma,0)\right|+\left|f(\sigma,0)\right|\right)d_{q_{k-1}}\sigma\, d_{q_{k-1}}s\right.\\
&\left.+\left|I_k\left(x(t_k)\right)-I_k(0)\right|+\left|I_k(0)\right|\right)\\
&+\frac{T}{T-\eta}\sum_{k=j+1}^{m}\left(\int_{t_{k-1}}^{t_k}\left(\left|f\left(s,x(s)\right)-f(s,0)\right|+\left|f(s,0)\right|\right)d_{q_{k-1}}s\right.\\
&\left.+\left|I_k^*\left(x(t_k)\right)-I_k^*(0)\right|+\left|I_k^*(0)\right|\right)(T-t_k)\\
&+\frac{T}{T-\eta}\int_{t_j}^{\eta}\int_{t_j}^{s}\left(\left|f\left(\sigma,x(\sigma)\right)-f(\sigma,0)\right|+\left|f(\sigma,0)\right|\right)d_{q_j}\sigma\, d_{q_j}s\\
&+\frac{T}{T-\eta}\int_{t_m}^{T}\int_{t_m}^{s}\left(\left|f\left(\sigma,x(\sigma)\right)-f(\sigma,0)\right|+\left|f(\sigma,0)\right|\right)d_{q_m}\sigma\, d_{q_m}s\\
&+\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\left(\left|f\left(\sigma,x(\sigma)\right)-f(\sigma,0)\right|+\left|f(\sigma,0)\right|\right)d_{q_{k-1}}\sigma\, d_{q_{k-1}}s\right.\\
&\left.+\left|I_k\left(x(t_k)\right)-I_k(0)\right|+\left|I_k(0)\right|\right)\\
&+\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\left(\left|f\left(s,x(s)\right)-f(s,0)\right|+\left|f(s,0)\right|\right)d_{q_{k-1}}s\right.\\
&\left.+\left|I_k^*\left(x(t_k)\right)-I_k^*(0)\right|+\left|I_k^*(0)\right|\right)(T-t_k)\\
&+\int_{t_m}^{T}\int_{t_m}^{s}\left(\left|f\left(\sigma,x(\sigma)\right)-f(\sigma,0)\right|+\left|f(\sigma,0)\right|\right)d_{q_m}\sigma\, d_{q_m}s\\
\le{}& T\sum_{k=1}^{j}\left((t_k-t_{k-1})(L_1r+M_1)+L_3r+M_3\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{T}{T-\eta}\sum_{k=j+1}^{m}\left(\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}(L_1r+M_1)+L_2r+M_2\right)\\
&+\frac{T}{T-\eta}\sum_{k=j+1}^{m}\big((t_k-t_{k-1})(L_1r+M_1)+L_3r+M_3\big)(T-t_k)\\
&+\frac{T}{T-\eta}\left(\frac{(\eta-t_j)^2}{1+q_j}+\frac{(T-t_m)^2}{1+q_m}\right)(L_1r+M_1)\\
&+\sum_{k=1}^{m}\left(\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}(L_1r+M_1)+L_2r+M_2\right)\\
&+\sum_{k=1}^{m}\big((t_k-t_{k-1})(L_1r+M_1)+L_3r+M_3\big)(T-t_k)+\frac{(T-t_m)^2}{1+q_m}(L_1r+M_1)\\
={}&r\Lambda+\rho\le(\delta+1-\varepsilon)r\le r.
\end{aligned}
$$

It follows that $\mathcal{A}B_r\subset B_r$.

For $x,y\in PC(J,\mathbb{R})$ and for each $t\in J$, we have

$$
\begin{aligned}
&\|\mathcal{A}x-\mathcal{A}y\|\\
&\le\sup_{t\in J}\Bigg\{t\sum_{k=1}^{j}\left(\int_{t_{k-1}}^{t_k}\big|f(s,x(s))-f(s,y(s))\big|\,d_{q_{k-1}}s+\big|I_k^*(x(t_k))-I_k^*(y(t_k))\big|\right)\\
&\quad+\frac{t}{T-\eta}\sum_{k=j+1}^{m}\bigg(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\big|f(\sigma,x(\sigma))-f(\sigma,y(\sigma))\big|\,d_{q_{k-1}}\sigma\,d_{q_{k-1}}s\\
&\quad+\big|I_k(x(t_k))-I_k(y(t_k))\big|\bigg)\\
&\quad+\frac{t}{T-\eta}\sum_{k=j+1}^{m}\left(\int_{t_{k-1}}^{t_k}\big|f(s,x(s))-f(s,y(s))\big|\,d_{q_{k-1}}s+\big|I_k^*(x(t_k))-I_k^*(y(t_k))\big|\right)(T-t_k)\\
&\quad+\frac{t}{T-\eta}\int_{t_j}^{\eta}\int_{t_j}^{s}\big|f(\sigma,x(\sigma))-f(\sigma,y(\sigma))\big|\,d_{q_j}\sigma\,d_{q_j}s\\
&\quad+\frac{t}{T-\eta}\int_{t_m}^{T}\int_{t_m}^{s}\big|f(\sigma,x(\sigma))-f(\sigma,y(\sigma))\big|\,d_{q_m}\sigma\,d_{q_m}s\\
&\quad+\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\big|f(\sigma,x(\sigma))-f(\sigma,y(\sigma))\big|\,d_{q_{k-1}}\sigma\,d_{q_{k-1}}s+\big|I_k(x(t_k))-I_k(y(t_k))\big|\right)\\
&\quad+\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\big|f(s,x(s))-f(s,y(s))\big|\,d_{q_{k-1}}s+\big|I_k^*(x(t_k))-I_k^*(y(t_k))\big|\right)(t-t_k)\\
&\quad+\int_{t_k}^{t}\int_{t_k}^{s}\big|f(\sigma,x(\sigma))-f(\sigma,y(\sigma))\big|\,d_{q_k}\sigma\,d_{q_k}s\Bigg\}\\
&\le T\|x-y\|\sum_{k=1}^{j}\big[(t_k-t_{k-1})L_1+L_3\big]+\frac{T\|x-y\|}{T-\eta}\sum_{k=j+1}^{m}\left(\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}L_1+L_2\right)\\
&\quad+\frac{T\|x-y\|}{T-\eta}\sum_{k=j+1}^{m}\big((t_k-t_{k-1})L_1+L_3\big)(T-t_k)
\end{aligned}
$$

$$\begin{aligned}
&+ \frac{T\|x-y\|}{T-\eta}\left(\frac{(\eta-t_j)^2}{1+q_j} + \frac{(T-t_m)^2}{1+q_m}\right)L_1 \\
&+ \sum_{k=1}^{m}\left(\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}L_1 + L_2\right)\|x-y\| + \sum_{k=1}^{m}\big((t_k-t_{k-1})L_1 + L_3\big)(T-t_k)\|x-y\| \\
&+ \frac{(T-t_m)^2}{1+q_m}L_1\|x-y\| \\
&= \Lambda\|x-y\|.
\end{aligned}$$

As $\Lambda < 1$, $\mathcal{A}$ is a contraction. Hence, by Banach's contraction mapping principle, we find that $\mathcal{A}$ has a fixed point which is the unique solution of problem (1.1). □

Our next result is based on Krasnoselskii's fixed-point theorem.

Lemma 3.3 (Krasnoselskii's fixed-point theorem) [19] *Let M be a closed, bounded, convex and nonempty subset of a Banach space X. Let A, B be the operators such that* (a) $Ax + By \in M$ *whenever* $x, y \in M$; (b) *A is compact and continuous*; (c) *B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Further, we use the notation

$$\begin{aligned}
\theta_1 &= T\sum_{k=1}^{j}(t_k-t_{k-1}) + \frac{T}{T-\eta}\sum_{k=j+1}^{m}\frac{(t_k-t_{k-1})^2}{1+q_{k-1}} \\
&+ \frac{T}{T-\eta}\sum_{k=j+1}^{m}(T-t_k)(t_k-t_{k-1}) + \frac{T(\eta-t_j)^2}{(T-\eta)(1+q_j)} \\
&+ \frac{T(T-t_m)^2}{(T-\eta)(1+q_m)} + \sum_{k=1}^{m+1}\frac{(t_k-t_{k-1})^2}{1+q_{k-1}} + \sum_{k=1}^{m}(T-t_k)(t_k-t_{k-1}),
\end{aligned} \tag{3.13}$$

and

$$\theta_2 = jTN_2 + \frac{(m-j)TN_1}{T-\eta} + mN_1 + N_2\sum_{k=1}^{m}(T-t_k) + \frac{TN_2}{T-\eta}\sum_{j+1}^{m}(T-t_k). \tag{3.14}$$

Theorem 3.4 *Let $f : J \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that* (H_2) *holds and in addition suppose that*:

(H_3) $|f(t,x)| \le \mu(t)$, $\forall(t,x) \in J \times \mathbb{R}$, *and* $\mu \in C(J, \mathbb{R}^+)$.
(H_4) *There exist constants $N_1, N_2 > 0$ such that $|I_k(x)| \le N_1$ and $|I_k^*(x)| \le N_2$ for all $x \in \mathbb{R}$, for $k = 1, 2, \ldots, m$.*

Then the impulsive q_k-difference boundary value problem (1.1) *has at least one solution on J provided that*

$$jTL_3 + mL_2 + \frac{T(m-j)L_2}{T-\eta} + L_3\sum_{k=1}^{m}(T-t_k) < 1. \tag{3.15}$$

Proof Firstly, we define $\sup_{t\in J}|\mu(t)| = \|\mu\|$. Choosing a suitable ball $B_R = \{x \in PC(J,\mathbb{R}) : \|x\| \leq R\}$, where

$$R \geq \|\mu\|\theta_1 + \theta_2, \tag{3.16}$$

and θ_1, θ_2 are defined by (3.13), (3.14), respectively, we define the operators $\mathcal{S}_1$ and $\mathcal{S}_2$ on B_R by

$$\begin{aligned}
&(\mathcal{S}_1x)(t)\\
&= -t\sum_{k=1}^{j}\int_{t_{k-1}}^{t_k} f\big(s,x(s)\big)\,d_{q_{k-1}}s - \frac{t}{T-\eta}\sum_{k=j+1}^{m}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f\big(\sigma,x(\sigma)\big)\,d_{q_{k-1}}\sigma\,d_{q_{k-1}}s\\
&\quad - \frac{t}{T-\eta}\sum_{k=j+1}^{m}(T-t_k)\int_{t_{k-1}}^{t_k} f\big(s,x(s)\big)\,d_{q_{k-1}}s + \frac{t}{T-\eta}\int_{t_j}^{\eta}\int_{t_j}^{s} f\big(\sigma,x(\sigma)\big)\,d_{q_j}\sigma\,d_{q_j}s\\
&\quad - \frac{t}{T-\eta}\int_{t_m}^{T}\int_{t_m}^{s} f\big(\sigma,x(\sigma)\big)\,d_{q_m}\sigma\,d_{q_m}s + \sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f\big(\sigma,x(\sigma)\big)\,d_{q_{k-1}}\sigma\,d_{q_{k-1}}s\\
&\quad + \sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k} f\big(s,x(s)\big)\,d_{q_{k-1}}s + \int_{t_k}^{t}\int_{t_k}^{s} f\big(\sigma,x(\sigma)\big)\,d_{q_k}\sigma\,d_{q_k}s, \quad t\in[0,T],
\end{aligned}$$

and

$$\begin{aligned}
(\mathcal{S}_2x)(t) &= -t\sum_{k=1}^{j} I_k^*\big(x(t_k)\big) - \frac{t}{T-\eta}\sum_{k=j+1}^{m} I_k\big(x(t_k)\big) - \frac{t}{T-\eta}\sum_{k=j+1}^{m}(T-t_k)I_k^*\big(x(t_k)\big)\\
&\quad + \sum_{0<t_k<t} I_k\big(x(t_k)\big) + \sum_{0<t_k<t}(t-t_k)I_k^*\big(x(t_k)\big), \quad t\in[0,T].
\end{aligned}$$

For any $x, y \in B_R$, we have

$$\begin{aligned}
\|\mathcal{S}_1x + \mathcal{S}_2y\| &\leq \|\mu\|\Bigg[T\sum_{k=1}^{j}(t_k - t_{k-1}) + \frac{T}{T-\eta}\sum_{k=j+1}^{m}\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}\\
&\quad + \frac{T}{T-\eta}\sum_{k=j+1}^{m}(T-t_k)(t_k-t_{k-1}) + \frac{T(\eta-t_j)^2}{(T-\eta)(1+q_j)}\\
&\quad + \frac{T(T-t_m)^2}{(T-\eta)(1+q_m)} + \sum_{k=1}^{m+1}\frac{(t_k-t_{k-1})^2}{1+q_{k-1}} + \sum_{k=1}^{m}(T-t_k)(t_k-t_{k-1})\Bigg]\\
&\quad + jTN_2 + \frac{(m-j)TN_1}{T-\eta} + mN_1 + N_2\sum_{k=1}^{m}(T-t_k) + \frac{TN_2}{T-\eta}\sum_{j+1}^{m}(T-t_k)\\
&= \|\mu\|\theta_1 + \theta_2\\
&\leq R.
\end{aligned}$$

Hence, $\mathcal{S}_1x + \mathcal{S}_2y \in B_R$.

To show that $\mathcal{S}_2$ is a contraction, for $x, y \in PC(J, \mathbb{R})$, we have

$$\begin{aligned}
\|\mathcal{S}_2 x - \mathcal{S}_2 y\| &\le T \sum_{k=1}^{j} \left| I_k^*\big(x(t_k)\big) - I_k^*\big(y(t_k)\big)\right| + \frac{T}{T-\eta} \sum_{k=j+1}^{m} \left| I_k\big(x(t_k)\big) - I_k\big(y(t_k)\big)\right| \\
&\quad + \sum_{k=1}^{m} \left| I\big(x(t_k)\big) - I_k\big(y(t_k)\big)\right| + \sum_{k=1}^{m} (t - t_k)\left| I_k^*\big(x(t_k)\big) - I_k^*\big(y(t_k)\big)\right| \\
&\le \left[jTL_3 + mL_2 + \frac{T(m-j)L_2}{T-\eta} + L_3 \sum_{k=1}^{m} (T - t_k)\right] \|x - y\|.
\end{aligned}$$

From (3.15), it follows that $\mathcal{S}_2$ is a contraction.

Next, the continuity of f implies that the operator $\mathcal{S}_1$ is continuous. Further, $\mathcal{S}_1$ is uniformly bounded on B_R by

$$\|\mathcal{S}_1 x\| \le \|\mu\| \theta_1.$$

Now we shall prove the compactness of $\mathcal{S}_1$. Setting $\sup_{(t,x)\in J\times B_R} |f(t,x)| = f^* < \infty$, then for each $\tau_1, \tau_2 \in (t_l, t_{l+1})$ for some $l \in \{0, 1, \ldots, m\}$ with $\tau_2 > \tau_1$, we have

$$\begin{aligned}
&\left|(\mathcal{S}_1 x)(\tau_2) - (\mathcal{S}_1 x)(\tau_1)\right| \\
&\quad\le |\tau_2 - \tau_1| \sum_{k=1}^{j} \int_{t_{k-1}}^{t_k} \left| f\big(s, x(s)\big)\right| d_{q_{k-1}} s \\
&\qquad + \frac{|\tau_2 - \tau_1|}{T-\eta} \sum_{k=j+1}^{m} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} \left| f\big(\sigma, x(\sigma)\big)\right| d_{q_{k-1}}\sigma \, d_{q_{k-1}} s \\
&\qquad + \frac{|\tau_2 - \tau_1|}{T-\eta} \sum_{k=j+1}^{m} (T - t_k) \int_{t_{k-1}}^{t_k} \left| f\big(s, x(s)\big)\right| d_{q_{k-1}} s \\
&\qquad + \frac{|\tau_2 - \tau_1|}{T-\eta} \int_{t_j}^{\eta} \int_{t_j}^{s} \left| f\big(\sigma, x(\sigma)\big)\right| d_{q_j}\sigma \, d_{q_j} s \\
&\qquad + \frac{|\tau_2 - \tau_1|}{T-\eta} \int_{t_m}^{T} \int_{t_m}^{s} \left| f\big(\sigma, x(\sigma)\big)\right| d_{q_m}\sigma \, d_{q_m} s + |\tau_2 - \tau_1| \sum_{k=1}^{l} \int_{t_{k-1}}^{t_k} \left| f\big(s, x(s)\big)\right| d_{q_{k-1}} s \\
&\qquad + \left| \int_{t_l}^{\tau_2} \int_{t_l}^{s} \left| f\big(\sigma, x(\sigma)\big)\right| d_{q_l}\sigma \, d_{q_l} s - \int_{t_l}^{\tau_1} \int_{t_l}^{s} \left| f\big(\sigma, x(\sigma)\big)\right| d_{q_l}\sigma \, d_{q_l} s \right| \\
&\quad\le |\tau_2 - \tau_1| f^* \left[\sum_{k=1}^{j} (t_k - t_{k-1}) + \frac{1}{T-\eta} \sum_{k=j+1}^{m} \frac{(t_k - t_{k-1})^2}{1 + q_{k-1}} + \frac{(\eta - t_j)^2}{(T-\eta)(1+q_j)} \right. \\
&\qquad + \frac{(T - t_m)^2}{(T-\eta)(1+q_m)} + \frac{1}{T-\eta} \sum_{k=j+1}^{m} (T - t_k)(t_k - t_{k-1}) \\
&\qquad \left. + \sum_{k=1}^{l} (t_k - t_{k-1}) + \frac{(\tau_1 + \tau_2 + 2t_l)}{1 + q_l} \right].
\end{aligned}$$

As $\tau_1 \to \tau_2$, the right hand side above (which is independent of x) tends to zero. Therefore, the operator $\mathcal{S}_1$ is equicontinuous. Since $\mathcal{S}_1$ maps bounded subsets into relatively compact subsets, it follows that $\mathcal{S}_1$ is relative compact on B_R. Hence, by the Arzelá-Ascoli theorem, $\mathcal{S}_1$ is compact on B_R. Thus all the assumptions of Lemma 3.3 are satisfied. Hence, by the conclusion of Lemma 3.3, the impulsive q_k-difference boundary value problem (1.1) has at least one solution on J. □

4 Examples

Example 4.1 Consider the following nonlinear second-order impulsive q_k-difference equation with three-point boundary condition:

$$\begin{cases} D^2_{\frac{4}{5+k}}x(t) = \frac{e^{-\cos^2 t}|x(t)|}{(6+t)^2(1+|x(t)|)}, & t \in J = [0,1], t \neq t_k = \frac{k}{10}, \\ \Delta x(t_k) = \frac{|x(t_k)|}{8(7+|x(t_k)|)}, & k = 1,2,\ldots,9, \\ D_{\frac{4}{5+k}}x(t_k^+) - D_{\frac{4}{4+k}}x(t_k) = \frac{1}{6}\tan^{-1}(\frac{1}{8}x(t_k)), & k = 1,2,\ldots,9, \\ x(0) = 0, \quad x(1) = x(\frac{1}{4}). \end{cases} \tag{4.1}$$

Here $q_k = 4/(5+k)$ for $k = 0,1,2,\ldots,9$, $m = 9$, $T = 1$, $\eta = 1/4$, $j = 2$, $f(t,x) = (e^{-\cos^2 t}|x|)/((6+t)^2(1+|x|))$, $I_k(x) = |x|/(8(7+|x|))$ and $I_k^*(x) = (1/6)\tan^{-1}(x/8)$. Since

$$\bigl|f(t,x) - f(t,y)\bigr| \leq (1/36)|x-y|,$$

$$\bigl|I_k(x) - I_k(y)\bigr| \leq (1/56)|x-y| \quad \text{and} \quad \bigl|I_k^*(x) - I_k^*(y)\bigr| \leq (1/48)|x-y|,$$

then (H_1) and (H_2) are satisfied with $L_1 = (1/36)$, $L_2 = (1/56)$, $L_3 = (1/48)$. We can show that

$$\Lambda \approx 0.5730986482 < 1.$$

Hence, by Theorem 3.2, the three-point impulsive q_k-difference boundary value problem (4.1) has a unique solution on $[0,1]$.

Example 4.2 Consider the following nonlinear second-order impulsive q_k-difference equation with three-point boundary condition:

$$\begin{cases} D^2_{\frac{3}{6+k}}x(t) = \frac{\sin^2(\pi t)}{(t+4)^2}\frac{|x(t)|}{(1+|x(t)|)}, & t \in J = [0,1], t \neq t_k = \frac{k}{10}, \\ \Delta x(t_k) = \frac{|x(t_k)|}{9(7+|x(t_k)|)}, & k = 1,2,\ldots,9, \\ D_{\frac{3}{6+k}}x(t_k^+) - D_{\frac{3}{5+k}}x(t_k) = \frac{|x(t_k)|}{4(5+|x(t_k)|)}, & k = 1,2,\ldots,9, \\ x(0) = 0, \quad x(1) = x(\frac{9}{20}). \end{cases} \tag{4.2}$$

Set $q_k = 3/(6+k)$ for $k = 0,1,2,\ldots,9$, $m = 9$, $T = 1$, $\eta = 9/20$, $j = 4$, $f(t,x) = (\sin^2(\pi t)|x|)/((t+4)^2(1+|x|))$, $I_k(x) = |x|/(9(7+|x|))$ and $I_k^*(x) = |x|/(4(5+|x|))$. Since

$$\bigl|I_k(x) - I_k(y)\bigr| \leq (1/63)|x-y| \quad \text{and} \quad \bigl|I_k^*(x) - I_k^*(y)\bigr| \leq (1/20)|x-y|,$$

then (H_2) is satisfied with $L_2 = (1/63)$, $L_3 = (1/20)$. It is easy to verify that $|f(t,x)| \leq \mu(t) \equiv 1$, $I_k(x) \leq N_1 = 1/9$ and $I_k^*(x) \leq N_2 = 1/4$ for all $t \in [0,1]$, $x \in \mathbb{R}$, $k = 1,\ldots,m$. Thus (H_3) and (H_4) are satisfied. We can show that

$$jTL_3 + mL_2 + \frac{T(m-j)L_2}{T-\eta} + L_3\sum_{k=1}^{m}(T-t_k) = \frac{19741}{27720} < 1.$$

Hence, by Theorem 3.3, the three-point impulsive q_k-difference boundary value problem (4.2) has at least one solution on $[0,1]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally in this article. They read and approved the final manuscript.

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