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# A fixed-point approach to the stability of a functional equation on quadratic forms

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**Abstract**

Using the fixed-point method, we prove the generalized Hyers-Ulam stability of the functional equation

$$f(x+y, z+w) + f(x-y, z-w) = 2f(x, z) + 2f(y, w).$$

The quadratic form $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(x, y) = ax^2 + bxy + cy^2$ is a solution of the above functional equation.

**Keywords:** alternative of fixed point, functional equation, quadratic form, stability

## 1. Introduction

In 1940, S. M. Ulam [1] gave a wide-ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms:

*Let $G_1$ be a group and let $G_2$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?*

The case of approximately additive mappings was solved by D. H. Hyers [2] under the assumption that $G_1$ and $G_2$ are Banach spaces. Thereafter, many authors investigated solutions or stability of various functional equations (see [3-11]).

Let $X$ be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on $X$ if $d$ satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Note that the only substantial difference of the generalized metric from the metric is that the range of generalized metric includes the infinity.

Throughout this paper, let $X$ and $Y$ be two real vector spaces and let $\phi : X \times X \times X \times X \to [0, \infty)$ be a function. For a mapping $f : X \times X \to Y$, consider the functional equation:

$$f(x+y, z+w) + f(x-y, z-w) = 2f(x, z) + 2f(y, w). \tag{1.1}$$

The quadratic form $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(x, y) := ax^2 + bxy + cy^2$ is a solution of the Equation 1.1.

The authors [12] acquired the general solution and proved the stability of the functional Equation 1.1 for the case that $X$ and $Y$ are real vector spaces as follows.

**Theorem A**. *A mapping $f : X \times X \to Y$ satisfies the Equation* 1.1 *for all $x, y, z, w \in X$ if and only if there exist two symmetric bi-additive mappings $S, T : X \times X \to Y$ and a bi-additive mapping $B : X \times X \to Y$ such that*

$$f(x, y) = S(x, x) + B(x, y) + T(y, y)$$

*for all $x, y \in X$.*

From now on, let $Y$ be a complete normed space.

**Theorem B**. *Assume that $\phi$ satisfies the condition*

$$\tilde{\varphi}(x, y, z, w) := \sum_{j=0}^{\infty} \frac{1}{4^{j+1}} \varphi(2^j x, 2^j y, 2^j z, 2^j w) < \infty$$

*for all $x, y, z, w \in X$. Let $f : X \times X \to Y$ be a mapping such that*

$$\| f(x + y, z + w) + f(x - y, z - w) - 2f(x, z) - 2f(y, w) \| \le \varphi(x, y, z, w)$$

*for all $x, y, z, w \in X$. Then, there exists a unique mapping $F : X \times X \to Y$ satisfying the Equation* 1.1 *such that*

$$\| f(x, y) - F(x, y) \| \le \tilde{\varphi}(x, x, y, y) \tag{1.2}$$

*for all $x, y \in X$. The mapping $F$ is given by*

$$F(x, y) := \lim_{j \to \infty} \frac{1}{4^j} f(2^j x, 2^j y)$$

*for all $x, y \in X$.*

In this paper, we prove the stability of the Equation 1.1 using the fixed-point method.

## 2. Stability using the alternative of fixed point

In this section, we investigate the stability of the functional Equation 1.1 using the alternative of fixed point. Before proceeding the proof, we will state the theorem, the alternative of fixed point.

**Theorem 2.1**. (*The alternative of fixed point* [13,14]). *Suppose that we are given a complete generalized metric space $(\Omega, d)$ and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then, for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1} x) = \infty \text{ for all } n \ge 0,$$

*or*

*there exists a positive integer $n_0$ such that*

- *$d(T^n x, T^{n+1} x) < \infty$ for all $n \ge n_0$;*
- *the sequence $(T^n x)$ is convergent to a fixed point $y^*$ of $T$;*
- *$y^*$ is the unique fixed point of $T$ in the set $\Delta = \{y \in \Omega \mid d(T^{n_0} x, y) < \infty\}$;*
- *$d(y, y^*) \le \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.*

**Lemma 2.2**. *Let* $\psi : X \times X \to [0, \infty)$ *be a function given by*

$$\psi(x, y) := \varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{y}{2}, \frac{y}{2}\right)$$

*for all* $x, y \in X$. *Consider the set* $\Omega := \{g \mid g : X \times X \to Y, g(0, 0) = 0\}$ *and the generalized metric* $d$ *on* $\Omega$ *given by*

$$d(g, h) = d_\psi(g, h) := \inf S_\psi(g, h),$$

*where* $S_\psi(g, h) := \{K \in [0, \infty] \mid ||g(x, y) - h(x, y)|| \le K\psi(x, y) \text{ for all } x, y \in X\}$ *for all* $g, h \in \Omega$. *Then,* $(\Omega, d)$ *is complete.*

**Proof**. Let $\{g_n\}$ be a Cauchy sequence in $(\Omega, d)$. Then, given $\varepsilon > 0$, there exists $N$ such that $d(g_n, g_k) < \varepsilon$ if $n, k \ge N$. Let $n, k \ge N$. Since $d(g_n, g_k) = \inf S_\psi(g_n, g_k) < \varepsilon$, there exists $K \in [0, \varepsilon)$ such that

$$\| g_n(x, y) - g_k(x, y) \| \le K\psi(x, y) \le \varepsilon\psi(x, y) \tag{2.1}$$

for all $x, y \in X$. So, for each $x, y \in X$, $\{g_n(x, y)\}$ is a Cauchy sequence in $Y$. Since $Y$ is complete, for each $x, y \in X$, there exists $g(x, y) \in Y$ such that $g_n(x, y) \to g(x, y)$ as $n \to \infty$ and $g(0, 0) = 0$. Thus, we have $g \in \Omega$. By (2.1), we obtain that

$$\begin{aligned} n \ge N &\Rightarrow \| g_n(x, y) - g(x, y) \| \le \varepsilon\psi(x, y) \text{ for all } x, y \in X \\ &\Rightarrow \varepsilon \in S_\psi(g_n, g) \\ &\Rightarrow d(g_n, g) = \inf S_\psi(g_n, g) \le \varepsilon. \end{aligned}$$

Hence, $g_n \to g \in \Omega$ as $n \to \infty$.

By using an idea of Cădariu and Radu (see [15]), we will prove the Hyers-Ulam stability of the functional equation related to quadratic forms.

**Theorem 2.3**. *Assume that* $\phi$ *satisfies the condition*

$$\lim_{n\to\infty} \frac{1}{4^n}\varphi(2^n x, 2^n y, 2^n z, 2^n w) = 0$$

*for all* $x, y, z, w \in X$. *Suppose that a mapping* $f : X \times X \to Y$ *satisfies the functional inequality*

$$\| f(x + y, z + w) + f(x - y, z - w) - 2f(x, z) - 2f(y, w) \| \le \varphi(x, y, z, w) \tag{2.2}$$

*for all* $x, y, z, w \in X$ *and* $f(0, 0) = 0$. *If there exists* $L < 1$ *such that the function* $\psi$ *given in Lemma* 2.2 *has the property*

$$\psi(x, y) \le 4L\psi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{2.3}$$

*for all* $x, y \in X$, *then there exists a unique mapping* $F : X \times X \to Y$ *satisfying* (1.1) *such that the inequality*

$$\| f(x, y) - F(x, y) \| \le \frac{L}{1 - L}\psi(x, y) \tag{2.4}$$

*holds for all* $x, y \in X$.

**Proof**. Consider the complete generalized metric space $(\Omega, d)$ given in Lemma 2.2. Now we define a mapping $T : \Omega \to \Omega$ by

$$Tg(x,y) := \frac{1}{4}g(2x,2y)$$

for all $g \in \Omega$ and all $x, y \in X$. Observe that, for all $g, h \in \Omega$,

$$\begin{aligned} & K' \in S_\psi(g,h) \text{ and } K' < K \\ & \quad \Rightarrow \| g(x,y) - h(x,y) \| \le K'\psi(x,y) \le K\psi(x,y) \text{ for all } x, y \in X \\ & \quad \Rightarrow K \in S_\psi(g,h). \end{aligned}$$

Let $g, h \in \Omega$ and $\varepsilon \in (0, \infty]$. Then, there is a $K' \in S_\psi(g, h)$ such that $K' < d(g, h) + \varepsilon$. By the above observation, we gain $d(g, h) + \varepsilon \in S_\psi(g, h)$. So we get $||g(x, y) - h(x, y)|| \le (d(g, h) + \varepsilon)\, \psi(x, y)$ for all $x, y \in X$. Thus, we have

$$\| \frac{1}{4}g(2x,2y) - \frac{1}{4}h(2x,2y) \| \le \frac{1}{4}(d(g,h) + \varepsilon)\psi(2x,2y)$$

for all $x, y \in X$. By (2.3), we obtain that

$$\| \frac{1}{4}g(2x,2y) - \frac{1}{4}h(2x,2y) \| \le L(d(g,h) + \varepsilon)\psi(x,y)$$

for all $x, y \in X$. Hence, $d(Tg, Th) \le L\,(d(g, h) + \varepsilon)$. Now we obtain that

$$d(Tg,Th) \le L(d(g,h) + \varepsilon)$$

for all $\varepsilon \in (0, \infty]$. Taking the limit as $\varepsilon \to 0^+$ in the above inequality, we get

$$d(Tg,Th) \le Ld(g,h)$$

for all $g, h \in \Omega$, that is, $T$ is a strictly contractive mapping of $\Omega$ with Lipschitz constant $L$.

Putting $y = x$ and $w = z$ in (2.2), by (2.3), we have the inequality

$$\left\| f(x,z) - \frac{1}{4}f(2x,2z) \right\| \le \frac{1}{4}\varphi(x,x,z,z) = \frac{1}{4}\psi(2x,2z) \le L\psi(x,z) \tag{2.5}$$

for all $x, z \in X$. Thus, we obtain that

$$d(f,Tf) \le L < \infty. \tag{2.6}$$

Applying the alternative of fixed point, we see that there exists a fixed point $F$ of $T$ in $\Omega$ such that

$$F(x,y) = \lim_{n\to\infty} \frac{1}{4^n}f(2^n x, 2^n y)$$

for all $x, y \in X$. Replacing $x, y, z, w$ by $2^n x$, $2^n y$, $2^n z$, $2^n w$ in (2.2), respectively, and dividing by $4^n$, we have

$$\begin{aligned} & \| F(x+y,z+w) + F(x-y,z-w) - 2F(x,z) - 2F(y,w) \| \\ & \quad = \lim_{n\to\infty} \frac{1}{4^n} \| f(2^n(x+y), 2^n(z+w)) + f(2^n(x-y), 2^n(z-w)) \\ & \qquad -2f(2^n x, 2^n z) - 2f(2^n y, 2^n w) \| \\ & \quad \le \lim_{n\to\infty} \frac{1}{4^n}\varphi(2^n x, 2^n y, 2^n z, 2^n w) = 0 \end{aligned}$$

for all $x, y, z, w \in X$. Thus, the mapping $F$ satisfies the Equation 1.1. By (2.3) and (2.5), we obtain that

$$\begin{aligned}\| T^n f(x,y) - T^{n+1} f(x,y) \| &= \frac{1}{4^n} \| f(2^n x, 2^n y) - \frac{1}{4} f(2^{n+1} x, 2^{n+1} y) \| \\ &\le \frac{L}{4^n} \psi(2^n x, 2^n y) \le \cdots \le \frac{L}{4^n} (4L)^n \psi(x,y) \\ &= L^{n+1} \psi(x,y)\end{aligned}$$

for all $x, y \in X$ and all $n \in \mathbb{N}$, that is, $d(T^n f, T^{n+1} f) \le L^{n+1} < \infty$ for all $n \in \mathbb{N}$. By the fixed-point alternative, there exists a natural number $n_0$ such that the mapping $F$ is the unique fixed point of $T$ in the set $\Delta = \{g \in \Omega | d(T^{n_0} f, g) < \infty\}$. So we have $d(T^{n_0} f, F) < \infty$. Since

$$d(f, T^{n_0} f) \le d(f, Tf) + d(Tf, T^2 f) + \cdots + d(T^{n_0 - 1} f, T^{n_0} f) < \infty,$$

we get $f \in \Delta$. Thus, we have $d(f, F) \le d(f, T^{m_0} f) + d(T^{m_0} f, F) < \infty$. Hence, we obtain

$$\|f(x,y) - F(x,y)\| \le K\psi(x,y)$$

for all $x \in X$ and some $K \in [0, \infty)$. Again using the fixed-point alternative, we have

$$d(f, F) \le \frac{1}{1-L} d(f, Tf).$$

By (2.6), we may conclude that

$$d(f, F) \le \frac{L}{1-L},$$

which implies the inequality (2.4).

**Lemma 2.4**. *Let* $\psi : X \times X \to [0, \infty)$ *be a function given by*

$$\psi(x,y) := \varphi(2x, 2x, 2y, 2y)$$

*for all* $x, y \in X$. *Consider the set* $\Omega : = \{g \mid g : X \times X \to Y, g(0, 0) = 0\}$ *and the generalized metric d on* $\Omega$ *given by*

$$d(g,h) = d_\psi(g,h) := \inf S_\psi(g,h),$$

*where* $S_\psi(g, h) : = K \in [0, \infty] \mid ||g(x, y) - h(x, y)|| \le K\psi(x, y)$ *for all* $x, y \in X\}$ *for all* $g, h \in \Omega$. *Then,* $(\Omega, d)$ *is complete.*

**Proof**. The proof is similar to the proof of Lemma 2.2.

**Theorem 2.5**. *Assume that* $\phi$ *satisfies the condition*

$$\lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}, \frac{w}{2^n}\right) = 0$$

*for all* $x, y, z, w \in X$. *Suppose that a mapping* $f : X \times X \to Y$ *satisfies the functional inequality* (2.2) *for all* $x, y, z, w \in X$ *and* $f(0, 0) = 0$. *If there exists* $L < 1$ *such that the function* $\psi$ *given in Lemma 2.4 has the property*

$$\psi(x,y) \le \frac{L}{4} \psi(2x, 2y) \tag{2.7}$$

*for all $x, y \in X$, then there exists a unique mapping $F : X \times X \to Y$ satisfying* (1.1) *such that the inequality*

$$\| f(x,y) - F(x,y) \| \leq \frac{L^2}{16(1-L)}\psi(x,y) \tag{2.8}$$

*holds for all $x, y \in X$.*

**Proof**. Consider the complete generalized metric space $(\Omega, d)$ given in Lemma 2.4. Now we define a mapping $T : \Omega \to \Omega$ by

$$Tg(x,y) := 4g\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $g \in \Omega$ and all $x, y \in X$. By the same argument of the proof of Theorem 2.3, $T$ is a strictly contractive mapping of $\Omega$ with Lipschitz constant $L$.

Replacing $x, y, z, w$ by $\frac{x}{2}, \frac{x}{2}, \frac{z}{2}, \frac{z}{2}$ in (2.2), respectively, and using (2.7), we have the inequality

$$\left\| f(x,z) - 4f\left(\frac{x}{2}, \frac{z}{2}\right) \right\| \leq \varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{z}{2}, \frac{z}{2}\right) = \psi\left(\frac{x}{4}, \frac{z}{4}\right) \leq \frac{L}{4}\psi\left(\frac{x}{2}, \frac{z}{2}\right) \leq \frac{L^2}{16}\psi(x,y) \tag{2.9}$$

for all $x, z \in X$. Thus, we obtain that

$$d(f, Tf) \leq \frac{L^2}{16} < \infty. \tag{2.10}$$

Applying the alternative of fixed point, we see that there exists a fixed point $F$ of $T$ in $\Omega$ such that

$$F(x,y) = \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)$$

for all $x, y \in X$. Replacing $x, y, z, w$ by $\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}, \frac{w}{2^n}$ in (2.2), respectively, and multiplying by $4^n$, we have

$$\begin{aligned} &\| F(x+y, z+w) + F(x-y, z-w) - 2F(x,z) - 2F(y,w) \| \\ &= \lim_{n\to\infty} 4^n \left\| f\left(\frac{x+y}{2^n}, \frac{z+w}{2^n}\right) + f\left(\frac{x-y}{2^n}, \frac{z-w}{2^n}\right) - 2f\left(\frac{x}{2^n}, \frac{z}{2^n}\right) - 2f\left(\frac{y}{2^n}, \frac{w}{2^n}\right) \right\| \\ &\leq \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}, \frac{w}{2^n}\right) = 0 \end{aligned}$$

for all $x, y, z, w \in X$. Thus, the mapping $F$ satisfies the Equation 1.1. By (2.7) and (2.9), we obtain that

$$\begin{aligned} ||T^n f(x,y) - T^{n+1} f(x,y)|| &= 4^n \left\| f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) - 4f\left(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}}\right) \right\| \\ &\leq 4^{n-2} L^2 \psi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \leq 4^{n-3} L^3 \psi\left(\frac{x}{2^{n-1}}, \frac{y}{2^{n-1}}\right) \\ &\leq \cdots \leq \frac{L^{n+2}}{16}\psi(x,y) \end{aligned}$$

for all $x, y \in X$ and all $n \in \mathbb{N}$, that is, $d(T^n f, T^{n+1} f) \leq \frac{L^{n+2}}{16} < \infty$ for all $n \in \mathbb{N}$. By the same reasoning of the proof of Theorem 2.3, we have

$$d(f,F) \le \frac{1}{1-L} d(f,Tf).$$

By (2.10), we may conclude that

$$d(f,F) \le \frac{L^2}{16(1-L)},$$

which implies the inequality (2.8).

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**Authors' contributions**
Both authors contributed equally to this work. The authors read and approved the final manuscript.

**Competing interests**
The authors declare that they have no competing interests.

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