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On best proximity point of ψ-Geraghty contractions

Erdal Karapınar[*]

[*]Correspondence:
erdalkarapinar@yahoo.com
Department of Mathematics, Atilim
University, Incek, Ankara 06836,
Turkey

Abstract

Very recently, Caballero, Harjani and Sadarangani (Fixed Point Theory Appl. 2012:231, 2012) observed some best proximity point results for Geraghty contractions by using the *P*-property. In this paper, we introduce the notion of ψ-Geraghty contractions and show the existence and uniqueness of the best proximity point of such contractions in the setting of a metric space. We state examples to illustrate our result.

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1 Introduction and preliminaries

In nonlinear functional analysis, fixed point theory and best proximity point theory play a crucial role in the establishment of the existence of certain differential and integral equations. As a consequence, fixed point theory is very useful for various quantitative sciences that involve such equations. To list a few, certain branches of computer sciences, engineering and economics are well-known examples in which fixed point theory is used.

The most remarkable paper in this field was reported by Banach [1] in 1922. In this paper, Banach proved that every contraction in a complete metric space has a unique fixed point. Following this outstanding paper, many authors have extended, generalized and improved this remarkable fixed point theorem of Banach by changing either the conditions of the mappings or the construction of the space. In particular, one of the notable generalizations of Banach fixed point theorem was reported by Geraghty [2].

Theorem 1.1 (Geraghty [2]) *Let (X,d) be a complete metric space and $T : X \to X$ be an operator. Suppose that there exists $\beta : (0,\infty) \to [0,1)$ satisfying the condition*

$$\beta(t_n) \to 1 \quad implies \quad t_n \to 0.$$

If T satisfies the following inequality:

$$d(Tx, Ty) \leq \beta\big(d(x,y)\big)d(x,y) \quad for\ any\ x,y \in X, \tag{1}$$

then T has a unique fixed point.

It is very natural that some mappings, especially non-self-mappings defined on a complete metric space (X, d), do not necessarily possess a fixed point, that is, $d(x, Tx) > 0$ for all $x \in X$. In such situations, it is reasonable to search for the existence (and uniqueness) of a point $x^* \in X$ such that $d(x^*, Tx^*)$ is an approximation of an $x \in X$ such that $d(x, Tx) = 0$. In other words, one speculates to determine an approximate solution x^* that is optimal in the sense that the distance between x^* and Tx^* is minimum. Here, the point $x^* \in X$ is called a best proximity point.

This research subject has attracted attention of a number of authors; for example, see [2–23]. In this paper we generalize and improve certain results of Caballero *et al.* in [6]. Notice also that in the best proximity point theory, we usually consider a non-self-mapping. In fixed point theory, almost all maps are self-mappings. For the sake of completeness, we recall some basic definitions and fundamental results on the best proximity theory.

Let (X, d) be a metric space and A and B be nonempty subsets of a metric space (X, d). A mapping $T : A \to B$ is called a k-contraction if there exists $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for any $x, y \in A$. It is clear that a k-contraction coincides with the celebrated Banach fixed point theorem if one takes $A = B$, where A is a complete subset of X.

Let A and B be two nonempty subsets of a metric space (X, d). We denote by A_0 and B_0 the following sets:

$$
\begin{aligned}
A_0 &= \big\{ x \in A : d(x, y) = d(A, B) \text{ for some } y \in B \big\}, \\
B_0 &= \big\{ y \in B : d(x, y) = d(A, B) \text{ for some } x \in A \big\},
\end{aligned} \tag{2}
$$

where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$.

In [13], the authors presented sufficient conditions which determine when the sets A_0 and B_0 are nonempty. In [19], the author introduced the following definition.

Definition 1.1 Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the *P*-property if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$
d(x_1, y_1) = d(A, B) \quad \text{and} \quad d(x_2, y_2) = d(A, B) \quad \Rightarrow \quad d(x_1, x_2) = d(y_1, y_2). \tag{3}
$$

It can be easily seen that for any nonempty subset A of (X, d), the pair (A, A) has the *P*-property. In [19], the author proved that any pair (A, B) of nonempty closed convex subsets of a real Hilbert space H satisfies the *P*-property. Now, we introduce the class F of those functions $\beta : (0, \infty) \to [0, 1)$ satisfying the following condition:

$$
\beta(t_n) \to 1 \quad \text{implies} \quad t_n \to 0. \tag{4}
$$

Definition 1.2 (See [6]) Let A, B be two nonempty subsets of a metric space (X, d). A mapping $T : A \to B$ is said to be a Geraghty-contraction if there exists $\beta \in F$ such that

$$
d(Tx, Ty) \leq \beta\big(d(x, y)\big) \cdot d(x, y) \quad \text{for any } x, y \in A. \tag{5}
$$

Theorem 1.2 (See [6]) *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $T : A \to B$ be a continuous Geraghty-contraction*

satisfying $T(A_0) \subseteq B_0$. *Suppose that the pair* (A, B) *has the P-property. Then there exists a unique* x^* *in A such that* $d(x^*, Tx^*) = d(A, B)$.

In the following section, we improve the theorem above by using a distance function ψ in Definition 1.2. In particular, we introduce Definition 2.1 and broaden the scope of Theorem 1.2 to ψ-Geraghty-contractions.

2 Main results

We start this section with the definition of the following auxiliary function. Let Ψ denote the class of functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:

(a) ψ is nondecreasing;
(b) ψ is subadditive, that is, $\psi(s + t) \leq \psi(s) + \psi(t)$;
(c) ψ is continuous;
(d) $\psi(t) = 0 \Leftrightarrow t = 0$.

We introduce the following contraction.

Definition 2.1 Let A, B be two nonempty subsets of a metric space (X, d). A mapping $T : A \to B$ is said to be a ψ-Geraghty contraction if there exists $\beta \in F$ such that

$$\psi\big(d(Tx, Ty)\big) \leq \beta\big(\psi\big(d(x, y)\big)\big)\psi\big(d(x, y)\big) \quad \text{for any } x, y \in A. \tag{6}$$

Remark 2.1 Notice that since $\beta : (0, \infty) \to [0, 1)$, we have

$$\psi\big(d(Tx, Ty)\big) \leq \beta\big(\psi\big(d(x, y)\big)\big)\psi\big(d(x, y)\big) < \psi\big(d(x, y)\big) \quad \text{for any } x, y \in A \text{ with } x \neq y. \tag{7}$$

We are now ready to state and prove our main theorem.

Theorem 2.1 *Let* (A, B) *be a pair of nonempty closed subsets of a complete metric space* (X, d) *such that* A_0 *is nonempty. Let* $T : A \to B$ *be a* ψ-*Geraghty contraction satisfying* $T(A_0) \subseteq B_0$. *Suppose that the pair* (A, B) *has the P-property. Then there exists a unique* x^* *in A such that* $d(x^*, Tx^*) = d(A, B)$.

Proof Regarding that A_0 is nonempty, we take $x_0 \in A_0$. Since $Tx_0 \in T(A_0) \subseteq B_0$, we can find $x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$. Analogously, regarding the assumption $Tx_1 \in T(A_0) \subseteq B_0$, we determine $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A, B)$. Recursively, we obtain a sequence $\{x_n\}$ in A_0 satisfying

$$d(x_{n+1}, Tx_n) = d(A, B) \quad \text{for any } n \in \mathbb{N}. \tag{8}$$

Since (A, B) has the P-property, we derive that

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \quad \text{for any } n \in \mathbb{N}. \tag{9}$$

If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$, then the proof is completed. Indeed,

$$0 = d(x_{n_0}, x_{n_0+1}) = d(Tx_{n_0-1}, Tx_{n_0}), \tag{10}$$

and consequently, $Tx_{n_0-1} = Tx_{n_0}$. On the other hand, due to (8) we have

$$d(x_{n_0}, Tx_{n_0-1}) = d(A, B).$$

Therefore, we conclude that

$$d(A, B) = d(x_{n_0}, Tx_{n_0-1}) = d(x_{n_0}, Tx_{n_0}). \tag{11}$$

For the rest of the proof, we suppose that $d(x_n, x_{n+1}) > 0$ for any $n \in \mathbb{N}$. Since T is a ψ-Geraghty contraction, for any $\mathbb{N}$, we have that

$$\begin{aligned}
\psi\big(d(x_n, x_{n+1})\big) &= \psi\big(d(Tx_{n-1}, Tx_n)\big) \\
&\leq \beta\big(\psi\big(d(x_{n-1}, x_n)\big)\big)\psi\big(d(x_{n-1}, x_n)\big) \\
&< \psi\big(d(x_{n-1}, x_n)\big).
\end{aligned} \tag{12}$$

Consequently, $\{\psi(d(x_n, x_{n+1}))\}$ is a nonincreasing sequence and bounded below, and so $\lim_{n\to\infty} \psi(d(x_n, x_{n+1})) = L$ exists. Let $\lim_{n\to\infty} \psi(d(x_n, x_{n+1})) = L \geq 0$. Assume that $L > 0$. Then, from (6), we have

$$\frac{\psi(d(x_{n+1}, x_{n+2}))}{\psi(d(x_n, x_{n+1}))} \leq \beta\big(\psi\big(d(x_n, x_{n+1})\big)\big) \leq 1$$

for each $n \geq 1$, which implies that

$$\lim_{n\to\infty} \beta\big(\psi\big(d(x_n, x_{n+1})\big)\big) = 1.$$

On the other hand, since $\beta \in F$, we conclude $L = 0$, that is,

$$\lim_{n\to\infty} \psi\big(d(x_n, x_{n+1})\big) = 0. \tag{13}$$

Notice that since $d(x_{n+1}, Tx_n) = d(A, B)$ for any $n \in \mathbb{N}$, for fixed $p, q \in \mathbb{N}$, we have $d(x_p, Tx_{p-1}) = d(x_q, Tx_{q-1}) = d(A, B)$, and since (A, B) satisfies the P-property, $d(x_p, x_q) = d(Tx_{p-1}, Tx_{q-1})$. In what follows, we prove that $\{x_n\}$ is a Cauchy sequence. On the contrary, assume that we have

$$\varepsilon = \limsup_{m,n\to\infty} d(x_n, x_m) > 0. \tag{14}$$

By using the triangular inequality,

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+1}) + d(x_{m+1}, x_m). \tag{15}$$

By (12) and since $d(x_{n+1}, x_{m+1}) = d(Tx_n, Tx_m)$, by the comment mentioned above, regarding the discussion on the P-property above together with (12), (15) and the property of the

function ψ, we derive that

$$\psi\big(d(x_n,x_m)\big) \leq \psi\big(d(x_n,x_{n+1}) + d(Tx_n,Tx_m) + d(x_{m+1},x_m)\big)$$

$$\leq \psi\big(d(x_n,x_{n+1})\big) + \psi\big(d(Tx_n,Tx_m)\big) + \psi\big(d(x_{m+1},x_m)\big)$$

$$\leq \psi\big(d(x_n,x_{n+1})\big) + \beta\big(\psi\big(d(x_n,x_m)\big)\big)\psi\big(d(x_n,x_m)\big) + \psi\big(d(x_{m+1},x_m)\big). \quad (16)$$

By a simple manipulation, (16) yields that

$$\psi\big(d(x_n,x_m)\big) \leq \big(1 - \beta\big(\psi\big(d(x_n,x_m)\big)\big)\big)^{-1}\big[\psi\big(d(x_n,x_{n+1})\big) + \psi\big(d(x_{m+1},x_m)\big)\big]. \quad (17)$$

By taking the properties of the function ψ into account, together with (13) and $\limsup_{m,n\to\infty} d(x_n,x_m) = \varepsilon > 0$ and $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$, the last inequality yields

$$\limsup_{m,n\to\infty}\big(1 - \beta\big(\psi\big(d(x_n,x_m)\big)\big)\big)^{-1} = \infty. \quad (18)$$

Therefore $\limsup_{m,n\to\infty} \beta(\psi(d(x_n,x_m))) = 1$. By taking the fact $\beta \in F$ into account, we get

$$\limsup_{m,n\to\infty} \psi\big(d(x_n,x_m)\big) = 0.$$

Regarding the properties of the function ψ, the limit above contradicts the assumption (14). Therefore, $\{x_n\}$ is a Cauchy sequence.

Since $(x_n) \subset A$ and A is a closed subset of the complete metric space (X,d), we can find $x^* \in A$ such that $x_n \to x^*$.

We claim that $Tx_n \to Tx^*$. Suppose, on the contrary, that $Tx_n \nrightarrow Tx^*$. This means that we can find $\varepsilon > 0$ such that for each $n \in \mathbb{N}$, there exists p_n with

$$\varepsilon < d\big(Tx_{p_n}, Tx^*\big). \quad (19)$$

Due to the properties of ψ, we get

$$0 < \psi(\varepsilon) \leq \psi\big(d\big(Tx_{p_n}, Tx^*\big)\big). \quad (20)$$

Using the fact that T is a ψ-Geraghty contraction, we have

$$\psi(\varepsilon) \leq \psi\big(d\big(Tx_{p_n}, Tx^*\big)\big) \leq \beta\big(\psi\big(d\big(x_{p_n}, x^*\big)\big)\big)\psi\big(d\big(x_{p_n}, x^*\big)\big) \quad (21)$$

for any $n \in \mathbb{N}$. Since $x_{p_n} \to x^*$ and $\psi(d(x_{p_n}, x_{p_{n+1}})) \to 0$, we can find n_0 such that for $n \geq n_0$

$$\psi\big(d\big(x_{p_n}, x^*\big)\big) < \psi(\varepsilon) \quad \text{and} \quad \psi\big(d\big(x_{p_n}, x_{p_{n+1}}\big)\big) < \psi(\varepsilon). \quad (22)$$

Consequently, for $n \geq n_0$ we have

$$0 < \psi(\varepsilon) \leq \psi\big(d\big(Tx_{p_n}, Tx^*\big)\big) \leq \beta\big(\psi\big(d\big(x_{p_n}, x^*\big)\big)\big)\psi\big(d\big(x_{p_n}, x^*\big)\big)$$

$$\leq \psi\big(d\big(x_{p_n}, x^*\big)\big) < \psi(\varepsilon), \quad (23)$$

a contradiction. Therefore, $Tx_n \to Tx^*$.

Regarding the fact that the sequence $\{d(x_{n+1}, Tx_n)\}$ is a constant sequence with value $d(A, B)$, we derive

$$d(x^*, Tx^*) = d(A, B) \tag{24}$$

which is equivalent to saying that x^* is the best proximity point of T. This completes the proof of the existence of a best proximity point.

We shall show the uniqueness of the best proximity point of T. Suppose that x^* and y^* are two distinct best proximity points of T, that is, $x^* \neq y^*$. This implies that

$$d(x^*, Tx^*) = d(A, B) = d(y^*, Ty^*). \tag{25}$$

Using the P-property, we have

$$d(x^*, x_2) = d(Tx^*, Ty^*). \tag{26}$$

Using the fact that T is a ψ-Geraghty contraction, we have

$$\psi\big(d(x^*, y^*)\big) = \psi\big(d(Tx^*, Ty^*)\big) \leq \beta\big(\psi\big(d(x^*, y^*)\big)\big)\psi\big(d(x^*, y^*)\big) < \psi\big(d(x^*, y^*)\big), \tag{27}$$

a contradiction. This completes the proof. □

Notice that the pair (A, A) satisfies the P-property for any nonempty subset A of X. Consequently, we have the following corollary.

Corollary 2.1 *Let (X, d) be a complete metric space and A be a nonempty closed subset of X. Let $T : A \to A$ be a ψ-Geraghty-contraction. Then T has a unique fixed point.*

Proof Apply Theorem 2.1 with $A = B$. □

If we take $\psi(t) = t$ we obtain Theorem 1.2 as a corollary of Theorem 2.1.

Corollary 2.2 *Let (X, d) be a complete metric space and A be a nonempty closed subset of X. Let $T : A \to A$ be a Geraghty-contraction. Then T has a unique fixed point.*

Proof Apply Theorem 2.1 with $A = B$ and $\psi(t) = t$. □

In order to illustrate our results, we present the following example.

Example 2.1 Suppose that $X = \mathbb{R}^2$ with the metric

$$d\big((x, y), (x', y')\big) = \max\big\{|x - x'|, |y - y'|\big\}$$

and consider the closed subsets

$$A = \big\{(0, x) : 0 \leq x < \infty\big\},$$
$$B = \big\{(1, x) : 0 \leq x < \infty\big\}$$

and $\psi(t) = \frac{t}{2}$ and $\beta(t) = \frac{\ln(1+t)}{t}$.

Set $T : A \to B$ to be the mapping defined by

$$T\big((0,x)\big) = \big(1, \ln(1+x)\big).$$

Since $d(A, B) = 1$, the pair (A, B) has the P-property.

Notice that $A_0 = A$ and $B_0 = B$ and $T(A_0) \subseteq B_0$.

Without loss of generality, we assume that $x > x'$. Moreover,

$$d\big(T(0,x), T(0,x')\big) = \big|\ln(1+x) - \ln\big(1+x'\big)\big| = \left|\ln\left(\frac{1+x}{1+x'}\right)\right|$$

and

$$d\big((0,x), (0,x')\big) = \big|x - x'\big|.$$

We have

$$
\begin{aligned}
\psi\big(d\big(T(0,x), T(0,x')\big)\big) &= \frac{1}{2}\left|\ln\left(\frac{1+x}{1+x'}\right)\right| = \frac{1}{2}\left|\ln\left(\frac{1+x'+x-x'}{1+x'}\right)\right| \\
&= \frac{1}{2}\left|\ln\left(1 + \frac{x-x'}{1+x'}\right)\right| \\
&\le \frac{1}{2}\ln\big(1 + |x - x'|\big) \quad (\text{since } \ln(1+t) \text{ is strictly increasing}) \\
&\le \ln\left(1 + \frac{1}{2}|x - x'|\right) \\
&\le \frac{\ln(1 + \frac{1}{2}|x-x'|)}{\frac{1}{2}|x-x'|}\frac{1}{2}|x - x'| \\
&= \beta\big(\psi\big(d\big((0,x), (0,x')\big)\big)\big)\psi\big(d\big((0,x), (0,x')\big)\big),
\end{aligned}
$$

where $\beta : [0, \infty) \to [0, 1)$ is defined as $\beta(t) = \frac{\ln(1+t)}{t}$.

Therefore, T is a ψ-Geraghty-contraction. Notice that the pair (A, B) satisfies the P-property. Indeed, if

$$d\big((0,x), (1,x')\big) = \max\big\{1, \big|x - x'\big|\big\} = d(A, B) = 1,$$
$$d\big((0,y), (1,y')\big) = \max\big\{1, \big|y - y'\big|\big\} = d(A, B) = 1$$

then $x = x'$ and $y = y'$ and hence

$$|x - y| = d\big((0,x), (0,y)\big) = d\big((1,x'), (1,y')\big) = \big|x' - y'\big|.$$

Therefore, since the assumptions of Theorem 2.1 are satisfied, by Theorem 2.1 there exists a unique $(0, x^*) \in A$ such that

$$d\big((0, x^*), T\big(0, x^*\big)\big) = 1 = d(A, B).$$

More precisely, the point $(0, 0) \in A$ is the best proximity point of T.

Example 2.2 Let $X = \mathbb{R}$ and $d(x,y) = |x - y|$ be a metric on X. Suppose $A = [0,1]$ and $B = [15/8, 2]$ are two closed subsets of $\mathbb{R}$. Define $T : A \to B$ by $Tx = -\frac{1}{8}x + 2$. Define $\beta : (0, \infty) \to [0,1)$ by $\beta(t) = \frac{1}{1+t}$ and $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = \frac{1}{2}t$. Clearly, $d(A, B) = 7/8$. Now we have

$$A_0 = \big\{ x \in A : d(x,y) = d(A,B) = 7/8 \text{ for some } y \in B \big\} = \{1\},$$

$$B_0 = \big\{ y \in B : d(x,y) = d(A,B) = 7/8 \text{ for some } x \in A \big\} = \{15/8\}.$$

Also, $T(A_0) = B_0$. Further, clearly, the pair (A, B) has the P-property. Let $x, y \in A$. Note that, if $x = y$, then condition (6) holds. Hence, we assume that $x \neq y$. We shall show that (6) holds. Suppose, on the contrary, there exist $x_0, y_0 \in A$ such that

$$\psi\big(d(Tx_0, Ty_0)\big) > \beta\big(\psi\big(d(x_0, y_0)\big)\big) \psi\big(d(x_0, y_0)\big),$$

and so

$$\frac{1}{2} \times \frac{1}{8} |x_0 - y_0| > \frac{\frac{1}{2}|x_0 - y_0|}{1 + \frac{1}{2}|x_0 - y_0|},$$

which yields that $14 < |x_0 - y_0|$, a contradiction. Therefore condition (6) holds for all $x, y \in A$. Hence, the conditions of Theorem 2.1 hold and T has a unique best proximity point. Here, $x = 1$ is the best proximity point of T.

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