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Proximinality in Banach space valued Musielak-Orlicz spaces

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Abstract

Let $(A, \mathcal{A}, \mu)$ be a σ-finite complete measure space and let Y be a subspace of a Banach space X. Let φ be a generalized Φ-function on $(A, \mathcal{A}, \mu)$. Denote by $L^{\varphi}(A, Y)$ and $L^{\varphi}(A, X)$ the Musielak-Orlicz spaces whose functions take values in Y and X, respectively. Firstly, let $f \in L^{\varphi}(A, X)$, we characterize the distance of f from $L^{\varphi}(A, Y)$. Then, if Y is weakly $\mathcal{K}$-analytic and proximinal in X, we show that $L^{\varphi}(A, Y)$ is proximinal in $L^{\varphi}(A, X)$. Finally, we give the connection between the proximinality of $L^{\varphi}(A, Y)$ in $L^{\varphi}(A, X)$ and the proximinality of $L^1(A, Y)$ in $L^1(A, X)$.

Keywords: proximinality; Musielak-Orlicz space; best approximation; weakly $\mathcal{K}$-analytic

1 Introduction

It is well known that Musielak-Orlicz spaces include many spaces as special spaces, such as Lebesgue spaces, weighted Lebesgue spaces, variable Lebesgue spaces and Orlicz spaces; see [1]. Especially, in recent decades, variable exponent function spaces, such as Lebesgue, Sobolev, Besov, Triebel-Lizorkin, Hardy, Morrey, and Herz spaces with variable exponents, have attracted much attention; see [2–16] and references therein. Cheng and the author discussed geometric properties of Banach space valued Bochner-Lebesgue and Bochner-Sobolev spaces with a variable exponent in [17]. Very recently, Musielak-Orlicz-Hardy spaces have been systemically developed; see, for example, [18–22]. These spaces have many applications in various fields such as PDE, electrorheological fluids, and image restoration; see [6, 23–25].

In recent years, proximinality in Banach space valued Bochner-Lebesgue spaces with constant exponent have been extensively studied; see [26–33]. Proximinality in Banach space valued Bochner-Lebesgue spaces with variable exponent was discussed by the author in [34]. In fact, we generalized those results in [29, 31] to Banach space valued Bochner-Lebesgue spaces with a variable exponent. Khandaqji, Khalil and Hussein considered proximinality in Orlicz-Bochner function spaces on the unit interval in [35], and Al-Minawi and Ayesh consider the same problem on finite measures in [36]. The best simultaneous approximation in Banach space valued Orlicz spaces was discussed in [37, 38]. Micherda discussed proximinality of subspaces of vector-valued Musielak-Orlicz spaces via modular in [39]. However, as usual, one considers the best approximation via the norm, so in this paper, we will discuss proximinality of subspaces of vector-valued Musielak-Orlicz spaces via the norm. To proceed, we need to recall some definitions. Our results will be given in the next section.

In what follows, $(A, \mathcal{A}, \mu)$ will be a σ-finite complete measure space. Suppose D is a subset of A, let χ_D be the indicator function on D. Let $(X, \|\cdot\|)$ be a Banach space. The dual space of X is the vector space X^* of all continuous linear mappings from X to $\mathbb{R}$ or $\mathbb{C}$. To avoid a double definition we let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$.

Definition 1 A convex, left-continuous function $\varphi : [0,\infty) \to [0,\infty]$ with $\varphi(0) = 0$, $\lim_{t\to 0^+} \varphi(t) = 0$, $\lim_{t\to\infty} \varphi(t) = \infty$ is called a Φ-function. It is called positive if $\varphi(t) > 0$ for all $t > 0$.

It is easy to see that if φ is a Φ-function, then it is nondecreasing on $[0,\infty)$.

Definition 2 Let $(A, \mathcal{A}, \mu)$ be a σ-finite complete measure space. A real function $\varphi : A \times [0,\infty) \to [0,\infty]$ is called a generalized Φ-function on $(A, \mathcal{A}, \mu)$ if

(a) $\varphi(y, \cdot)$ is a Φ-function for all $y \in A$,
(b) $y \mapsto \varphi(y, t)$ is measurable for all $t \geq 0$.

If φ is a generalized Φ-function on $(A, \mathcal{A}, \mu)$, we write $\varphi \in \Phi(A, \mu)$.

Definition 3 Let $\varphi \in \Phi(A, \mu)$. Define

$$\varrho_\varphi(f) := \int_A \varphi\big(y, \|f(y)\|\big)\, \mathrm{d}\mu(y)$$

for strongly μ-measurable functions $f : A \to X$. Then the Bochner-Musielak-Orlicz space $L^\varphi(A, X)$ is the collection of all strongly μ-measurable functions $f : A \to X$ endowed with the norm:

$$\|f\|_{L^\varphi(A,X)} := \inf\big\{\lambda > 0, \varrho_\varphi(f/\lambda) \leq 1\big\}.$$

Let

$$E^\varphi(A, X) := \big\{f \in L^\varphi(A, X) : \rho_\varphi(\lambda f) < \infty \text{ for all } \lambda > 0\big\}.$$

Definition 4 Let $\varphi \in \Phi(A, \mu)$. The function φ is said to obey the Δ_2-condition if there exists $K \geq 2$ such that

$$\varphi(s, 2t) \leq K\varphi(s, t)$$

for all $s \in A$ and all $t \geq 0$.

When X is $\mathbb{R}$ or $\mathbb{C}$, we simply denote $L^\varphi(A, X)$ by $L^\varphi(A)$, and $E^\varphi(A, X)$ by $E^\varphi(A)$. Usually, $E^\varphi(A, X)$ is a proper subspace of $L^\varphi(A, X)$. But when the φ satisfies the Δ_2-condition, they are the same. It is easy to see that $E^\varphi(A, X) = L^\varphi(A, X)$ is equivalent to $E^\varphi(A) = L^\varphi(A)$, this means that the equality depends only on φ.

We remark that ρ_φ is a semimodular on the space of all X-valued strongly μ-measurable functions on A. For a semimodular, we recommend the reader reference [6]. Let ρ be a semimodular on vector space E, $E_\rho = \{x \in E : \rho(x/\lambda) < \infty \text{ for some } \lambda > 0\}$, $\|x\|_\rho = \inf\{\lambda > 0 : \rho(x/\lambda) \leq 1\}$. We will use the following elementary result for a semimodular, which is Corollary 2.1.15 in [6].

Lemma 1 *Let ρ be a semimodular on E, $x \in E_\rho$.*

(i) *If $\|x\|_\rho \leq 1$, then $\rho(x) \leq \|x\|_\rho$.*

(ii) *If $1 < \|x\|_\rho$, then $\|x\|_\rho \leq \rho(x)$.*

Let X be a Banach space and let Y be a closed subspace of X. Then Y is called proximinal in X if for any $x \in X$ there exists $y \in Y$ such that

$$\|x - y\| = \operatorname{dist}(x, Y) = \inf\{\|x - u\| : u \in Y\}.$$

In this case y is called a best approximation of x in Y. If this best approximation is unique for any $x \in X$, then Y is said to be Chebyshev.

For simplicity, we denote $\|\cdot\|_{L^\varphi(A,X)}$ or $\|\cdot\|_{L^\varphi(A)}$ by $\|\cdot\|_\varphi$. For $f \in L^\varphi(A,X)$, $Y \subset X$, let

$$\operatorname{dist}_\varphi\bigl(f, L^\varphi(A,Y)\bigr) := \inf\bigl\{\|f - g\|_\varphi : g \in L^\varphi(A,Y)\bigr\}.$$

2 Main results

Firstly, we estimate $\operatorname{dist}_\varphi(f, L^\varphi(A,Y))$.

Theorem 1 *Let Y be a subspace of Banach space X. Suppose $\varphi \in \Phi(A,\mu)$. For $f \in L^\varphi(A,X)$, define $\phi : A \to \mathbb{R}$ by $\phi(s) := \operatorname{dist}(f(s), Y)$. Then*

(i) *$\phi \in L^\varphi(A)$ and $\operatorname{dist}_\varphi(f, L^\varphi(A,Y)) \geq \|\phi\|_\varphi$;*

(ii) *$\operatorname{dist}_\varphi(f, L^\varphi(A,Y)) = \|\phi\|_\varphi$ for $f \in E^\varphi(A,X)$.*

Proof (i) Given $f \in L^\varphi(A,X)$, we see that there exists a sequence of simple functions $\{f_n\}$ which converges to f almost everywhere and in $L^\varphi(A,X)$. Since the distance function $d(x,Y)$ is a continuous function of $x \in X$, $\|f_n(s) - f(s)\| \to 0$ implies that $|\operatorname{dist}(f_n(s), Y) - \operatorname{dist}(f(s), Y)| \to 0$. Moreover, each function $\phi_n : A \to \mathbb{R}$ defined by $\phi_n(s) := \operatorname{dist}(f_n(s), Y)$ is a simple function; therefore we conclude that ϕ is measurable. Now, for any $g \in L^\varphi(A,Y)$ and any $\lambda > 0$,

$$\begin{aligned}\rho_\varphi\bigl(\lambda(f - g)\bigr) &= \int_A \varphi\bigl(s, \lambda\bigl\|f(s) - g(s)\bigr\|\bigr)\,\mathrm{d}\mu(s)\\ &\geq \int_A \varphi\bigl(s, \lambda\operatorname{dist}\bigl(f(s), Y\bigr)\bigr)\,\mathrm{d}\mu(s)\\ &= \rho_\varphi(\lambda\phi).\end{aligned}$$

Thus, we have

$$\|f - g\|_\varphi \geq \|\phi\|_\varphi.$$

This implies $\phi \in L^\varphi(A)$ and, by taking an infimum on $g \in L^\varphi(A,Y)$, we have

$$\operatorname{dist}_\varphi\bigl(f, L^\varphi(A,Y)\bigr) \geq \|\phi\|_\varphi.$$

(ii) We first assume that f is a simple function. Let $f(s) := \sum_{i=1}^m \chi_{A_i} x_i$ where $\{A_i\}_{i=1}^m$ are disjoint measurable subsets in A such that $0 < \mu(A_i) < \infty$ and $0 \neq x_i \in X$ for $i \in \{1,\ldots,m\}$. Without loss of generality, we suppose that $\operatorname{dist}_\varphi(f, L^\varphi(A,Y)) = 1$. Let $0 < \epsilon < 1$. Since $\phi(s) \leq$

$\|f(s)\|$, we have $\rho_\varphi(\lambda\phi) \leq \rho_\varphi(\lambda f) < \infty$ for any $\lambda > 0$. Then, by the dominated convergence theorem, we find that there exists $\delta > 0$ such that

$$\int_{A_i} \varphi\bigl(s, \operatorname{dist}(x_i, Y) + \delta\bigr)\,\mathrm{d}\mu(s) \leq \int_{A_i} \varphi\bigl(s, \operatorname{dist}(x_i, Y)\bigr)\,\mathrm{d}\mu(s) + \frac{\epsilon}{m}, \quad \forall i \in \{1, \ldots, m\}.$$

Now take $y_i \in Y$ such that $\|x_i - y_i\| < \operatorname{dist}(x_i, Y) + \delta$ for $i \in \{1, \ldots, m\}$. Let $g(s) = \sum_{i=1}^m \chi_{A_i} y_i$. Therefore $\|f - g\|_\varphi \geq \operatorname{dist}_\varphi(f, L^\varphi(A, Y)) = 1$. By Lemma 1, we see that

$$\begin{aligned}
1 &\leq \|f - g\|_\varphi \leq \rho_\varphi(f - g) = \sum_{i=1}^m \int_{A_i} \varphi\bigl(s, \|x_i - y_i\|\bigr)\,\mathrm{d}\mu(s) \\
&\leq \sum_{i=1}^m \int_{A_i} \varphi\bigl(s, \operatorname{dist}(x_i, Y) + \delta\bigr)\,\mathrm{d}\mu(s) \\
&\leq \sum_{i=1}^m \left(\int_{A_i} \varphi\bigl(s, \operatorname{dist}(x_i, Y)\bigr)\,\mathrm{d}\mu(s) + \frac{\epsilon}{m} \right) \\
&= \int_A \varphi\bigl(s, \operatorname{dist}\bigl(f(s), Y\bigr)\bigr)\,\mathrm{d}\mu(s) + \epsilon.
\end{aligned}$$

Thus, $\rho_\varphi(\phi) \geq 1 - \epsilon$. Since ϵ is arbitrary, we have $\rho_\varphi(\phi) \geq 1$. By Lemma 1 again, we have $\|\phi\|_\varphi \geq 1$. This means that $\|\operatorname{dist}(f(\cdot), Y)\|_\varphi \geq \operatorname{dist}_\varphi(f, L^\varphi(A, Y))$. Therefore, we have proved that $\|\operatorname{dist}(f(\cdot), Y)\|_\varphi = \operatorname{dist}_\varphi(f, L^\varphi(A, Y))$ for simple functions.

Finally, let $f \in E^\varphi(A, X)$, there exists a sequence of simple functions $\{g_n\}_{n\in\mathbb{N}}$ convergent to f μ-almost everywhere, $\|g_n(s)\| \leq \|f(s)\|$ μ-almost everywhere and $\|f - g_n\|_\varphi \to 0$ as n tends to ∞. Let $\phi_n(s) = \operatorname{dist}(g_n(s), Y)$. From the previous proof, we have

$$\|\phi_n\|_\varphi = \operatorname{dist}_\varphi\bigl(g_n, L^\varphi(A, Y)\bigr).$$

It is easy to see that $\operatorname{dist}_\varphi(g_n, L^\varphi(A, Y)) \to \operatorname{dist}_\varphi(f, L^\varphi(A, Y))$ as $n \to \infty$. Since $\phi_n(s) \leq \|g_n(s)\| \leq \|f(s)\|$ μ-almost everywhere, and $\phi_n(s) \to \phi(s)$ μ-almost everywhere as n tends to ∞, by Lemma 2.3.16(c) in [6], we conclude that $\phi_n \to \phi$ in $L^\varphi(A)$. Hence, letting $n \to \infty$, we see that

$$\|\phi\|_\varphi = \operatorname{dist}_\varphi\bigl(f, L^\varphi(A, Y)\bigr),$$

which completes the proof of Theorem 1. □

Corollary 1 *Let Y be a closed subspace of a Banach space X. Suppose $\varphi \in \Phi(A, \mu)$. An element g of $L^\varphi(A, Y)$ is a best approximation to an element f in $E^\varphi(A, X)$ if and only if $g(s)$ is a best approximation in Y to $f(s)$ for almost every $s \in A$. Furthermore, if φ satisfies the Δ_2-condition, then an element g of $L^\varphi(A, Y)$ is a best approximation to an element f in $L^\varphi(A, X)$ if and only if $g(s)$ is a best approximation in Y to $f(s)$ for almost every $s \in A$.*

Corollary 2 *Let Y be a Chebyshev subspace of a Banach space X. Suppose $\varphi \in \Phi(A, \mu)$ satisfies the Δ_2-condition. If $L^\varphi(A, Y)$ is proximinal in $L^\varphi(A, X)$, then it is a Chebyshev subspace of $L^\varphi(A, X)$.*

Remark 1 Our results Theorem 1, Corollaries 1 and 2 cover the results for vector Orlicz spaces in [36]. Indeed, in [36] the authors only considered vector Orlicz spaces on finite measures. Analogous results for best simultaneous approximation were obtained in [37, 38] for vector Orlicz spaces on finite measures.

Next, we transfer the proximinality of Y in X to $L^{\varphi}(A,Y)$ in $L^{\varphi}(A,X)$. To do so, we need some preliminaries.

Lemma 2 *Let $(A,\mathcal{A},\mu)$ be a σ-finite complete measure space. Suppose $\varphi \in \Phi(A,\mu)$. Let Y be a proximinal subspace of a Banach space X. Suppose $f \in L^{\varphi}(A,X)$ and g is a strongly μ-measurable function such that $g(s)$ is a best approximation to $f(s)$ from Y for almost everywhere $s \in A$. Then g is a best approximation to f from $L^{\varphi}(A,Y)$.*

Proof Since $0 \in Y$, it follows that $\|g(s)\| \leq 2\|f(s)\|$ μ-almost everywhere. Thus, $g \in L^{\varphi}(A,Y)$. For each $h \in L^{\varphi}(A,Y)$, by assumption we know that $\|f(s) - g(s)\| \leq \|f(s) - h(s)\|$ μ-almost everywhere. So $\rho_{\varphi}(\lambda(f-g)) \leq \rho_{\varphi}(\lambda(f-h))$ for any $\lambda > 0$. Thus, $\|f-g\|_{\varphi} \leq \|f-h\|_{\varphi}$. This ends the proof. □

Definition 5 Let (T,τ) be a Polish space (*i.e.* a topological space which is separable and completely metrizable). A set $Q \subset T$ is analytic if it is empty or if there exists a continuous mapping $f : \mathbb{N}^{\mathbb{N}} \to T$ satisfying $f(\mathbb{N}^{\mathbb{N}}) = Q$, where $\mathbb{N}^{\mathbb{N}}$ denotes the space of all infinite sequences of natural numbers endowed with the Tychonoff topology.

Definition 6 Let (T,τ) be a Polish space, H a topological space and denote by $\sigma(\mathcal{A})$ the smallest σ-algebra containing all analytic subsets of T. Then a mapping $f : T \to H$ is said to be analytic measurable if $f^{-1}(C) \in \sigma(\mathcal{A})$ for every $C \in \mathcal{B}(H)$, where $\mathcal{B}(H)$ is for the Borel sets of H.

Definition 7 Let H, T be topological spaces. Then a multifunction $F : H \to 2^{T}$ is said to be upper semi-continuous if for every $x \in H$ and for every open set U satisfying $F(x) \subset U$, there exists an open neighborhood V of x such that $F(V) \subset U$.

Definition 8 A subset C of a topological space T is $\mathcal{K}$-analytic if it can be written as $C = \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} F(\sigma)$ for some upper semi-continuous mapping $F : \mathbb{N}^{\mathbb{N}} \to 2^{T}$ with compact values. In the case when T is a Banach space endowed with its weak topology, C is said to be weakly $\mathcal{K}$-analytic.

For the theory of $\mathcal{K}$-analytic sets, we recommend [40]. Specially, all reflexive and all separable Banach spaces are weakly $\mathcal{K}$-analytic. The following lemma is just Theorem 3.3 in [31].

Lemma 3 *Let $(X, \|\cdot\|)$ be a real Banach space and let Y be a proximinal, weakly $\mathcal{K}$-analytic convex subset of X. Then, for each closed and separable set $M \subset X$, there exists an analytic measurable mapping $h : M \to Y$ such that $h(M)$ is separable in Y and $h(x)$ is a best approximation of x in Y for any $x \in M$.*

Thus, following the argument of the proof of (i) → (ii) in [39, p.185], we have the following conclusion, the details being omitted.

Theorem 2 *Let $(A, \mathcal{A}, \mu)$ be a σ-finite complete measure space. Suppose $\varphi \in \Phi(A, \mu)$, and Y is a weakly $\mathcal{K}$-analytic linear subspace of a real Banach space X. If Y is proximinal in X, then $L^\varphi(A, Y)$ is proximinal in $L^\varphi(A, X)$.*

Theorem 3 *Let $(A, \mathcal{A}, \mu)$ be a σ-finite measure space. Suppose $\varphi \in \Phi(A, \mu)$ such that $E^\varphi(A) = L^\varphi(A)$. Let Y be a linear subspace of a real Banach space X. If $L^\varphi(A, Y)$ is proximinal in $L^\varphi(A, X)$, then Y is proximinal in X.*

Proof Since the measure μ is σ-finite, let us choose positive measure set Q such that $\chi_Q \in L^\varphi(A)$. For any $x \in X$, let $f(t) := \chi_Q(t) \cdot x$, $t \in A$. Then $f \in L^\varphi(A, X)$. By the assumption, we know that there is a g in $L^\varphi(A, Y)$ which is a best approximation element of f. Consequently, $g(s)$ is a best approximation to $f(s)$ in Y for almost every $s \in A$ by Corollary 1. Therefore, there is a best approximation element of x in Y. Thus, Y is proximinal in X. □

From Theorems 2 and 3, we deduce the following corollary.

Corollary 3 *Let $(A, \mathcal{A}, \mu)$ be a σ-finite complete measure space. Suppose $\varphi \in \Phi(A, \mu)$ such that $E^\varphi(A) = L^\varphi(A)$. Let Y be a weakly $\mathcal{K}$-analytic linear subspace of a real Banach space X. Then the following conditions are equivalent:*
(i) *Y is proximinal in X;*
(ii) *$L^\varphi(A, Y)$ is proximinal in $L^\varphi(A, X)$.*

Remark 2 An analog to Corollary 3 in terms of a modular was obtained in [39].

Finally, we give a characterization of proximinity of $L^\varphi(A, Y)$ in $L^\varphi(A, X)$ via the proximinity of $L^1(A, Y)$ in $L^1(A, X)$. When $L^\varphi(A, X)$ is a Bochner-Lebesgue space, which was obtained in [29] and [32, 33] on finite measure spaces and σ finite measure spaces, respectively. $L^\varphi(A, X)$ is a Bochner-Orlicz space, which was discussed in [35].

Theorem 4 *Let $(A, \mathcal{A}, \mu)$ be a σ-finite complete measure space. Suppose $\varphi \in \Phi(A, \mu)$ such that the set of simple functions, $S(A, \mu)$, satisfies $S(A, \mu) \subset L^{\varphi^*}(A, \mu)$, where φ^* is the conjugate function of φ (see* [6]*). Let Y be a closed subspace of a Banach space X. If $L^1(A, Y)$ is proximinal in $L^1(A, X)$, then $L^\varphi(A, Y)$ is proximinal in $L^\varphi(A, X)$.*

Proof Since A is σ-finite, we may write $A = \bigcup_{i=1}^\infty A_i$, where $\{A_i\}$ is a sequence of disjoint measurable sets each of finite measure. Let $f \in L^\varphi(A, X)$. For any $n \in \mathbb{N}$, since $\mu(A_n) < \infty$, then $\chi_{A_n} \in L^{\varphi^*}(A)$. Thus, by the norm conjugate formula (see Corollary 2.7.5 in [6]), we find that $f\chi_{A_n} \in L^1(A, X)$. By assumption, we know that there exists $g_n \in L^1(A, Y)$ such that

$$\|f\chi_{A_n} - g_n\|_{L^1} \le \|f\chi_{A_n} - h\|_{L^1}, \quad \forall h \in L^1(A, Y).$$

By Corollary 1, we have, for all $y \in Y$,

$$\bigl\|f(t)\chi_{A_n} - g_n(t)\bigr\| \le \bigl\|f(t)\chi_{A_n} - y\bigr\|$$

μ-almost everywhere. Therefore $g_n(t) = 0$ μ-almost every $t \in A_n^c$. Let $g = \sum_{n=1}^\infty g_n$. Since $f = \sum_{n=1}^\infty f\chi_{A_n}$, it follows that for all $h \in L^\varphi(A, Y)$,

$$\bigl\|f(t) - g(t)\bigr\| \le \bigl\|f(t) - h(t)\bigr\|$$

μ-almost everywhere. Because $0 \in Y$, it follows that $\|g(t)\| \leq 2\|f(t)\|$. Thus, $g \in L^\varphi(A, Y)$ and

$$\|f - g\|_\varphi \leq \|f - h\|_\varphi$$

for all $h \in L^\varphi(A, Y)$. This finishes the proof. □

Theorem 5 *Let $(A, \mathcal{A}, \mu)$ be a σ-finite complete measure space. Suppose $\varphi \in \Phi(A, \mu)$ satisfies $E^\varphi(A) = L^\varphi(A)$ and, for each $t \in A$, $\varphi(t, \cdot)$ is strictly increasing. Let Y be a closed subspace of a Banach space X. If $L^\varphi(A, Y)$ is proximinal in $L^\varphi(A, X)$, then $L^1(A, Y)$ is proximinal in $L^1(A, X)$.*

Proof We use the idea from [35]. Indeed in [35] the authors only considered Banach space valued Orlicz spaces on the unit interval. Since, for each $t \in A$, $\varphi(t, \cdot)$ is strictly increasing, let $\varphi^{-1}(t, \cdot)$ be its inverse function, which means, for each $s \in [0, \infty)$, $\varphi(t, \varphi^{-1}(t, s)) = s$. Define the map $J : L^1(A, X) \to L^\varphi(A, X)$ by setting

$$J(f)(t) := \begin{cases} \frac{\varphi^{-1}(t, \|f(t)\|)}{\|f(t)\|} f(t), & f(t) \neq 0; \\ 0, & f(t) = 0. \end{cases}$$

Then $\|J(f)(t)\| = \varphi^{-1}(t, \|f(t)\|)$. Therefore $\rho_\varphi(J(f)) = \|f\|_{L^1}$. So J is injective. Moreover, if $g \in L^\varphi(A, X)$, let

$$f(t) := \begin{cases} \frac{\varphi(t, \|g(t)\|)}{\|g(t)\|} g(t), & g(t) \neq 0; \\ 0, & g(t) = 0. \end{cases}$$

Then $f(t) \in X$ and $\|f(t)\| = \varphi(t, \|g(t)\|)$. Thus, $f \in L^1(A, X)$. In addition, for $g(t) \neq 0$,

$$J(f)(t) = \frac{\varphi^{-1}(t, \varphi(t, \|g(t)\|))}{\varphi(t, \|g(t)\|)} f(t) = \frac{\|g(t)\|}{\varphi(t, \|g(t)\|)} f(t) = g(t).$$

If $g(t) = 0$, then $f(t) = 0$ also, thus $J(f)(t) = 0 = g(t)$. Hence J is surjective and $J(L^1(A, Y)) = L^\varphi(A, Y)$ also.

Now, let $f \in L^1(A, X)$. Without loss of generality we may suppose that $f(t) \neq 0$ μ-almost everywhere, for otherwise we can restrict our measure to the support of f. Since $J(f) \in L^\varphi(A, X)$, by the assumption, we know that there exists some $g \in L^1(A, Y)$ such that

$$\|J(f) - J(g)\|_\varphi \leq \|J(f) - J(v)\|_\varphi$$

for all $v \in L^1(A, Y)$. By Corollary 1, we see that, for all $y \in Y$,

$$\|J(f)(t) - J(g)(t)\| \leq \|J(f)(t) - y\|$$

μ-almost everywhere. Multiplying both sides of the last inequality by $\frac{\|f(t)\|}{\varphi^{-1}(t, \|f(t)\|)}$, we obtain, for all $y \in Y$,

$$\left\| f(t) - \frac{\|f(t)\|}{\varphi^{-1}(t, \|f(t)\|)} \frac{\varphi^{-1}(t, \|g(t)\|)}{\|g(t)\|} g(t) \right\| \leq \|f(t) - y\|.$$

Let $h(t) = \frac{\|f(t)\|}{\varphi^{-1}(t,\|f(t)\|)} \frac{\varphi^{-1}(t,\|g(t)\|)}{\|g(t)\|} g(t)$. Since $h(t)$ is a best approximation of $f(t)$ in Y, and $0 \in Y$, it follows that $\|h(t)\| \le 2\|f(t)\|$. Therefore, $h \in L^1(A, Y)$. Thus, for all $w \in L^1(A, Y)$,

$$\|f(t) - h(t)\| \le \|f(t) - w(t)\|$$

μ-almost everywhere. Thus, by Corollary 1 h is a best approximation of f in $L^1(A, Y)$. This finishes the proof. □

From Theorems 4 and 5, we deduce the following corollary.

Corollary 4 *Let $(A, \mathcal{A}, \mu)$ be a σ-finite complete measure space. Suppose $\varphi \in \Phi(A, \mu)$ such that $E^\varphi(A) = L^\varphi(A)$, for each $t \in A$, $\varphi(t, \cdot)$ is strictly increasing and the set of simple functions $S(A, \mu)$ satisfies $S(A, \mu) \subset L^{\varphi^*}(A, \mu)$. Let Y be a closed subspace of a Banach space X. Then the following conditions are equivalent*:

(i) *$L^1(A, Y)$ is proximinal in $L^1(A, X)$*;

(ii) *$L^\varphi(A, Y)$ is proximinal in $L^\varphi(A, X)$*.

Remark 3 When (A, μ) is a finite measure and φ is a Orlicz function that satisfies the Δ_2-condition, the result of Corollary 4 was obtained in [36]. While (A, μ) is the unit interval and φ is a Young function that satisfies the Δ_2-condition, the result of Corollary 4 was obtained in [35].

Competing interests
The author declare that he has no competing interests.

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