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Multidimensional fixed point theorems for isotone mappings in partially ordered metric spaces

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Abstract

The purpose of this paper is to present some multidimensional fixed point theorems for isotone mappings in a complete metric space endowed with a partial order. Our results include the corresponding unidimensional, coupled, tripled, and k -dimensional fixed point theorems as particular cases.

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1 Introduction

A very recent trend in metrical fixed point theory, initiated by Ran and Reurings [1] and continued by Nieto and Rodríguez-López [2, 3], is to consider a partial order on the ambient metric space (X, d) and to transfer a part of the contractive property of the nonlinear operators into its monotonicity properties. Their results were extended in other contexts: L -spaces [4], probabilistic metric spaces [5], and metric spaces with a graph [6]. Some fixed point theorems are proved for a mixed monotone mapping in a metric space endowed with partial order, and the obtained results found important applications to the existence of solutions for matrix equations or ordinary differential equations and integral equations; see [7–20] and the references therein.

In 2006, Bhaskar and Lakshmikantham [7] introduced the notion of *coupled fixed point* and proved some fixed point theorems under certain conditions. Later, Berinde and Borcut [8] introduced the concept of *tripled fixed point* and proved some related theorems. These results were then extended and generalized by several authors in the last years; see [9–20] and the references therein. Amongst these generalizations, we refer to the one obtained by Berinde [9], who considered the more general contractive condition

$$\begin{aligned} & \varphi\left(\frac{d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))}{2}\right) \\ & \leq \varphi\left(\frac{d(x, u) + d(y, v)}{2}\right) - \psi\left(\frac{d(x, u) + d(y, v)}{2}\right), \end{aligned}$$

where $\varphi, \psi : [0, \infty) \rightarrow [0, \infty)$ are functions satisfying some appropriate conditions, and $x \geq u, y \leq v$.

Currently, the authors of [21–23] obtained some k -dimensional fixed point theorems for mixed monotone mappings, which extended the corresponding coupled, tripled and quadrupled fixed point theorems. In order to guarantee the existence of k -dimensional fixed points, the authors proved that more than one sequences are simultaneously Cauchy. Besides, some authors focused on obtaining some fixed point theorems for a monotone mapping in the context of ordered metric space; see [24, 25] and the references therein. In 2010, Harjani and Sadarangani [24] proved some fixed point theorems for a one-variable monotone mapping in a partially ordered metric space.

It is well known that the fixed point problems for isotone mappings are easier than that of mixed monotone mappings. So, the above results bring us to the following natural question: can we obtain some multidimensional fixed point theorems for a k -variable isotone mapping, which include the unidimensional, coupled, tripled, and k -dimensional fixed point theorems as particular cases?

Motivated and inspired by the above results, we establish some fixed point theorems for a k -variable isotone mapping, which include the unidimensional, coupled, tripled, and k -dimensional fixed point theorems as particular cases. Moreover, we show that some control conditions in the recent corresponding literature are not necessary.

2 Preliminaries

In order to fix the framework needed to state our main results, we recall the following notions. For simplicity, we denote from now on $\underbrace{X \times X \cdots X}_k \times X$ by X^k , where $k \in \mathbb{N}$ and X is a non-empty set. If elements x, y of a partially ordered set (X, \leq) are comparable (i.e. $x \leq y$ or $y \leq x$ holds), we will write $x \asymp y$. Let $\{A, B\}$ be a partition of the set $\Lambda_k = \{1, 2, \dots, k\}$, that is, $A \cup B = \Lambda_k$ and $A \cap B = \emptyset$, $\Omega_{A,B} = \{\sigma : \Lambda_k \rightarrow \Lambda_k : \sigma(A) \subseteq A \text{ and } \sigma(B) \subseteq B\}$ and $\Omega'_{A,B} = \{\sigma : \Lambda_k \rightarrow \Lambda_k : \sigma(A) \subseteq B \text{ and } \sigma(B) \subseteq A\}$. Henceforth, let $\sigma_1, \sigma_2, \dots, \sigma_k$ be k mappings from Λ_k into itself, and let Υ be the k -tuple $(\sigma_1, \sigma_2, \dots, \sigma_k)$.

Definition 2.1 ([12]) Let (R, \leq) be a partially ordered set and d be a metric on R . We say that (R, d, \leq) is regular if the following conditions hold:

- (i) if a non-decreasing sequence $\{x_n\}$ is such that $x_n \rightarrow x$, then $x_n \leq x$ for all n ,
- (ii) if a non-increasing sequence $\{y_n\}$ is such that $y_n \rightarrow y$, then $y_n \geq y$ for all n .

Definition 2.2 (Altering distance function [26]) A function $\psi : [0, +\infty) \rightarrow [0, +\infty)$ is called an altering distance function if the following properties are satisfied:

- (i) ψ is continuous and non-decreasing,
- (ii) $\psi(t) = 0$ if and only if $t = 0$.

Definition 2.3 ([7]) Let (X, \leq) be a partially ordered set and $F : X^2 \rightarrow X$. We say F has the mixed monotone property if F is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad x_1 \leq x_2 \quad \text{implies} \quad F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad y_1 \leq y_2 \quad \text{implies} \quad F(x, y_1) \geq F(x, y_2).$$

Definition 2.4 ([8]) Let (X, \leq) be a partially ordered set and $F : X^3 \rightarrow X$. We say that F has the mixed monotone property if $F(x, y, z)$ is monotone non-decreasing in x , it is monotone non-increasing in y and is monotone non-decreasing in z , that is, for any $x, y, z \in X$,

$$\begin{aligned} x_1, x_2 \in X, \quad x_1 \leq x_2 &\Rightarrow F(x_1, y, z) \leq F(x_2, y, z), \\ y_1, y_2 \in X, \quad y_1 \leq y_2 &\Rightarrow F(x, y_1, z) \geq F(x, y_2, z) \end{aligned}$$

and

$$z_1, z_2 \in X, \quad z_1 \leq z_2 \Rightarrow F(x, y, z_1) \leq F(x, y, z_2).$$

Definition 2.5 ([22]) Let (X, \leq) be a partially ordered space. We say that F has the mixed g -monotone property if F is monotone non-decreasing in arguments of A and monotone non-increasing in arguments of B , i.e., for all $x_1, x_2, \dots, x_n, y, z \in X$ and all i

$$y \leq z \Rightarrow F(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) \leq_i F(x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_n).$$

Definition 2.6 Let $F : X^k \rightarrow X$ be a mapping. A point $(x_1, x_2, \dots, x_k) \in X^k$ is:

- (i) a coupled fixed point [7] if $k = 2$, $F(x_1, x_2) = x_1$ and $F(x_2, x_1) = x_2$;
- (ii) a tripled fixed point [8] if $k = 3$, $F(x_1, x_2, x_3) = x_1$, $F(x_2, x_1, x_2) = x_2$ and $F(x_3, x_2, x_1) = x_3$;
- (iii) a Υ -fixed point [23] of F if $F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \dots, x_{\sigma_i(k)}) = x_i$ for $i \in \Lambda_k$.

Let (X, \leq) be a partially ordered set and d be a metric on X such that (X, d) is a metric space. We use the next notation from [22]:

$$x \leq_i y \Leftrightarrow \begin{cases} x \leq y, & \text{if } i \in A, \\ x \geq y, & \text{if } i \in B. \end{cases}$$

The product space X^k is endowed with the following natural partial order: for $(y_1, y_2, \dots, y_i, \dots, y_k), (v_1, v_2, \dots, v_i, \dots, v_k) \in X^k$,

$$(y_1, y_2, \dots, y_i, \dots, y_k) \leq (v_1, v_2, \dots, v_i, \dots, v_k) \Leftrightarrow y_i \leq_i v_i, \tag{2.1}$$

which will be denoted in the sequel, for convenience, by \leq , also. Obviously, (X^k, \leq) is a partially ordered set. In particular, we denote by A the odd numbers in Λ_k and by B its even numbers if $k = 2$ or 3 . The mapping $\rho_k : X^k \times X^k \rightarrow [0, +\infty)$, given by

$$\rho_k(Y, V) = \frac{1}{k} [d(y_1, v_1) + d(y_2, v_2) + \dots + d(y_k, v_k)], \tag{2.2}$$

where $Y = (y_1, y_2, \dots, y_k), V = (v_1, v_2, \dots, v_k) \in X^k$, defines a metric on X^k . It is easy to see that

$$\rho_k(Y_n, Y) \rightarrow 0 \quad (\text{as } n \rightarrow \infty) \Leftrightarrow d(y_i^n, y_i) \rightarrow 0 \quad (\text{as } n \rightarrow \infty), i = 1, 2, \dots, k, \tag{2.3}$$

where $Y_n = (y_1^n, y_2^n, \dots, y_k^n), Y = (y_1, y_2, \dots, y_k) \in X^k$.

Let Φ denote the set of all continuous and strictly increasing functions $\varphi : [0, \infty) \rightarrow [0, \infty)$, and Ψ denote the set of all functions $\psi : [0, \infty) \rightarrow [0, \infty)$ such that $\lim_{t \rightarrow r} \psi(t) > 0$ for all $r > 0$.

Inspired by Definitions 2.3-2.6, we give Definitions 2.7 and 2.9.

Definition 2.7 Let (X, \leq) be a partially ordered set and T be a self-mapping on X^k . It is said that T is an isotone property if, for any $Y_1, Y_2 \in X^k$,

$$Y_1 \leq Y_2 \quad \Rightarrow \quad T(Y_1) \leq T(Y_2).$$

Remark 2.8 Note that if $k = 1$ in Definition 2.7, then T is a non-decreasing mapping (see [2]).

Definition 2.9 An element $Y \in X^k$ is called a fixed point of the mapping $T : X^k \rightarrow X^k$ if $T(Y) = Y$.

In order to prove our main results, we need the following lemmas.

Lemma 2.10 ([27]) *Let (X, \leq) be a partially ordered set and d be a metric on X . If (X, \leq, d) is regular, then (X^k, \leq, ρ_k) is regular.*

3 Main results

Now, we state and prove our main results.

Theorem 3.1 *Let (X, \leq) be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T : X^k \rightarrow X^k$ be an isotone mapping for which there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that, for all $Y, V \in X^k$ with $Y \geq V$,*

$$\varphi(\rho_k(T(Y), T(V))) \leq \varphi(\rho_k(Y, V)) - \psi(\rho_k(Y, V)), \tag{3.1}$$

where ρ_k is defined by (2.2). Suppose either

- (a) T is continuous or
- (b) (X, \leq, d) is regular.

If there exists $Z_0 \in X^k$ such that $Z_0 \asymp T(Z_0)$, then T has a fixed point.

Proof Consider the Picard iteration associated to T and the initial approximation Z_0 , that is, the sequence $\{Z_n\} \subset X^k$ defined by $Z_{n+1} = T(Z_n)$ for $n \geq 0$. Obviously, if $Z_{n_0+1} = Z_{n_0}$ for some $n_0 \geq 0$, then Z_{n_0} is a fixed point of T . So, we may assume that $Z_{n+1} \neq Z_n$ for every $n \geq 0$.

Since $Z_0 \asymp T(Z_0)$, without loss of generality, we assume that $Z_0 \leq T(Z_0)$ (the case $Z_0 \geq T(Z_0)$ is treated similarly), that is, $Z_0 \leq Z_1$. Since T is an isotone mapping, we see that the sequence $\{Z_n\}_{n=0}^\infty$ is non-decreasing. Taking $Y = Z_n$ and $V = Z_{n-1}$ in (3.1), we obtain

$$\begin{aligned} \varphi(\rho_k(Z_{n+1}, Z_n)) &= \varphi(\rho_k(T(Z_n), T(Z_{n-1}))) \\ &\leq \varphi(\rho_k(Z_n, Z_{n-1})) - \psi(\rho_k(Z_n, Z_{n-1})), \quad \forall n \geq 1. \end{aligned} \tag{3.2}$$

In view of the fact that $\psi \geq 0$,

$$\varphi(\rho_k(Z_{n+1}, Z_n)) \leq \varphi(\rho_k(Z_n, Z_{n-1})), \quad \forall n \geq 1.$$

Since φ is strictly increasing, we have

$$\rho_k(Z_{n+1}, Z_n) \leq \rho_k(Z_n, Z_{n-1}), \quad \forall n \geq 1.$$

Hence, the sequence $\{\delta_n\}_{n=0}^\infty$ given by $\delta_n = \rho_k(Z_{n+1}, Z_n)$ is monotone decreasing and bounded below. Therefore, there exists some $\delta \geq 0$ such that $\lim_{n \rightarrow \infty} \delta_n = \delta$. We shall prove that $\delta = 0$. Assume that $\delta > 0$. Then by letting $n \rightarrow \infty$ in (3.2) and using the properties of φ and ψ , we have

$$\varphi(\delta) \leq \varphi(\delta) - \lim_{n \rightarrow \infty} \psi(\delta_{n-1}) = \varphi(\delta) - \lim_{\delta_{n-1} \rightarrow \delta} \psi(\delta_{n-1}) < \varphi(\delta),$$

which is a contradiction. Thus,

$$\lim_{n \rightarrow \infty} \delta_n = 0. \tag{3.3}$$

We claim that $\{Z_n\}$ is a Cauchy sequence. Indeed, if not, then there would exist an $\epsilon > 0$ and subsequences $\{Z_{m(t)}\}$ and $\{Z_{n(t)}\}$ of $\{Z_n\}$ such that $n(t)$ is the minimal in the sense that $n(t) > m(t) \geq t$ and $\rho_k(Z_{m(t)}, Z_{n(t)}) > \epsilon$. Therefore, $\rho_k(Z_{m(t)}, Z_{n(t)-1}) \leq \epsilon$.

Using the triangle inequality, we obtain

$$\begin{aligned} \epsilon &< \rho_k(Z_{m(t)}, Z_{n(t)}) \leq \rho_k(Z_{m(t)}, Z_{m(t)-1}) + \rho_k(Z_{m(t)-1}, Z_{n(t)-1}) + \rho_k(Z_{n(t)-1}, Z_{n(t)}) \\ &\leq \rho_k(Z_{m(t)}, Z_{m(t)-1}) + \rho_k(Z_{m(t)-1}, Z_{m(t)}) + \rho_k(Z_{m(t)}, Z_{n(t)-1}) + \rho_k(Z_{n(t)-1}, Z_{n(t)}) \\ &\leq 2\rho_k(Z_{m(t)}, Z_{m(t)-1}) + \epsilon + \rho_k(Z_{n(t)-1}, Z_{n(t)}). \end{aligned}$$

Letting $t \rightarrow \infty$ in the above inequality and using (3.3), we get

$$\lim_{t \rightarrow \infty} \rho_k(Z_{m(t)}, Z_{n(t)}) = \lim_{t \rightarrow \infty} \rho_k(Z_{m(t)-1}, Z_{n(t)-1}) = \epsilon. \tag{3.4}$$

Since $n(t) > m(t)$, we have $Z_{m(t)-1} \leq Z_{n(t)-1}$. Hence, using (3.1) with $Y = Z_{n(t)-1}$ and $V = Z_{m(t)-1}$, we obtain

$$\varphi(\rho_k(Z_{n(t)}, Z_{m(t)})) \leq \varphi(\rho_k(Z_{n(t)-1}, Z_{m(t)-1})) - \psi(\rho_k(Z_{n(t)-1}, Z_{m(t)-1})).$$

Letting $t \rightarrow \infty$ in the above inequality and using (3.4), we have

$$\varphi(\epsilon) \leq \varphi(\epsilon) - \lim_{t \rightarrow \infty} \psi(r_t) = \varphi(\epsilon) - \lim_{r_t \rightarrow \epsilon} \psi(r_t) < \varphi(\epsilon),$$

where $r_t = \rho_k(Z_{n(t)-1}, Z_{m(t)-1})$, which is a contradiction. Hence, the sequence $\{Z_n\}_{n=0}^\infty$ is a Cauchy sequence in the metric space (X^k, ρ_k) . Since (X, d) is a complete metric space, by (2.2), we find that (X^k, ρ_k) is complete. Therefore, there exists $\bar{Z} \in X^k$ such that $\lim_{n \rightarrow \infty} Z_n = \bar{Z}$.

Now suppose that (a) holds. It follows from $Z_{n+1} = T(Z_n)$ that \bar{Z} is a fixed point of T , that is, $T(\bar{Z}) = \bar{Z}$.

Suppose that (b) holds. Using Lemma 2.10, we find that (X^k, \leq, ρ_k) is regular. Thus, by $\{Z_n\}_{n=0}^\infty$ is non-decreasing sequence that converges to \bar{Z} , we have $Z_n \leq \bar{Z}$ for all $n \geq 0$.

From (3.1) and the fact that $\psi \geq 0$, we obtain

$$\begin{aligned} \varphi(\rho_k(Z_{n+1}, T(\bar{Z}))) &= \varphi(\rho_k(T(Z_n), T(\bar{Z}))) \\ &\leq \varphi(\rho_k(Z_n, \bar{Z})) - \psi(\rho_k(Z_n, \bar{Z})) \leq \varphi(\rho_k(Z_n, \bar{Z})) \end{aligned} \quad (3.5)$$

for all $n \geq 0$. From (3.5) and the strict monotonicity of φ , we have

$$\rho_k(Z_{n+1}, T(\bar{Z})) \leq \rho_k(Z_n, \bar{Z}). \quad (3.6)$$

Letting $n \rightarrow \infty$ in (3.6) and using $\lim_{n \rightarrow \infty} Z_n = \bar{Z}$, we get $\rho_k(\bar{Z}, T(\bar{Z})) \leq \rho_k(\bar{Z}, \bar{Z}) = 0$ and so $\rho_k(\bar{Z}, T(\bar{Z})) = 0$, which implies $\bar{Z} = T(\bar{Z})$. \square

Taking $k = 1$ in Theorem 3.1, we obtain the following result immediately.

Corollary 3.2 *Let (X, \leq) be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T : X \rightarrow X$ be a non-decreasing mapping for which there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that, for all $y, v \in X$ with $y \geq v$,*

$$\varphi(d(T(y), T(v))) \leq \varphi(d(y, v)) - \psi(d(y, v)).$$

Suppose either

- (a) T is continuous or
- (b) (X, d, \leq) is regular.

If there exists $z_0 \in X$ such that $z_0 \asymp T(z_0)$, then T has a fixed point.

Remark 3.3 Corollary 3.2 includes Theorems 2.1 and 2.2 in [24] as particular cases. Note that in [24] the authors use only condition $z_0 \leq T(z_0)$, although the alternative assumption $z_0 \geq T(z_0)$ is also acceptable. In addition, the condition in [24], $\varphi(t) = 0$ if and only if $t = 0$, is not necessary in Corollary 3.2. Moreover, the control condition $\psi \in \Psi$ is weaker than the condition in [24] (i.e., ψ is an altering distance function).

Taking $k = 2$ and $T(Y) = (F(x, y), F(y, x))$ for $Y = (x, y) \in X^2$ in Theorem 3.1, we obtain the following result.

Corollary 3.4 *Let (X, \leq) be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^2 \rightarrow X$ be a mixed monotone mapping for which there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that, for all $x, y, u, v \in X$ with $x \geq u$, $y \leq v$,*

$$\begin{aligned} \varphi\left(\frac{d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))}{2}\right) \\ \leq \varphi\left(\frac{d(x, u) + d(y, v)}{2}\right) - \psi\left(\frac{d(x, u) + d(y, v)}{2}\right). \end{aligned} \quad (3.7)$$

Suppose either

- (a) F is continuous or
- (b) (X, d, \leq) is regular.

If there exist $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0, y_0) \quad \text{and} \quad y_0 \geq F(y_0, x_0) \tag{3.8}$$

or

$$x_0 \geq F(x_0, y_0) \quad \text{and} \quad y_0 \leq F(y_0, x_0), \tag{3.9}$$

then there exist $\bar{x}, \bar{y} \in X$ such that $\bar{x} = F(\bar{x}, \bar{y})$ and $\bar{y} = F(\bar{y}, \bar{x})$.

Proof Now we use the result of Theorem 3.1 ($k = 2$) to prove Corollary 3.4. For simplicity, we denote $Y = (x, y)$, $V = (u, v)$, $\tilde{Z} = (\tilde{x}, \tilde{y})$, and $Z_n = (x_n, y_n)$ for all $n \geq 0$. The product space X^2 is endowed with the following partial order:

$$\text{for } Y, V \in X^2, \quad Y \leq V \iff x \leq u, \quad y \geq v. \tag{3.10}$$

Consider the function $\rho_2 : X^2 \times X^2 \rightarrow [0, +\infty)$ defined by

$$\rho_2(Y, V) = \frac{1}{2} [d(x, u) + d(y, v)], \quad \forall Y, V \in X^2. \tag{3.11}$$

Obviously, (X^2, \leq) and ρ_2 are two particular cases of (X^k, \leq) and ρ_k defined by (2.1) and (2.2), respectively.

Let the operator $T : X^2 \rightarrow X^2$ be defined by

$$T(Y) = (F(x, y), F(y, x)), \quad \forall Y \in X^2. \tag{3.12}$$

Now we show that T is an isotone mapping. Suppose that $Z_1 \leq Z_2$ for $Z_1, Z_2 \in X^2$. From (3.10), we have $x_1 \leq x_2, y_1 \geq y_2$. Thus, by the mixed monotonicity of F , we have

$$F(x_1, y_1) \leq F(x_2, y_2) \quad \text{and} \quad F(y_1, x_1) \geq F(y_2, x_2) \tag{3.13}$$

for any $Z_1 \leq Z_2$. From (3.10), (3.12), and (3.13), we have

$$T(Z_1) = (F(x_1, y_1), F(y_1, x_1)) \leq (F(x_2, y_2), F(y_2, x_2)) = T(Z_2)$$

for any $Z_1 \leq Z_2$. Therefore, T is an isotone mapping.

From (3.11) and (3.12), we have

$$\frac{d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))}{2} = \rho_2(T(Y), T(V))$$

and

$$\frac{d(x, u) + d(y, v)}{2} = \rho_2(Y, V)$$

for $Y, V \in X^2$. Thus, by (3.7), we have $\varphi(\rho_2(T(Y), T(V))) \leq \varphi(\rho_2(Y, V)) - \psi(\rho_2(Y, V))$, $\forall Y \geq V$. By (3.8)-(3.10), there exists $Z_0 \in X^2$ such that $Z_0 \asymp T(Z_0)$.

Now suppose that (a) holds. It is easy to see that T is continuous. Indeed, let $\{Z_n\}$ be a sequence in X^2 such that $\rho_2(Z_n, \tilde{Z}) \rightarrow 0$ (as $n \rightarrow \infty$), where $\tilde{Z} \in X^2$. From (3.11) and (2.3), we have $d(x_n, \tilde{x}) \rightarrow 0$ and $d(y_n, \tilde{y}) \rightarrow 0$ as $n \rightarrow \infty$. Since F is continuous, this implies $d(F(x_n, y_n), F(\tilde{x}, \tilde{y})) \rightarrow 0$ and $d(F(y_n, x_n), F(\tilde{y}, \tilde{x})) \rightarrow 0$ as $n \rightarrow \infty$. Thus, by (3.11) and (3.12), we have

$$\rho_2(T(Z_n), T(\tilde{Z})) = \frac{1}{2} [d(F(x_n, y_n), F(\tilde{x}, \tilde{y})) + d(F(y_n, x_n), F(\tilde{y}, \tilde{x}))] \rightarrow 0 \quad (n \rightarrow \infty),$$

which implies that T is continuous.

Hence, there is no doubt that all conditions of Theorem 3.1 are satisfied. Hence, T has a fixed point $\tilde{Z} = (\tilde{x}, \tilde{y}) \in X^2$. From (3.12), we have $\tilde{x} = F(\tilde{x}, \tilde{y})$ and $\tilde{y} = F(\tilde{y}, \tilde{x})$. This completes the proof. \square

Remark 3.5 It is worth pointing out that we omit the control conditions: $\varphi(t) < t$ for all $t > 0$, $\varphi(t + s) \leq \varphi(t) + \varphi(s)$ for all $t, s \in [0, \infty)$ and $\lim_{t \rightarrow 0^+} \psi(t) = 0$, which are necessary in the proof of Theorem 2 in Berinde [9].

Taking $k = 3$ and $T(Y) = (F(x, y, z), F(y, x, y), F(z, y, x))$ for $Y = (x, y, z) \in X^3$ in Theorem 3.1, by a similar argument to the proof of Corollary 3.4, we obtain the following result.

Corollary 3.6 Let (X, \leq) be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^3 \rightarrow X$ be a mixed monotone mapping for which there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that, for all $x, y, z, u, v, w \in X$ with $x \geq u, y \leq v, z \geq w$,

$$\begin{aligned} & \varphi \left(\frac{d(F(x, y, z), F(u, v, w)) + d(F(y, x, y), F(v, u, v)) + d(F(z, y, x), F(w, v, u))}{3} \right) \\ & \leq \varphi \left(\frac{d(x, u) + d(y, v) + d(z, w)}{3} \right) - \psi \left(\frac{d(x, u) + d(y, v) + d(z, w)}{3} \right). \end{aligned}$$

Suppose either

- (a) F is continuous or
- (b) (X, \leq, d) is regular.

If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0, y_0, z_0), \quad y_0 \geq F(y_0, x_0, y_0) \quad \text{and} \quad z_0 \leq F(z_0, y_0, x_0) \tag{3.14}$$

or

$$x_0 \geq F(x_0, y_0, z_0), \quad y_0 \leq F(y_0, x_0, y_0) \quad \text{and} \quad z_0 \geq F(z_0, y_0, x_0), \tag{3.15}$$

then there exist $x, y, z \in X$ such that

$$F(x, y, z) = x, \quad F(y, x, y) = y \quad \text{and} \quad F(z, y, x) = z,$$

that is, F has a tripled fixed point.

Theorem 3.7 *Let (X, \leq) be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T : X^k \rightarrow X^k$ be an isotone mapping for which there exists a constant $\alpha \in [0, 1)$ such that, for all $Y, V \in X^k$ with $Y \geq V$,*

$$\rho_k(T(Y), T(V)) \leq \alpha \rho_k(Y, V), \tag{3.16}$$

where ρ_k is defined by (2.2). Suppose either T is continuous or (X, \leq, d) is regular. If there exists $Z_0 \in X^k$ such that $Z_0 \succ T(Z_0)$, then T has a fixed point.

Proof Taking $\varphi(t) = t$ and $\psi(t) = (1 - \alpha)t$ for $t \in [0, \infty)$ and $\alpha \in [0, 1)$ in Theorem 3.1, we obtain the above result immediately. \square

Remark 3.8 The metric ρ_k in Theorem 3.1 can be replaced by some other metrics on X^k , for example, by the next one:

$$\bar{\rho}_k(Y, V) = \max_{1 \leq i \leq k} d(y_i, v_i), \tag{3.17}$$

where $Y = (y_1, y_2, \dots, y_k)$ and $V = (v_1, v_2, \dots, v_k) \in X^k$, and the result will also be true.

Using a similar argument to the proof of Corollaries 3.4 and 3.6, we deduce the following corollaries from Theorem 3.7, respectively.

Corollary 3.9 *Let (X, \leq) be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^2 \rightarrow X$ be a mixed monotone mapping for which there exists a constant $\alpha \in [0, 1)$ such that, for each $x \geq u, y \leq v$,*

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leq \alpha [d(x, u) + d(y, v)].$$

Suppose either F is continuous or (X, \leq, d) is regular. If there exist $x_0, y_0 \in X$ such that (3.8) or (3.9), then there exist $\bar{x}, \bar{y} \in X$ such that $\bar{x} = F(\bar{x}, \bar{y})$ and $\bar{y} = F(\bar{y}, \bar{x})$.

Corollary 3.10 *Let (X, \leq) be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^3 \rightarrow X$ be a mixed monotone mapping for which there exists a constant $\alpha \in [0, 1)$ such that, for each $x \geq u, y \leq v, z \geq w$,*

$$d(F(x, y, z), F(u, v, w)) + d(F(y, x, y), F(v, u, v)) + d(F(z, y, x), F(w, v, u)) \leq \alpha [d(x, u) + d(y, v) + d(z, w)].$$

Suppose either F is continuous or (X, \leq, d) is regular. If there exist $x_0, y_0, z_0 \in X$ such that (3.14) or (3.15), then F has a tripled fixed point.

Now we give a k -dimensional fixed point theorem for mixed monotone mappings.

Corollary 3.11 *Let (X, \leq) be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $\Upsilon = (\sigma_1, \sigma_2, \dots, \sigma_k)$ be a k -tuple of mapping*

from Λ_k into itself such that $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$. Let $F : X^k \rightarrow X$ be a mixed monotone mapping. Assume that there exists $\alpha \in [0, 1)$ verifying

$$d(F(x_1, x_2, \dots, x_k), F(y_1, y_2, \dots, y_k)) \leq \alpha \max_{1 \leq i \leq k} d(x_i, y_i) \tag{3.18}$$

for which $x_i \leq_i y_i$ for $i \in \Lambda_k$. Suppose either F is continuous or (X, \leq, d) is regular. If there exist $x_0^1, x_0^2, \dots, x_0^k \in X$ verifying

$$x_0^i \leq_i F(x_0^{\sigma_i(1)}, x_0^{\sigma_i(2)}, \dots, x_0^{\sigma_i(k)}) \quad \text{for } i \in \Lambda_k, \tag{3.19}$$

then F has one Υ -fixed point.

Proof For brevity, (y_1, y_2, \dots, y_k) , (v_1, v_2, \dots, v_k) , and $(x_0^1, x_0^2, \dots, x_0^k)$ will be denoted by Y , V and X_0 , respectively. Consider the mapping $T : X^k \rightarrow X^k$ defined by

$$T(Y) = (F(y_{\sigma_1(1)}, y_{\sigma_1(2)}, \dots, y_{\sigma_1(k)}), \dots, F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \dots, y_{\sigma_i(k)}), \dots, F(y_{\sigma_k(1)}, y_{\sigma_k(2)}, \dots, y_{\sigma_k(k)})) \tag{3.20}$$

for $Y \in X^k$.

Now, we show that T is an isotone mapping. Indeed, suppose that $Y \leq V$ for $Y, V \in X^k$. By (2.1), we have $y_t \leq v_t$ when $t \in A$ and $y_t \geq v_t$ when $t \in B$. For each $i \in A$, we have $\sigma_i \in \Omega_{A,B}$. So, $y_{\sigma_i(t)} \leq v_{\sigma_i(t)}$, $\forall t \in A$ and $y_{\sigma_i(t)} \geq v_{\sigma_i(t)}$, $\forall t \in B$. Thus, by the mixed monotonicity of F , we have, for fixed $i \in A$,

$$F(y_{\sigma_i(1)}, \dots, y_{\sigma_i(t-1)}, y_{\sigma_i(t)}, y_{\sigma_i(t+1)}, \dots, y_{\sigma_i(k)}) \leq F(y_{\sigma_i(1)}, \dots, y_{\sigma_i(t-1)}, v_{\sigma_i(t)}, y_{\sigma_i(t+1)}, \dots, y_{\sigma_i(k)}) \tag{3.21}$$

when $t \in A$. Similarly, if $t \in B$, then the inequality (3.21) holds for fixed $i \in A$. So, for fixed $i \in A$, inequality (3.21) holds for $t \in \Lambda_k$. From this, we have

$$\begin{aligned} F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \dots, y_{\sigma_i(k)}) &\leq F(v_{\sigma_i(1)}, y_{\sigma_i(2)}, y_{\sigma_i(3)}, \dots, y_{\sigma_i(k)}) \\ &\leq F(v_{\sigma_i(1)}, v_{\sigma_i(2)}, y_{\sigma_i(3)}, \dots, y_{\sigma_i(k)}) \\ &\dots \\ &\leq F(v_{\sigma_i(1)}, v_{\sigma_i(2)}, v_{\sigma_i(3)}, \dots, v_{\sigma_i(k)}) \end{aligned} \tag{3.22}$$

for $i \in A$. Similarly, we have

$$F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \dots, y_{\sigma_i(k)}) \geq F(v_{\sigma_i(1)}, v_{\sigma_i(2)}, v_{\sigma_i(3)}, \dots, v_{\sigma_i(k)}) \tag{3.23}$$

for $i \in B$. From (3.20), (3.22), and (3.23), we deduce that T is an isotone mapping.

Suppose that $Y \leq V$ for $Y, V \in X^k$. For fixed $i \in A$, we have $y_{\sigma_i(t)} \leq v_{\sigma_i(t)}$ for $t \in \Lambda_k$. From (3.18), we have

$$d(F(y_{\sigma_i(1)}, \dots, y_{\sigma_i(k)}), F(v_{\sigma_i(1)}, \dots, v_{\sigma_i(k)})) \leq \alpha \max_{1 \leq i \leq k} d(y_i, v_i) \tag{3.24}$$

for all $i \in A$. Similarly, for fixed $i \in B$, we have $y_{\sigma_i(t)} \geq_t v_{\sigma_i(t)}$ for $t \in \Lambda_k$. It follows from (3.18) that

$$\begin{aligned} & d(F(y_{\sigma_i(1)}, \dots, y_{\sigma_i(k)}), F(v_{\sigma_i(1)}, \dots, v_{\sigma_i(k)})) \\ &= d(F(v_{\sigma_i(1)}, \dots, v_{\sigma_i(k)}), F(y_{\sigma_i(1)}, \dots, y_{\sigma_i(k)})) \\ &\leq \alpha \max_{1 \leq i \leq k} d(y_i, v_i) \end{aligned} \tag{3.25}$$

for all $i \in B$. By (2.1), (3.20), (3.24), and (3.25), we have $\bar{\rho}_k(T(Y), T(V)) \leq \alpha \bar{\rho}_k(Y, V)$, where $\bar{\rho}_k$ is defined by (3.17). From (3.19), there exists $X_0 \in X^k$ such that $X_0 \leq T(X_0)$. Using the same argument as in the proof of Corollary 3.4, we deduce that T is continuous. Using Theorem 3.7, we see that T has a fixed point. So, F has a Υ -fixed point. \square

Remark 3.12 In order to obtain the multidimensional fixed point theorems, the authors of [21–23] constructed some Cauchy sequences using the properties of multidimensional mixed monotone mappings. To prove that more than one sequence is simultaneously Cauchy does not seem so easy. It is also known that the fixed point problems for isotone mappings are easier than for mixed monotone mappings. Using the properties of isotone mappings, we obtain some multidimensional fixed point theorems by constructing only one Cauchy sequence. As an application, we give a simple proof of Corollary 3.11, which is similar to Theorem 9 in [22].

Theorem 3.13 *In addition to the hypotheses of Theorem 3.1, suppose that, for all fixed points $\bar{Y}, Y^* \in X^k$ of T , there exists $Z \in X^k$ such that Z is comparable to \bar{Y} and to Y^* . Then T has a unique fixed point.*

Proof From Theorem 3.1, the set of fixed points of T is non-empty. Assume that \bar{Y} and Y^* are two fixed points of T . Put $Z_0 = Z$ and $Z_{n+1} = T(Z_n)$ for $n \geq 0$. Since Z is comparable to \bar{Y} , we may assume $Z \leq \bar{Y}$. Since T is an isotone mapping, we obtain inductively $Z_n \leq \bar{Y}$ for $n \geq 0$. Therefore, by (3.1), we have

$$\varphi(\rho_k(\bar{Y}, Z_{n+1})) = \varphi(\rho_k(T(\bar{Y}), T(Z_n))) \leq \varphi(\rho_k(\bar{Y}, Z_n)) - \psi(\rho_k(\bar{Y}, Z_n)), \tag{3.26}$$

which, by the fact that $\psi \geq 0$, implies

$$\varphi(\rho_k(\bar{Y}, Z_{n+1})) \leq \varphi(\rho_k(\bar{Y}, Z_n)). \tag{3.27}$$

By (3.27) and the strict monotonicity of φ , we see that the sequence $\{\Delta_n\}$ defined by $\Delta_n = \rho_k(\bar{Y}, Z_n)$ is non-increasing. Hence, there exists $\beta \geq 0$ such that $\lim_{n \rightarrow \infty} \Delta_n = \beta$. We shall prove that $\beta = 0$. Suppose, conversely, that $\beta > 0$. Letting $n \rightarrow \infty$ in (3.26), we get

$$\varphi(\beta) \leq \varphi(\beta) - \lim_{\Delta_n \rightarrow \beta} \psi(\Delta_n) < \varphi(\beta),$$

which is a contradiction. Thus $\beta = 0$, that is,

$$\lim_{n \rightarrow \infty} \rho_k(\bar{Y}, Z_n) = 0. \tag{3.28}$$

Similarly, we obtain

$$\lim_{n \rightarrow \infty} \rho_k(Y^*, Z_n) = 0. \quad (3.29)$$

Combining (3.28) and (3.29) yields $Y^* = \bar{Y}$. □

Using a similar argument to the proof of Corollary 3.4, we deduce the following corollaries from Theorem 3.13, respectively.

Corollary 3.14 *In addition to the hypotheses of Corollary 3.4 (or Corollary 3.9), suppose that, for all coupled fixed points $\bar{Y}, Y^* \in X^2$ of F , there exists $Z \in X^2$ such that Z is comparable to \bar{Y} and to Y^* . Then F has a unique coupled fixed point.*

Corollary 3.15 *In addition to the hypotheses of Corollary 3.6 (or Corollary 3.10), suppose that, for all tripled fixed points $\bar{Y}, Y^* \in X^3$ of F , there exists $Z \in X^3$ such that Z is comparable to \bar{Y} and to Y^* . Then F has a unique tripled fixed point.*

Corollary 3.16 *In addition to the hypotheses of Corollary 3.11, suppose that, for all Υ -fixed points $\bar{Y}, Y^* \in X^k$ of F , there exists $Z \in X^k$ such that Z is comparable to \bar{Y} and to Y^* . Then F has a unique tripled fixed point.*

Remark 3.17 In this paper, the results show how to extend unidimensional fixed point results to the multidimensional case and give a simple and unified approach to some coupled, tripled, and k -dimensional fixed points. It is worth pointing out that we extend the interesting technique of proof in [10] from $k = 2$ to k arbitrary. The results in [19] and [20] are the unidimensional and coupled cases, respectively.

Competing interests

The author declares that she has no competing interests.

Author's contributions

SW completed the paper herself. The author read and approved the final manuscript.

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