

RESEARCH

Open Access

Singular integral equation involving a multivariable analog of Mittag-Leffler function

Sebastien Gaboury^{1*} and Mehmet Ali Özarslan²*Correspondence:
s1gabour@uqac.ca¹Department of Mathematics and
Computer Science, University of
Quebec at Chicoutimi, Quebec,
G7H 2B1, CanadaFull list of author information is
available at the end of the article**Abstract**

Motivated by the recent work of the second author (Özarslan in *Appl. Math. Comput.* 229:350-358, 2014), we present, in this paper, some fractional calculus formulas for a mild generalization of the multivariable Mittag-Leffler function, a Schläfli's type contour integral representation, some multilinear and mixed multilateral generating functions; and, finally, we consider a singular integral equation with the function $E_{(\rho_r), \lambda}^{(\gamma_r), (1)}(x_1, \dots, x_r)$ in the kernel and we provide its solution.

MSC: 26A33; 33E12**Keywords:** fractional integrals and derivatives; Mittag-Leffler function; contour integral representation; generating functions; singular integral equation; Laplace transform

1 Introduction

The celebrated Mittag-Leffler function [1, 2] is defined by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \quad (1.1)$$
$$(\alpha \in \mathbb{C}; \Re(\alpha) > 0; z \in \mathbb{C}),$$

where \mathbb{C} denotes the set of complex numbers.

The Mittag-Leffler function arises naturally in the solution of fractional integral equations [3]. A generalization of the Mittag-Leffler function $E_{\alpha}(z)$ has been investigated by Wiman [4]. He studied the following function:

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (1.2)$$
$$(\alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0; \Re(\beta) > 0; z \in \mathbb{C}).$$

Other generalizations of the Mittag-Leffler functions were given in [5, 6]. Let us recall the one given by Srivastava and Tomovski [6]:

$$E_{\alpha,\beta}^{\gamma,K}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{Kk}}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!} \tag{1.3}$$

$$(\alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > \max\{0, \Re(K) - 1\}; \Re(K) > 0; \Re(\beta) > 0; z \in \mathbb{C}),$$

where $(\lambda)_\kappa$ denotes the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_\kappa := \frac{\Gamma(\lambda + \kappa)}{\Gamma(\lambda)} = \begin{cases} \lambda(\lambda + 1) \cdots (\lambda + \kappa - 1) & (\kappa = n \in \mathbb{N}; \lambda \in \mathbb{C}), \\ 1 & (\kappa = 0; \lambda \in \mathbb{C} \setminus \{0\}), \end{cases} \tag{1.4}$$

where \mathbb{N} denotes the set of positive integers.

Multivariable analog of the Mittag-Leffler function has been introduced and investigated by Saxena *et al.* [7, p.536, Eq. (1.14)] in the following form:

$$\begin{aligned} E_{(\rho_r),\lambda}^{(\gamma_r)}(z_1, \dots, z_r) &= E_{(\rho_1, \dots, \rho_r),\lambda}^{(\gamma_1, \dots, \gamma_r)}(z_1, \dots, z_r) \\ &= \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(\gamma_1)_{k_1} \cdots (\gamma_r)_{k_r}}{\Gamma(k_1 \rho_1 + \cdots + k_r \rho_r + \lambda)} \frac{z_1^{k_1} \cdots z_r^{k_r}}{k_1! \cdots k_r!} \end{aligned} \tag{1.5}$$

$$(\lambda, z_j, \gamma_j, \rho_j \in \mathbb{C}; \Re(\rho_j) > 0; j = 1, 2, \dots, r).$$

This function is, in fact, a special case of the generalized Lauricella series in several variables, introduced by Srivastava and Daoust [8] and Srivastava and Karlsson [9].

A mild generalization of the multivariable analog of the Mittag-Leffler function, which will play an important role in this paper, has been given by Saxena *et al.* [7, p.547, Eq. (7.1)]:

$$E_{(\rho_r),\lambda}^{(\gamma_r),(l_r)}(z_1, \dots, z_r) = \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(\gamma_1)_{k_1 l_1} \cdots (\gamma_r)_{k_r l_r}}{\Gamma(k_1 \rho_1 + \cdots + k_r \rho_r + \lambda)} \frac{z_1^{k_1} \cdots z_r^{k_r}}{k_1! \cdots k_r!} \tag{1.6}$$

$$(\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-; \gamma_j, \rho_j, l_j \in \mathbb{C}; \Re(\rho_j) > 0; \Re(l_j) > 0; j = 1, 2, \dots, r).$$

Recently, the second author in [10] introduced a class of polynomials suggested by the multivariate Laguerre polynomials in the following form:

$$\begin{aligned} Z_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r; \rho_1, \dots, \rho_r) \\ = \frac{\Gamma(\rho_1 n_1 + \cdots + \rho_r n_r + \alpha + 1)}{n_1! \cdots n_r!} \sum_{k_1, \dots, k_r}^{n_1, \dots, n_r} \frac{(-n_1)_{k_1} \cdots (-n_r)_{k_r} x_1^{\rho_1 k_1} \cdots x_r^{\rho_r k_r}}{\Gamma(\rho_1 k_1 + \cdots + \rho_r k_r + \alpha + 1) k_1! \cdots k_r!} \end{aligned} \tag{1.7}$$

$$(\alpha, \rho_1, \dots, \rho_r \in \mathbb{C}; \Re(\rho_j) > 0 (j = 1, 2, \dots, r)).$$

It is easy to see that the following relation between the class of polynomials given by (1.7) and the generalized multivariable Mittag-Leffler function (1.6) exists:

$$\begin{aligned} Z_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r; \rho_1, \dots, \rho_r) \\ = \frac{\Gamma(\rho_1 n_1 + \cdots + \rho_r n_r + \alpha + 1)}{n_1! \cdots n_r!} E_{(\rho_1, \dots, \rho_r), \alpha + 1}^{(-n_1, \dots, -n_r), (1, \dots, 1)}(x_1^{\rho_1}, \dots, x_r^{\rho_r}). \end{aligned} \tag{1.8}$$

Note that by further specializing the several parameters involved, we can obtain many well-known classes of polynomials such as the Laguerre polynomials of r variables defined by Erdélyi [11] and the Konhauser polynomials [12].

Another interesting generalization of the polynomials $Z_{n_1, \dots, n_r}^{(\alpha)}(x_1, \dots, x_r; \rho_1, \dots, \rho_r)$ is given by

$$\begin{aligned}
 & Z_{n_1, \dots, n_r}^{(\alpha; N_1, \dots, N_r)}(x_1, \dots, x_r; \rho_1, \dots, \rho_r) \\
 &= \frac{\Gamma(\rho_1 n_1 + \dots + \rho_r n_r + \alpha + 1)}{n_1! \dots n_r!} \sum_{k_1, \dots, k_r=0}^{\lfloor \frac{n_1}{N_1} \rfloor, \dots, \lfloor \frac{n_r}{N_r} \rfloor} \frac{(-n_1)_{N_1 k_1} \dots (-n_r)_{N_r k_r} x_1^{\rho_1 k_1} \dots x_r^{\rho_r k_r}}{\Gamma(\rho_1 k_1 + \dots + \rho_r k_r + \alpha + 1) k_1! \dots k_r!} \quad (1.9) \\
 & (\alpha, \rho_1, \dots, \rho_r \in \mathbb{C}, \Re(\rho_i) > 0, N_i \in \mathbb{N} (i = 1, \dots, r)).
 \end{aligned}$$

Obviously, setting $N_i = 1 (i = 1, \dots, r)$ leads to (1.8).

In this paper, we obtain a Schläfli's type contour integral representation for the multi-variable polynomials given in (1.9). Next, we give some multilinear and mixed multilateral generating functions. We also recall the fractional order integral of the generalized multivariable Mittag-Leffler function. Finally, we consider a singular integral equation with $E_{(\rho_r), \lambda}^{(\gamma_r), (1)}(x_1, \dots, x_r)$ in the kernel and we give its solution. Throughout this paper, the variables x_1, \dots, x_r are assumed to be real variables.

2 Schläfli's type contour integral representation of $Z_{n_1, \dots, n_r}^{(\alpha; N_1, \dots, N_r)}(x_1, \dots, x_r; \rho_1, \dots, \rho_r)$

Let us define the following polynomials set:

$$\begin{aligned}
 & P_{n_1, \dots, n_r}^{(N_1, \dots, N_r)}(x_1, \dots, x_r; \rho_1, \dots, \rho_r) \\
 &:= \sum_{k_1, \dots, k_r=0}^{\lfloor \frac{n_1}{N_1} \rfloor, \dots, \lfloor \frac{n_r}{N_r} \rfloor} (-n_1)_{N_1 k_1} \dots (-n_r)_{N_r k_r} \frac{x_1^{\rho_1 k_1} \dots x_r^{\rho_r k_r}}{k_1! \dots k_r!} \quad (2.1) \\
 & (\rho_j \in \mathbb{C}; \Re(\rho_j) > 0; N_j \in \mathbb{N} (j = 1, \dots, r)).
 \end{aligned}$$

The Schläfli's type contour integral representation of $Z_{n_1, \dots, n_r}^{(\alpha; N_1, \dots, N_r)}(x_1, \dots, x_r; \rho_1, \dots, \rho_r)$ in terms of $P_{n_1, \dots, n_r}^{(N_1, \dots, N_r)}(x_1, \dots, x_r; \rho_1, \dots, \rho_r)$ is given in the next theorem.

Theorem 2.1 *Let $\alpha, \rho_j \in \mathbb{C}$ with $\Re(\rho_j) > 0 (j = 1, \dots, r)$ and let $N_j \in \mathbb{N} (j = 1, \dots, r)$. Then the following integral representation holds true:*

$$\begin{aligned}
 & Z_{n_1, \dots, n_r}^{(\alpha; N_1, \dots, N_r)}(x_1, \dots, x_r; \rho_1, \dots, \rho_r) \\
 &= \frac{\Gamma(\rho_1 n_1 + \dots + \rho_r n_r + \alpha + 1)}{n_1! \dots n_r!} \frac{1}{2\pi i} \\
 & \times \int_{-\infty}^{(0+)} P_{n_1, \dots, n_r}^{(N_1, \dots, N_r)}\left(\frac{x_1}{t}, \dots, \frac{x_r}{t}; \rho_1, \dots, \rho_r\right) t^{-\alpha-1} e^t dt. \quad (2.2)
 \end{aligned}$$

Proof We have

$$\begin{aligned} & \frac{1}{2\pi i} \int_{-\infty}^{(0+)} P_{n_1, \dots, n_r}^{(N_1, \dots, N_r)} \left(\frac{x_1}{t}, \dots, \frac{x_r}{t}; \rho_1, \dots, \rho_r \right) t^{-\alpha-1} e^t dt \\ &= \frac{1}{2\pi i} \int_{-\infty}^{(0+)} \sum_{k_1, \dots, k_r=0}^{\lfloor \frac{n_1}{N_1} \rfloor, \dots, \lfloor \frac{n_r}{N_r} \rfloor} \frac{(-n_1)_{N_1 k_1} \cdots (-n_r)_{N_r k_r}}{k_1! \cdots k_r!} x_1^{\rho_1 k_1} \cdots x_r^{\rho_r k_r} \frac{e^t}{t^{\rho_1 k_1 + \cdots + \rho_r k_r + \alpha + 1}} dt. \end{aligned} \tag{2.3}$$

With the help of Hankel’s formula [13]

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} t^{-z} e^t dt, \tag{2.4}$$

we find from (2.3) and (2.4) the result asserted by Theorem 2.1. □

3 Multilinear and multilateral generating functions

We begin this section by proving a linear generating function for the polynomials $Z_{n_1, \dots, n_j}^{(\alpha; N_1, \dots, N_j)}(x_1, \dots, x_j; \rho_1, \dots, \rho_j)$ by means of the mild generalization of the multivariate analog of Mittag-Leffler functions.

Theorem 3.1 *We have*

$$\begin{aligned} & \sum_{n_1, \dots, n_j=0}^{\infty} \frac{(\gamma_1)_{n_1} \cdots (\gamma_j)_{n_j} Z_{n_1, \dots, n_j}^{(\alpha; N_1, \dots, N_j)}(x_1, \dots, x_j; \rho_1, \dots, \rho_j)}{\Gamma(\rho_1 n_1 + \cdots + \rho_j n_j + \alpha + 1)} t_1^{n_1} \cdots t_j^{n_j} \\ &= \prod_{i=1}^j (1 - t_i)^{-\gamma_i} E_{\rho_1, \dots, \rho_j, \alpha + 1}^{(\gamma_j), (N_j)} \left(\frac{x_1^{\rho_1} (-t_1)^{N_1}}{(1 - t_1)^{N_1}}, \dots, \frac{x_j^{\rho_j} (-t_j)^{N_j}}{(1 - t_j)^{N_j}} \right), \end{aligned}$$

where $|t_i| < 1$ ($i = 1, \dots, j$).

Proof Direct calculations yield

$$\begin{aligned} & \sum_{n_1, \dots, n_j=0}^{\infty} \frac{(\gamma_1)_{n_1} \cdots (\gamma_j)_{n_j} Z_{n_1, \dots, n_j}^{(\alpha; N_1, \dots, N_j)}(x_1, \dots, x_j; \rho_1, \dots, \rho_j)}{\Gamma(\rho_1 n_1 + \cdots + \rho_j n_j + \alpha + 1)} t_1^{n_1} \cdots t_j^{n_j} \\ &= \sum_{n_1, \dots, n_j=0}^{\infty} \sum_{k_1, \dots, k_j=0}^{\lfloor \frac{n_1}{N_1} \rfloor, \dots, \lfloor \frac{n_j}{N_j} \rfloor} \frac{(-1)^{N_1 k_1 + \cdots + N_j k_j} (\gamma_1)_{n_1} \cdots (\gamma_j)_{n_j} x_1^{\rho_1 k_1} \cdots x_j^{\rho_j k_j} t_1^{n_1} \cdots t_j^{n_j}}{\Gamma(\rho_1 k_1 + \cdots + \rho_j k_j + \alpha + 1) (n_1 - N_1 k_1)! \cdots (n_j - N_j k_j)! k_1! \cdots k_j!} \\ &= \sum_{n_1, \dots, n_j=0}^{\infty} \sum_{k_1, \dots, k_j=0}^{\infty} \frac{(-1)^{N_1 k_1 + \cdots + N_j k_j} (\gamma_1)_{n_1 + N_1 k_1} \cdots (\gamma_j)_{n_j + N_j k_j} x_1^{\rho_1 k_1} \cdots x_j^{\rho_j k_j} t_1^{n_1 + N_1 k_1} \cdots t_j^{n_j + N_j k_j}}{\Gamma(\rho_1 k_1 + \cdots + \rho_j k_j + \alpha + 1) n_1! \cdots n_j! k_1! \cdots k_j!} \\ &= \sum_{k_1, \dots, k_j=0}^{\infty} \frac{(\gamma_1)_{N_1 k_1} \cdots (\gamma_j)_{N_j k_j} (x_1^{\rho_1} (-t_1)^{N_1})^{k_1} \cdots (x_j^{\rho_j} (-t_j)^{N_j})^{k_j}}{\Gamma(\rho_1 k_1 + \cdots + \rho_j k_j + \alpha + 1) k_1! \cdots k_j!} \\ &\quad \times \sum_{n_1, \dots, n_j=0}^{\infty} \frac{(\gamma_1 + N_1 k_1)_{n_1} \cdots (\gamma_j + N_j k_j)_{n_j}}{n_1! \cdots n_j!} t_1^{n_1} \cdots t_j^{n_j} \\ &= \prod_{i=1}^j (1 - t_i)^{-\gamma_i} E_{\rho_1, \dots, \rho_j, \alpha + 1}^{(\gamma_j), (N_j)} \left(\frac{x_1^{\rho_1} (-t_1)^{N_1}}{(1 - t_1)^{N_1}}, \dots, \frac{x_j^{\rho_j} (-t_j)^{N_j}}{(1 - t_j)^{N_j}} \right), \end{aligned}$$

where we have interchanged the order of summations which is guaranteed because of the uniform convergence of the series under the conditions $|t_i| < 1$ ($i = 1, \dots, j$). \square

Now let $(\gamma) := (\gamma_1, \dots, \gamma_j)$, $(\lambda) := (\lambda_1, \dots, \lambda_j)$, $(\eta) := (\eta_1, \dots, \eta_j)$, $(\psi) := (\psi_1, \dots, \psi_j)$, $(\rho) := (\rho_1, \dots, \rho_j)$, $(N) := (N_1, \dots, N_j)$ be complex j -tuples. By making use of the above theorem we have the following.

Theorem 3.2 *Corresponding to an identically non-vanishing function $\Omega_{(\eta)}(\xi_1, \dots, \xi_s)$ of complex variables ξ_1, \dots, ξ_s ($s \in \mathbb{N}$), let*

$$\begin{aligned} &\Lambda_{(\eta),(\psi)}(\xi_1, \dots, \xi_s; \varsigma_1, \dots, \varsigma_j) \\ &:= \sum_{k_1, \dots, k_j=0}^{\infty} a_{k_1, \dots, k_j} \Omega_{\eta_1 + \psi_1 k_1, \dots, \eta_j + \psi_j k_j}(\xi_1, \dots, \xi_s) \varsigma_1^{k_1} \dots \varsigma_j^{k_j} \quad (a_{k_1, \dots, k_j} \neq 0). \end{aligned} \tag{3.1}$$

Suppose also that

$$\begin{aligned} &\Theta_{n_1, \dots, n_j; q_1, \dots, q_j}^{(\gamma), (\lambda), (\eta), (\psi), \alpha, (N)}(\xi_1, \dots, \xi_s; x_1, \dots, x_j; (\rho); \varsigma_1, \dots, \varsigma_j) \\ &= \sum_{k_1, \dots, k_j=0}^{\lfloor \frac{n_1}{q_1} \rfloor, \dots, \lfloor \frac{n_j}{q_j} \rfloor} a_{k_1, \dots, k_j} \Omega_{\eta_1 + \psi_1 k_1, \dots, \eta_j + \psi_j k_j}(\xi_1, \dots, \xi_s) \\ &\quad \times \frac{(\gamma_1 + \lambda_1 k_1)_{n_1 - q_1 k_1} \dots (\gamma_j + \lambda_j k_j)_{n_j - q_j k_j} Z_{n_1 - q_1 k_1, \dots, n_j - q_j k_j}^{(\alpha; N_1, \dots, N_j)}(x_1, \dots, x_j; \rho_1, \dots, \rho_j)}{\Gamma(\rho_1(n_1 - q_1 k_1) + \dots + \rho_j(n_j - q_j k_j) + \alpha + 1)} \\ &\quad \times \varsigma_1^{k_1} \dots \varsigma_j^{k_j} \quad (q_1, \dots, q_j \in \mathbb{N}). \end{aligned} \tag{3.2}$$

Then

$$\begin{aligned} &\sum_{n_1, \dots, n_j=0}^{\infty} \Theta_{n_1, \dots, n_j; q_1, \dots, q_j}^{(\gamma), (\lambda), (\eta), (\psi), \alpha, (N)}\left(\xi_1, \dots, \xi_s; x_1, \dots, x_j; (\rho); \frac{\varsigma_1}{t_1^{q_1}}, \dots, \frac{\varsigma_j}{t_j^{q_j}}\right) t_1^{n_1} \dots t_j^{n_j} \\ &= \prod_{i=1}^j (1 - t_i)^{-\gamma_i} \Lambda_{(\eta),(\psi)}\left(\xi_1, \dots, \xi_s; \frac{\varsigma_1}{(1 - t_1)^{\lambda_1}}, \dots, \frac{\varsigma_j}{(1 - t_j)^{\lambda_j}}\right) \\ &\quad \times E_{\rho_1, \dots, \rho_j, \alpha + 1}^{(\gamma), (N)}\left(\frac{x_1^{\rho_1} (-t_1)^{N_1}}{(1 - t_1)^{N_1}}, \dots, \frac{x_j^{\rho_j} (-t_j)^{N_j}}{(1 - t_j)^{N_j}}\right), \end{aligned} \tag{3.3}$$

provided that each member of (3.3) exists and $|t_i| < 1$ ($i = 1, \dots, j$).

Proof Following similar lines to [10], the proof is completed. \square

4 Fractional integrals and derivatives

In this section, we first recall the definitions of the Riemann-Liouville fractional integrals and derivatives. Next, we give the fractional integral and derivative of the generalized multivariable Mittag-Leffler function $E_{(\rho_r), \lambda}^{(\gamma_r), (l_r)}(x_1, \dots, x_r)$ where x_j are real variables for $j = 1, \dots, r$.

Definition 4.1 Let $\Omega = [a, b]$ be a finite interval of the real axis. The Riemann-Liouville fractional integral of order $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$ is defined by

$${}_x I_{a^+}^\alpha [f] = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t) dt}{(x-t)^{1-\alpha}} \quad (x > a). \tag{4.1}$$

It is well known [14, p.71] that

$${}_x I_{0^+}^\alpha [x^p] = \frac{\Gamma(1+p)}{\Gamma(1+p+\alpha)} x^{p+\alpha} \quad (\Re(\alpha) > 0; \Re(p) > -1). \tag{4.2}$$

Definition 4.2 Let $\Omega = [a, b]$ be a finite interval of the real axis. The Riemann-Liouville fractional derivative of order $\alpha \in \mathbb{C}$ with $\Re(\alpha) \geq 0$ is defined by

$${}_x D_{a^+}^\alpha [f] = \left(\frac{d}{dx}\right)_x^n I_{a^+}^{n-\alpha} [f] \quad (n = [\Re(\alpha)] + 1; x > a), \tag{4.3}$$

where $[\Re(\alpha)]$ denotes the integral part of $\Re(\alpha)$.

Using (4.2), we see easily that

$${}_x D_{0^+}^\alpha [x^p] = \frac{\Gamma(1+p)}{\Gamma(1+p-\alpha)} x^{p-\alpha} \quad (\Re(\alpha) \geq 0; \Re(p) > -1). \tag{4.4}$$

Now, let us give two fractional calculus formulas obtained by Jaimini and Gupta [15, p.145, Eqs. (1) and (2)] involving the generalized multivariable Mittag-Leffler function.

Theorem 4.3 Let $\alpha, \lambda, \rho_j, \gamma_j, l_j, \omega_j \in \mathbb{C}$ such that $\Re(\alpha) > 0; \Re(\lambda) > 0; \Re(\rho_j) > 0; \Re(l_j) > 0$ ($j = 1, \dots, r$). Then the following fractional calculus formulas:

$${}_x I_{0^+}^\alpha [x^{\lambda-1} E_{(\rho_r), \lambda}^{(\gamma_r), (l_r)}(\omega_1 x^{\rho_1}, \dots, \omega_r x^{\rho_r})] = x^{\lambda+\alpha-1} E_{(\rho_r), \lambda+\alpha}^{(\gamma_r), (l_r)}(\omega_1 x^{\rho_1}, \dots, \omega_r x^{\rho_r}) \tag{4.5}$$

and

$${}_x D_{0^+}^\alpha [x^{\lambda-1} E_{(\rho_r), \lambda}^{(\gamma_r), (l_r)}(\omega_1 x^{\rho_1}, \dots, \omega_r x^{\rho_r})] = x^{\lambda-\alpha-1} E_{(\rho_r), \lambda-\alpha}^{(\gamma_r), (l_r)}(\omega_1 x^{\rho_1}, \dots, \omega_r x^{\rho_r}) \tag{4.6}$$

hold true.

Setting $\lambda = \lambda + 1, l_j = \omega_j = 1$ ($j = 1, \dots, r$), replacing $\gamma_1, \dots, \gamma_r$, respectively, by $-n_1, \dots, -n_r$, where n_j ($j = 1, \dots, r$) are positive integers in (4.5) and (4.6), and making use of (1.8) yield the following special cases given by Özarslan [10, p.353, Theorem 6 and Theorem 8]:

$$\begin{aligned} &{}_x I_{0^+}^\alpha [x^\lambda Z_{n_1, \dots, n_r}^{(\lambda)}(x, \dots, x; \rho_1, \dots, \rho_r)] \\ &= \frac{\Gamma(\rho_1 n_1 + \dots + \rho_r n_r + \lambda + 1)}{\Gamma(\rho_1 n_1 + \dots + \rho_r n_r + \lambda + \alpha + 1)} x^{\lambda+\alpha} Z_{n_1, \dots, n_r}^{(\lambda+\alpha)}(x, \dots, x; \rho_1, \dots, \rho_r) \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} &{}_x D_{0^+}^\alpha [x^\lambda Z_{n_1, \dots, n_r}^{(\lambda)}(x, \dots, x; \rho_1, \dots, \rho_r)] \\ &= \frac{\Gamma(\rho_1 n_1 + \dots + \rho_r n_r + \lambda + 1)}{\Gamma(\rho_1 n_1 + \dots + \rho_r n_r + \lambda - \alpha + 1)} x^{\lambda-\alpha} Z_{n_1, \dots, n_r}^{(\lambda+\alpha)}(x, \dots, x; \rho_1, \dots, \rho_r). \end{aligned} \tag{4.8}$$

Further special cases of (4.5) and (4.6) can be obtained by suitably specializing the coefficients involved. For instance, if we set $l_j = 1$ ($j = 1, \dots, r$), then (4.5) and (4.6) reduce to two results obtained by Saxena *et al.* [7].

We end this section by giving a recurrence relation for the generalized multivariable Mittag-Leffler function $E_{(\rho_r), \lambda}^{(\gamma_r), (l_r)}(z_1, \dots, z_r)$.

Theorem 4.4 *Let $\lambda, \rho_j, \gamma_j, l_j \in \mathbb{C}$ such that $\Re(\lambda) > 0; \Re(\rho_j) > 0; \Re(l_j) > 0$ ($j = 1, \dots, r$). Then the following recurrence relation holds true:*

$$E_{(\rho_r), \lambda}^{(\gamma_r), (l_r)}(z_1, \dots, z_r) = \lambda E_{(\rho_r), \lambda+1}^{(\gamma_r), (l_r)}(z_1, \dots, z_r) + \sum_{i=1}^r \rho_i z_i \frac{\partial}{\partial z_i} E_{(\rho_r), \lambda+1}^{(\gamma_r), (l_r)}(z_1, \dots, z_r). \tag{4.9}$$

Proof From (1.6), we have

$$\begin{aligned} & \lambda E_{(\rho_r), \lambda+1}^{(\gamma_r), (l_r)}(z_1, \dots, z_r) + \sum_{i=1}^r \rho_i z_i \frac{\partial}{\partial z_i} E_{(\rho_r), \lambda+1}^{(\gamma_r), (l_r)}(z_1, \dots, z_r) \\ &= \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(\gamma_1)_{k_1 l_1} \cdots (\gamma_r)_{k_r l_r}}{\Gamma(k_1 \rho_1 + \cdots + k_r \rho_r + \lambda + 1)} [\lambda + \rho_1 k_1 + \cdots + \rho_r k_r] \frac{z_1^{k_1} \cdots z_r^{k_r}}{k_1! \cdots k_r!} \\ &= E_{(\rho_r), \lambda}^{(\gamma_r), (l_r)}(z_1, \dots, z_r). \end{aligned} \tag{4.10}$$

□

5 Singular integral equation

In this section, we solve a singular integral equation with the generalized multivariable Mittag-Leffler function in the kernel. To do so, we first find the Laplace transform of the function $E_{(\rho_r), \lambda}^{(\gamma_r), (l_r)}((\mu x)^{\rho_1}, \dots, (\mu x)^{\rho_r})$ and we compute an integral involving the product of two generalized multivariable Mittag-Leffler functions.

We denote the Laplace transform of a function f [16, p.218] by

$$\mathbb{L}[f(t)](p) = \tilde{f}(p) = \int_0^{\infty} e^{-pt} f(t) dt \quad (\Re(p) > 0). \tag{5.1}$$

Lemma 5.1 *Let $p, \lambda, \mu, \rho_j, \gamma_j, l_j \in \mathbb{C}$ such that $\Re(p) > 0; \Re(\mu) > 0; \Re(\lambda) > 0; \Re(\rho_j) > 0; \Re(l_j) > 0$ ($j = 1, \dots, r$), we have*

$$\begin{aligned} & \mathbb{L}\left[t^{\lambda-1} E_{(\rho_r), \lambda}^{(\gamma_r), (l_r)}((\mu t)^{\rho_1}, \dots, (\mu t)^{\rho_r})\right](p) \\ &= \frac{1}{p^\lambda} \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(\gamma_1)_{k_1 l_1} \cdots (\gamma_r)_{k_r l_r}}{k_1! \cdots k_r!} \left(\frac{\mu}{p}\right)^{\rho_1 k_1 + \cdots + \rho_r k_r}. \end{aligned} \tag{5.2}$$

Proof Using (5.1), we get

$$\begin{aligned} & \mathbb{L}\left[t^{\lambda-1}E_{(\rho_r),\lambda}^{(\gamma_r),(l_r)}((\mu t)^{\rho_1}, \dots, (\mu t)^{\rho_r})\right](p) \\ &= \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(\gamma_1)_{k_1 l_1} \cdots (\gamma_r)_{k_r l_r}}{\Gamma(k_1 \rho_1 + \cdots + k_r \rho_r + \lambda)} \frac{\mu^{\rho_1 k_1 + \cdots + \rho_r k_r}}{k_1! \cdots k_r!} \cdot \int_0^{\infty} e^{-pt} t^{\rho_1 k_1 + \cdots + \rho_r k_r + \lambda - 1} dt \\ &= \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(\gamma_1)_{k_1 l_1} \cdots (\gamma_r)_{k_r l_r}}{\Gamma(k_1 \rho_1 + \cdots + k_r \rho_r + \lambda)} \frac{\mu^{\rho_1 k_1 + \cdots + \rho_r k_r}}{k_1! \cdots k_r!} \cdot \frac{\Gamma(k_1 \rho_1 + \cdots + k_r \rho_r + \lambda)}{p^{k_1 \rho_1 + \cdots + k_r \rho_r + \lambda}} \\ &= \frac{1}{p^\lambda} \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(\gamma_1)_{k_1 l_1} \cdots (\gamma_r)_{k_r l_r}}{k_1! \cdots k_r!} \left(\frac{\mu}{p}\right)^{\rho_1 k_1 + \cdots + \rho_r k_r}, \end{aligned} \tag{5.3}$$

where we used the well-known formula [16, p.218, Eq. (3)]

$$\int_0^{\infty} e^{-pt} t^{\lambda-1} dt = \frac{\Gamma(\lambda)}{p^\lambda} \quad (\min\{\Re(\lambda), \Re(p)\} > 0). \tag{5.4}$$

□

Theorem 5.2 Let $p, \lambda, \mu, \nu, \rho_j, \gamma_j, l_j, \sigma_j, m_j \in \mathbb{C}$ such that $\Re(p) > 0; \Re(\mu) > 0; \Re(\nu) > 0; \Re(\lambda) > 0; \Re(\rho_j) > 0; \Re(\sigma_j) > 0; \Re(l_j) > 0; \Re(m_j) > 0$ ($j = 1, \dots, r$), we have

$$\begin{aligned} & \int_0^x (x-t)^{\lambda-1} E_{(\rho_r),\lambda}^{(\gamma_r),(l_r)}((\mu[x-t])^{\rho_1}, \dots, (\mu[x-t])^{\rho_r}) \\ & \quad \times t^{\nu-1} E_{(\rho_r),\nu}^{(\sigma_r),(m_r)}((\mu t)^{\rho_1}, \dots, (\mu t)^{\rho_r}) dt \\ &= t^{\lambda+\nu-1} E_{\rho_1, \dots, \rho_r, \rho_1, \dots, \rho_r, \lambda+\nu}^{\gamma_1, \dots, \gamma_r, \sigma_1, \dots, \sigma_r, l_1, \dots, l_r, m_1, \dots, m_r}((\mu t)^{\rho_1}, \dots, (\mu t)^{\rho_r}, (\mu t)^{\rho_1}, \dots, (\mu t)^{\rho_r}). \end{aligned} \tag{5.5}$$

Proof With the help of the convolution theorem for the Laplace transform (see [17])

$$\mathbb{L}\left[\int_0^x f(x-t)g(t) dt\right](p) = \mathbb{L}[f(x)](p)\mathbb{L}[g(x)](p), \tag{5.6}$$

we have

$$\begin{aligned} & \mathbb{L}\left[\int_0^x (x-t)^{\lambda-1} E_{(\rho_r),\lambda}^{(\gamma_r),(l_r)}((\mu[x-t])^{\rho_1}, \dots, (\mu[x-t])^{\rho_r}) \right. \\ & \quad \left. \times t^{\nu-1} E_{(\rho_r),\nu}^{(\sigma_r),(m_r)}((\mu t)^{\rho_1}, \dots, (\mu t)^{\rho_r}) dt\right](p) \\ &= \mathbb{L}\left[t^{\lambda-1} E_{(\rho_r),\lambda}^{(\gamma_r),(l_r)}((\mu t)^{\rho_1}, \dots, (\mu t)^{\rho_r})\right](p) \\ & \quad \cdot \mathbb{L}\left[t^{\nu-1} E_{(\rho_r),\nu}^{(\sigma_r),(m_r)}((\mu t)^{\rho_1}, \dots, (\mu t)^{\rho_r})\right](p). \end{aligned} \tag{5.7}$$

From Lemma 5.1, we have

$$\begin{aligned} & \mathbb{L}\left[\int_0^x (x-t)^{\lambda-1} E_{(\rho_r),\lambda}^{(\gamma_r),(l_r)}((\mu[x-t])^{\rho_1}, \dots, (\mu[x-t])^{\rho_r}) \right. \\ & \quad \left. \times t^{\nu-1} E_{(\rho_r),\nu}^{(\sigma_r),(m_r)}((\mu t)^{\rho_1}, \dots, (\mu t)^{\rho_r}) dt\right](p) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{p^\lambda} \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(\gamma_1)_{k_1 l_1} \cdots (\gamma_r)_{k_r l_r}}{k_1! \cdots k_r!} \left(\frac{\mu}{p}\right)^{\rho_1 k_1 + \cdots + \rho_r k_r} \\
 &\quad \cdot \frac{1}{p^\mu} \sum_{i_1, \dots, i_r=0}^{\infty} \frac{(\sigma_1)_{i_1 m_1} \cdots (\sigma_r)_{i_r m_r}}{i_1! \cdots i_r!} \left(\frac{\mu}{p}\right)^{\rho_1 i_1 + \cdots + \rho_r i_r} \\
 &= \frac{1}{p^{\lambda+\nu}} \sum_{k_1, \dots, k_r, i_1, \dots, i_r=0}^{\infty} \frac{(\gamma_1)_{k_1 l_1} \cdots (\gamma_r)_{k_r l_r} \cdot (\sigma_1)_{i_1 m_1} \cdots (\sigma_r)_{i_r m_r}}{k_1! \cdots k_r! \cdot i_1! \cdots i_r!} \left(\frac{\mu}{p}\right)^{\rho_1(k_1+i_1) + \cdots + \rho_r(k_r+i_r)} \\
 &= \mathbb{L}\left[t^{\lambda+\nu-1} E_{\rho_1, \dots, \rho_r, \rho_1, \dots, \rho_r, \lambda+\nu}^{\gamma_1, \dots, \gamma_r, \sigma_1, \dots, \sigma_1, l_1, \dots, l_r, m_1, \dots, m_r}((\mu t)^{\rho_1}, \dots, (\mu t)^{\rho_r}, (\mu t)^{\rho_1}, \dots, (\mu t)^{\rho_r})\right](p). \quad (5.8)
 \end{aligned}$$

Finally, taking the inverse Laplace transform on both sides of (5.8), the result follows. \square

Now, let us consider the following convolution equation involving the generalized multivariable Mittag-Leffler in the kernel:

$$\int_0^x (x-t)^{\lambda-1} E_{(\rho_r), \lambda}^{(\gamma_r), (1)}((\mu[x-t])^{\rho_1}, \dots, (\mu[x-t])^{\rho_r}) \cdot \phi(t) dt = \psi(x), \quad (5.9)$$

where $\Re(\alpha) > -1$.

Theorem 5.3 *The singular integral equation (5.9) admits a locally integrable solution*

$$\phi(x) = \int_0^x (x-t)^{\omega-\lambda-1} E_{(\rho_r), \omega-\lambda}^{(-\gamma_r), (1)}((\mu[x-t])^{\rho_1}, \dots, (\mu[x-t])^{\rho_r}) \cdot {}_t I_{0^+}^{-\omega} \psi(t) dt, \quad (5.10)$$

provided that ${}_t I_{0^+}^{-\omega} \psi(t)$ exists for $\Re(\omega) > \Re(\alpha + 1)$ and is locally integrable for $0 < t < \delta < \infty$.

Proof Applying the Laplace transform on both sides of (5.9), using the convolution theorem as well as Lemma 5.1, we find

$$\frac{1}{p^\lambda} \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(\gamma_1)_{k_1} \cdots (\gamma_r)_{k_r}}{k_1! \cdots k_r!} \left(\frac{\mu}{p}\right)^{\rho_1 k_1 + \cdots + \rho_r k_r} \cdot \mathbb{L}[\phi(t)](p) = \mathbb{L}[\psi(t)](p), \quad (5.11)$$

which under the assumptions that $|\frac{\mu}{p}| < 1$ can be rewritten as

$$\frac{1}{p^\lambda} \prod_{j=1}^r \left(1 - \left(\frac{\mu}{p}\right)^{\rho_j}\right)^{-\gamma_j} \cdot \mathbb{L}[\phi(t)](p) = \mathbb{L}[\psi(t)](p). \quad (5.12)$$

Therefore, we have

$$\mathbb{L}[\phi(t)](p) = \left\{ \prod_{j=1}^r \left(1 - \left(\frac{\mu}{p}\right)^{\rho_j}\right)^{\gamma_j} p^{-\omega+\lambda} \right\} \cdot \{p^\omega \mathbb{L}[\psi(t)](p)\}. \quad (5.13)$$

Taking the inverse Laplace transform on both sides of (5.13) and with the help of the following property [5, p.217, Eq. (3.8)]:

$$p^\mu \mathbb{L}[f(t)](p) = \mathbb{L}[{}_t I_{0^+}^{-\mu} f(t)](p) \quad (\mu, p \in \mathbb{C}; \Re(p) > 0), \quad (5.14)$$

which holds for suitable f , we thus obtain

$$\phi(x) = \int_0^x (x-t)^{\omega-\lambda-1} E_{(\rho_r), \omega-\lambda}^{(-\gamma_r), (1)}((\mu[x-t])^{\rho_1}, \dots, (\mu[x-t])^{\rho_r}) \cdot {}_t I_{0^+}^{-\omega} \psi(t) dt. \quad (5.15)$$

□

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors completed the paper together. All authors read and approved the final manuscript.

Author details

¹Department of Mathematics and Computer Science, University of Quebec at Chicoutimi, Quebec, G7H 2B1, Canada.

²Eastern Mediterranean University, Gazimagusa, TRNC, Mersin, 10, Turkey.

Received: 10 July 2014 Accepted: 3 September 2014 Published: 24 Sep 2014

References

1. Mittag-Leffler, GM: Sur la nouvelle fonction $E_\alpha(x)$. *C. R. Acad. Sci. Paris* **137**, 554-558 (1903)
2. Mittag-Leffler, GM: Sur la représentation analytique d'une fonction monogène (cinquième note). *Acta Math.* **29**, 101-181 (1905)
3. Saxena, RK, Mathai, AM, Haubold, HJ: On fractional kinetic equations. *Astrophys. Space Sci.* **282**(1), 281-287 (2002)
4. Wiman, A: Über den fundamental Satz in der Theorien der Funktionen $E_\alpha(x)$. *Acta Math.* **29**, 191-201 (1905)
5. Prabhakar, TR: A singular integral equation with a generalized Mittag-Leffler function in the kernel. *Yokohama Math. J.* **19**, 7-15 (1971)
6. Srivastava, HM, Tomovski, Z: Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel. *Appl. Math. Comput.* **211**(1), 198-210 (2009)
7. Saxena, RK, Kalla, SL, Saxena, R: Multivariable analogue of generalized Mittag-Leffler function. *Integral Transforms Spec. Funct.* **22**, 533-548 (2011)
8. Srivastava, HM, Daoust, MC: Certain generalized Neumann expansion associated with Kampé de Fériet function. *Proc. K. Ned. Akad. Wet., Ser. A, Indag. Math.* **31**, 449-457 (1969)
9. Srivastava, HM, Karlsson, PW: Multiple Gaussian Hypergeometric Series. Ellis Horwood, Chichester (1985)
10. Özarslan, MA: On a singular integral equation including a set of multivariate polynomials suggested by Laguerre polynomials. *Appl. Math. Comput.* **229**, 350-358 (2014)
11. Erdélyi, A: Beitrag zur theorie der konfluenten hypergeometrischen funktionen von mehreren veränderlichen. *Sitzungsber. Akad. Wiss. Wien, Math.-Naturw. Kl., Abt. Ila* **146**, 431-467 (1937)
12. Konhäuser, JDE: Biorthogonal polynomials suggested by the Laguerre polynomials. *Pac. J. Math.* **21**, 303-314 (1967)
13. Erdélyi, A, Magnus, W, Oberhettinger, F, Tricomi, F: Higher Transcendental Functions, vols. 1-3. McGraw-Hill, New York (1953)
14. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. North-Holland Mathematical Studies, vol. 204. Elsevier, Amsterdam (2006)
15. Jaimini, BB, Gupta, J: On certain fractional differential equations involving generalized multivariable Mittag-Leffler function. *Note Mat.* **32**(2), 141-156 (2012)
16. Srivastava, HM, Manocha, HL: A Treatise on Generating Functions. Ellis Horwood, Chichester (1984)
17. Titchmarsh, EC: Introduction to the Theory of Fourier Integrals, 3rd edn. Chelsea, New York (1986). The first edition in Oxford University Press, Oxford (1937)

10.1186/1687-1847-2014-252

Cite this article as: Gaboury and Özarslan: Singular integral equation involving a multivariable analog of Mittag-Leffler function. *Advances in Difference Equations* 2014, **2014**:252