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Global smooth solutions to 3D MHD with mixed partial dissipation and magnetic diffusion

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Abstract
In this paper, we prove the existence of global smooth solutions to the Cauchy problem of 3D incompressible magnetohydrodynamics (MHD) flows with mixed partial dissipation and magnetic diffusion if the initial condition is suitably small.

Keywords: incompressible magnetohydrodynamics; global smooth solution; partial dissipation and magnetic diffusion

1 Introduction

In this paper, we consider the following 3D incompressible MHD equations with mixed partial dissipation and magnetic diffusion (see [1]), *i.e.*,

$$u_t + u \cdot \nabla u = -\nabla p + \mu u_{xx} + \mu u_{yy} + b \cdot \nabla b, \tag{1}$$

$$b_t + u \cdot \nabla b = \eta b_{xx} + \eta b_{yy} + b \cdot \nabla u, \tag{2}$$

$$\operatorname{div} u = 0, \qquad \operatorname{div} b = 0, \tag{3}$$

associated with the initial data

$$u(0,x) = u_0, \qquad b(0,x) = b_0. \tag{4}$$

Here $u = (u_1(t,x), u_2(t,x), u_3(t,x))$ is the velocity field, $b = (b_1(t,x), b_2(t,x), b_3(t,x))$ is the magnetic field, $p = p(t,x)$ is scalar pressure, $\mu > 0$ is the kinematic viscosity, $\eta > 0$ is the magnetic diffusion. For more background, we refer the reader to [2] for MHD and [1, 3] for MHD with mixed partial dissipation and magnetic diffusion. Without loss of generality, we assume that $\mu = \eta = 1$ in the remainder of the paper.

To state the main results, we first introduce the following conventions and notations which will be used throughout this paper. Set

$$\int f \, dx \triangleq \int_{\mathbb{R}^3} f \, dx \, dy \, dz,$$

and that $\|\cdot\|$ is the L^2 norm, *i.e.*,

$$\|f\| = \left(\int f^2 \, \mathrm{d}x\right)^{\frac{1}{2}},$$

$$H^m \triangleq W^{m,2}(\mathbb{R}^3) = \left\{ \sum_{|\alpha|\le m} \int \left|D^\alpha u\right|^2 \mathrm{d}x \right\}^{1/2} \quad \text{with the norm } \|\cdot\|_{H^m}.$$

Our main result of this paper can be stated as follows.

Theorem 1.1 *Assume that $u_0 \in H^2$ and $b_0 \in H^2$ with $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$ and $\|u_0\|_{H^1} + \|b_0\|_{H^1} \le \varepsilon$, where ε is a sufficiently small positive number. Then* (1)-(4) *admit global smooth solutions.*

Remark 1.1 Theorem 1.1 is Theorem 1.2 in [1], which has not been proved in their paper. We would also emphasize that our proof of Theorem 1.1 is clearer for deducing the desired a priori estimates in Lemma 2.3 (see the next section) than that of Proposition 3.1 in [1].

The rest of the paper is organized as follows. In Section 2, we deduce the desired *a priori* estimates to complete the proof of Theorem 1.1. We finish the proof of Theorem 1.1 in Section 3 by the method of vanishing viscosities.

2 A priori estimates

In this section, we deduce the desired *a priori* estimates in order to finish the main result. Before we begin to prove the main theorem of this paper, we first state the following useful lemma that was deduced in [1].

Lemma 2.1 *Assume that f, g, h, f_x, g_y, h_z are all in $L^2(\mathbb{R}^3)$. Then we have*

$$\int |fgh| \, \mathrm{d}x \le C\|f\|^{\frac{1}{2}} \|g\|^{\frac{1}{2}} \|h\|^{\frac{1}{2}} \|f_x\|^{\frac{1}{2}} \|g_y\|^{\frac{1}{2}} \|h_z\|^{\frac{1}{2}}.$$

Clearly, the standard energy estimate shows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2 + \|b\|^2\right) + \|u_x\|^2 + \|u_y\|^2 + \|b_x\|^2 + \|b_y\|^2 = 0. \tag{5}$$

We denote that $\omega = \nabla \times u$ and $j = \nabla \times b$. Thus, applying the operator '$\nabla\times$' to (1) and (2), together with (3), we deduce that

$$\omega_t + u\cdot\nabla\omega - \omega\cdot\nabla u = \omega_{xx} + \omega_{yy} + b\cdot\nabla j - j\cdot\nabla b, \tag{6}$$

$$j_t + u\cdot\nabla j = j_{xx} + j_{yy} + b\cdot\nabla\omega + \varepsilon_{ijk}(\partial_j b_l\,\partial_l u_k - \partial_j u_l\,\partial_l b_k), \tag{7}$$

where ε_{ijk} is defined as follows:

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \text{ is an even permutation,} \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation,} \\ 0 & \text{others.} \end{cases}$$

The first key lemma is the following.

Lemma 2.2 *If* (u,b) *solves* (1)-(4) *and the initial data satisfies*

$$\|u_0\|^2 + \|b_0\|^2 \le \frac{1}{4C} \quad \text{and} \quad \|\omega_0\|^2 + \|j_0\|^2 \le \frac{1}{4}, \tag{8}$$

where C *is a suitable large number, then the vorticity* ω *and the current density* j *satisfy*

$$\|\omega\|^2 + \|j\|^2 \le 1, \qquad \int_0^t \big(\|\omega_x\|^2 + \|\omega_y\|^2 + \|j_x\|^2 + \|j_y\|^2\big)\,\mathrm{d}s \le 1 \quad \text{for all } t \ge 0.$$

Proof Multiplying (6) and (7) by ω and j, respectively, then integrating the resulting equations by parts, after adding the two equalities together, we finally deduce

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\omega\|^2 + \|j\|^2\big) + \|\omega_x\|^2 + \|\omega_y\|^2 + \|j_x\|^2 + \|j_y\|^2 \\
&\quad = \int \big(\omega \cdot \nabla\omega \cdot \omega - j \cdot \nabla b \cdot \omega + \varepsilon_{ijk}(\partial_j b_l\, \partial_l u_k - \partial_j u_l\, \partial_l b_k) j_i\big)\,\mathrm{d}x = I + J + K + L.
\end{aligned} \tag{9}$$

We have to estimate each term on the right-hand side of (9). Some of the terms are the same as in [1] and are proved here for completeness. First, I can be written as

$$\begin{aligned}
I &= \int \omega \cdot \nabla u \cdot \omega\,\mathrm{d}x = \int \omega_i\, \partial_i u_j \omega_j\,\mathrm{d}x \\
&= \int \omega_1\, \partial_x u_j \omega_j\,\mathrm{d}x + \int \omega_2\, \partial_y u_j \omega_j\,\mathrm{d}x + \int \omega_3\, \partial_z u_j \omega_j\,\mathrm{d}x = I_1 + I_2 + I_3.
\end{aligned}$$

With the help of Lemma 2.1, we deduce that

$$\begin{aligned}
I_1 &= \int \omega_1\, \partial_x u_j \omega_j\,\mathrm{d}x \le C\|\omega\|^{\frac{1}{2}}\|u_x\|^{\frac{1}{2}}\|\omega\|^{\frac{1}{2}}\|\omega_x\|^{\frac{1}{2}}\|\omega_y\|^{\frac{1}{2}}\|\omega_x\|^{\frac{1}{2}} \\
&\le \frac{1}{24}\|\omega_x\|^2 + \frac{1}{16}\|\omega_y\|^2 + C\|u_x\|^2\|\omega\|^4.
\end{aligned}$$

Similarly, we obtain that

$$I_2 = \int \omega_2\, \partial_y u_j \omega_j\,\mathrm{d}x \le \frac{1}{24}\|\omega_x\|^2 + \frac{1}{16}\|\omega_y\|^2 + C\|u_y\|^2\|\omega\|^4,$$

and

$$\begin{aligned}
I_3 &= \int \omega_3\, \partial_z u_j \omega_j\,\mathrm{d}x = \int (\partial_x u_2 - \partial_y u_1)\, \partial_z u_j \omega_j\,\mathrm{d}x \\
&\le \frac{1}{24}\|\omega_x\|^2 + \frac{1}{16}\|\omega_y\|^2 + C\big(\|u_x\|^2 + \|u_y\|^2\big)\|\omega\|^4.
\end{aligned}$$

In order to bound J, we rewrite the integrand explicitly as follows:

$$\begin{aligned}
J &= -\int j \cdot \nabla b \cdot \omega\,\mathrm{d}x = -\int j_i\, \partial_i b_j \omega_j\,\mathrm{d}x \\
&= -\int j_1\, \partial_x b_j \omega_j\,\mathrm{d}x - \int j_2\, \partial_y b_j \omega_j\,\mathrm{d}x - \int j_3\, \partial_z b_j \omega_j\,\mathrm{d}x = J_1 + J_2 + J_3.
\end{aligned}$$

Due to Lemma 2.1, we see that

$$\begin{aligned} J_1 &= -\int j_1\,\partial_x b_j \omega_j\,\mathrm{d}x \le C\|j\|^{\frac12}\|b_x\|^{\frac12}\|\omega\|^{\frac12}\|j_x\|^{\frac12}\|\partial_x b_z\|^{\frac12}\|\omega_y\|^{\frac12} \\ &\le \frac{1}{22}\|j_x\|^2 + \frac{1}{16}\|\omega_y\|^2 + C\|b_x\|^2\big(\|\omega\|^4 + \|j\|^4\big). \end{aligned}$$

Similarly,

$$\begin{aligned} J_2 &= -\int j_2\,\partial_y b_j \omega_j\,\mathrm{d}x \le \frac{1}{16}\|\omega_y\|^2 + \frac{1}{22}\|j_x\|^2 + \frac{1}{26}\|j_y\|^2 + C\|b_y\|^2\big(\|\omega\|^4 + \|j\|^4\big), \\ J_3 &= -\int j_3\,\partial_z b_j \omega_j\,\mathrm{d}x = -\int (\partial_x b_2 - \partial_y b_1)\,\partial_z b_j \omega_j\,\mathrm{d}x \\ &\le \frac{1}{24}\|\omega_x\|^2 + \frac{1}{22}\|j_x\|^2 + \frac{1}{26}\|j_y\|^2 + C\big(\|b_x\|^2 + \|b_y\|^2\big)\big(\|\omega\|^4 + \|j\|^4\big). \end{aligned}$$

Now, let us turn to bound L,

$$\begin{aligned} L &= -\int \varepsilon_{ijk}\,\partial_j u_l\,\partial_l u_k j_i\,\mathrm{d}x \\ &= -\int \varepsilon_{ijk}\,\partial_x u_l\,\partial_l u_k j_i\,\mathrm{d}x - \int \varepsilon_{ijk}\,\partial_y u_l\,\partial_l u_k j_i\,\mathrm{d}x - \int \varepsilon_{ijk}\,\partial_z u_l\,\partial_l u_k j_i\,\mathrm{d}x \\ &= L_1 + L_2 + L_3. \end{aligned}$$

By Lemma 2.1, we have that

$$\begin{aligned} L_1 &= -\int \varepsilon_{ijk}\,\partial_x u_l\,\partial_l u_k j_i\,\mathrm{d}x \le C\|u_x\|^{\frac12}\|j\|^{\frac12}\|j\|^{\frac12}\|j_x\|^{\frac12}\|j_y\|^{\frac12}\|\omega_x\|^{\frac12} \\ &\le \frac{1}{24}\|\omega_x\|^2 + \frac{1}{22}\|j_x\|^2 + \frac{1}{26}\|j_y\|^2 + C\|u_x\|^2\|j\|^4. \end{aligned}$$

Similarly,

$$L_2 = -\int \varepsilon_{ijk}\,\partial_y u_l\,\partial_l u_k j_i\,\mathrm{d}x \le \frac{1}{16}\|\omega_y\|^2 + \frac{1}{22}\|j_x\|^2 + \frac{1}{26}\|j_y\|^2 + C\|u_y\|^2\|j\|^4.$$

As for L_3, we should split it into three parts:

$$\begin{aligned} L_3 &= -\int \varepsilon_{ijk}\,\partial_z u_l\,\partial_l u_k j_i\,\mathrm{d}x \\ &= -\int \varepsilon_{ijk}\,\partial_z u_1\,\partial_x u_k j_i\,\mathrm{d}x - \int \varepsilon_{ijk}\,\partial_z u_2\,\partial_y u_k j_i\,\mathrm{d}x - \int \varepsilon_{ijk}\,\partial_z u_3\,\partial_z u_k j_i\,\mathrm{d}x \\ &= L_{31} + L_{32} + L_{33}, \\ L_{31} &= -\int \varepsilon_{ijk}\,\partial_z u_1\,\partial_x u_k j_i\,\mathrm{d}x \\ &\le C\|\omega\|^{\frac12}\|b_x\|^{\frac12}\|j\|^{\frac12}\|\omega_x\|^{\frac12}\|\partial_x b_z\|^{\frac12}\|j_y\|^{\frac12} \\ &\le \frac{1}{24}\|\omega_x\|^2 + \frac{1}{22}\|j_x\|^2 + \frac{1}{26}\|j_y\|^2 + C\|b_x\|^2\big(\|\omega\|^4 + \|j\|^4\big). \end{aligned}$$

Similarly,

$$L_{32} = -\int \varepsilon_{ijk}\,\partial_z u_2\,\partial_y u_k j_i\,\mathrm{d}x \le \frac{1}{24}\|\omega_x\|^2 + \frac{1}{26}\|j_y\|^2 + C\|b_y\|^2\big(\|\omega\|^4 + \|j\|^4\big).$$

To bound L_{33}, using the incompressibility condition $\operatorname{div} u = 0$, we deduce that

$$\begin{aligned} L_{33} &= -\int \varepsilon_{ijk}\,\partial_z u_3\,\partial_z u_k j_i\,\mathrm{d}x \\ &= \int \varepsilon_{ijk}(\partial_x u_1 + \partial_y u_2)\,\partial_z u_k j_i\,\mathrm{d}x = L_{33}^1 + L_{33}^2. \end{aligned}$$

We get the L_{33}^1 and L_{33}^2 as follows:

$$\begin{aligned} L_{33}^1 &= \int \varepsilon_{ijk}\,\partial_x u_1\,\partial_z u_k j_i\,\mathrm{d}x \le C\|u_x\|^{\frac{1}{2}}\|j\|^{\frac{1}{2}}\|j\|^{\frac{1}{2}}\|\partial_x u_z\|^{\frac{1}{2}}\|j_x\|^{\frac{1}{2}}\|j_y\|^{\frac{1}{2}} \\ &\le \frac{1}{24}\|\omega_x\|^2 + \frac{1}{22}\|j_x\|^2 + \frac{1}{26}\|j_y\|^2 + C\|u_x\|^2\|j\|^4, \end{aligned}$$

and

$$L_{33}^2 \le \frac{1}{16}\|\omega_y\|^2 + \frac{1}{22}\|j_x\|^2 + \frac{1}{26}\|j_y\|^2 + C\|u_y\|^2\|j\|^4.$$

To bound K, we should divide it into three parts:

$$\begin{aligned} K &= \int \varepsilon_{ijk}\,\partial_j b_l\,\partial_l u_k j_i\,\mathrm{d}x \\ &= \int \varepsilon_{i1k}\,\partial_x b_l\,\partial_l u_k j_i\,\mathrm{d}x + \int \varepsilon_{i2k}\,\partial_y b_l\,\partial_l u_k j_i\,\mathrm{d}x + \int \varepsilon_{i3k}\,\partial_z b_l\,\partial_l u_k j_i\,\mathrm{d}x \\ &= K_1 + K_2 + K_3. \end{aligned}$$

Similarly, we deduce that

$$\begin{aligned} K_1 &= \int \varepsilon_{i1k}\,\partial_x b_l\,\partial_l u_k j_i\,\mathrm{d}x \le C\|b_x\|^{\frac{1}{2}}\|\omega\|^{\frac{1}{2}}\|j\|^{\frac{1}{2}}\|\partial_x b_z\|^{\frac{1}{2}}\|\omega_x\|^{\frac{1}{2}}\|j_y\|^{\frac{1}{2}} \\ &\le \frac{1}{24}\|\omega_x\|^2 + \frac{1}{22}\|j_x\|^2 + \frac{1}{26}\|j_y\|^2 + C\|b_x\|^2\big(\|\omega\|^4 + \|j\|^4\big), \end{aligned}$$

and

$$K_2 = \int \varepsilon_{i2k}\,\partial_y b_l\,\partial_l u_k j_i\,\mathrm{d}x \le \frac{1}{24}\|\omega_x\|^2 + \frac{1}{26}\|j_y\|^2 + C\|b_y\|^2\big(\|\omega\|^4 + \|j\|^4\big).$$

For K_3, we have

$$\begin{aligned} K_3 &= \int \varepsilon_{i3k}\,\partial_z b_l\,\partial_l u_k j_i\,\mathrm{d}x \\ &= \int \varepsilon_{i3k}\,\partial_z b_1\,\partial_x u_k j_i\,\mathrm{d}x + \int \varepsilon_{i3k}\,\partial_z b_2\,\partial_y u_k j_i\,\mathrm{d}x + \int \varepsilon_{i3k}\,\partial_z b_3\,\partial_z u_k j_i\,\mathrm{d}x \\ &= K_{31} + K_{32} + K_{33}. \end{aligned}$$

Thus,

$$K_{31} = \int \varepsilon_{i3k}\, \partial_z b_1\, \partial_x u_k j_i \,\mathrm{d}x \le C\|j\|^{\frac{1}{2}} \|u_x\|^{\frac{1}{2}} \|j\|^{\frac{1}{2}} \|\omega_x\|^{\frac{1}{2}} \|j_x\|^{\frac{1}{2}} \|j_y\|^{\frac{1}{2}}$$
$$\le \frac{1}{24}\|\omega_x\|^2 + \frac{1}{22}\|j_x\|^2 + \frac{1}{26}\|j_y\|^2 + C\|u_x\|^2\|j\|^4,$$

and

$$K_{32} = \int \varepsilon_{i3k}\, \partial_z b_2\, \partial_y u_k j_i \,\mathrm{d}x \le \frac{1}{16}\|\omega_y\|^2 + \frac{1}{22}\|j_x\|^2 + \frac{1}{26}\|j_y\|^2 + C\|u_y\|^2\|j\|^4.$$

As for K_{33}, using $\operatorname{div} b = 0$, we obtain

$$K_{33} = \int \varepsilon_{i3k}\, \partial_z b_3\, \partial_z u_k j_i \,\mathrm{d}x = -\int \varepsilon_{i3k}(\partial_x b_1 + \partial_y b_2)\, \partial_z u_k j_i \,\mathrm{d}x$$
$$\le \frac{1}{24}\|\omega_x\|^2 + \frac{1}{26}\|j_y\|^2 + C\big(\|b_x\|^2 + \|b_y\|^2\big)\big(\|\omega\|^4 + \|j\|^4\big).$$

Substituting all the above estimates into (9), we conclude that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\omega\|^2 + \|j\|^2\big) + \frac{1}{2}\big(\|\omega_x\|^2 + \|\omega_y\|^2 + \|j_x\|^2 + \|j_y\|^2\big)$$
$$\le C\big(\|u_x\|^2 + \|u_y\|^2 + \|b_x\|^2 + \|b_y\|^2\big)\big(\|\omega\|^4 + \|j\|^4\big).$$

Let $\|\omega\|^2 + \|j\|^2 \le 1$, we deduce, with the assumption (8) on the initial data, that

$$\|\omega\|^2 + \|j\|^2 + \int_0^t \big(\|\omega_x\|^2 + \|\omega_y\|^2 + \|j_x\|^2 + \|j_y\|^2\big)\,\mathrm{d}s \le 1.$$

Thus, the proof of Lemma 2.2 is completed. □

Now, we turn to deduce the higher order estimates about the solution.

Lemma 2.3 *If* (u, b) *is the solution of* (1)-(4), *then*

$$\|\nabla\omega\|^2 + \|\nabla j\|^2 + \int_0^t \big(\|\nabla\omega_x\|^2 + \|\nabla\omega_y\|^2 + \|\nabla j_x\|^2 + \|\nabla j_y\|^2\big)\,\mathrm{d}s \le C. \tag{10}$$

Proof Multiplying (6) and (7) by $\Delta\omega$ and Δj, respectively, then integrating the resultant equations by parts, after adding the two equalities together, we finally obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\nabla\omega\|^2 + \|\nabla j\|^2\big) + \big(\|\nabla\omega_x\|^2 + \|\nabla\omega_y\|^2 + \|\nabla j_x\|^2 + \|\nabla j_y\|^2\big)$$
$$= \int \big[u\cdot\nabla\omega\cdot\Delta\omega + u\cdot\nabla j\cdot\Delta j - \omega\cdot\nabla\omega\cdot\Delta\omega + j\cdot\nabla b\cdot\Delta\omega - b\cdot\nabla j\cdot\Delta\omega$$
$$- b\cdot\nabla\omega\cdot\Delta j - \varepsilon_{ijk}(\partial_j b_l\, \partial_l u_k - \partial_j u_l\, \partial_l b_k)\Delta j_i\big]\,\mathrm{d}x$$
$$= M + N + P + Q + R + S. \tag{11}$$

Now, we turn to bound each term on the right-hand side of (11). Similar as the proof of Lemma 2.2, keeping in mind Lemma 2.1 and the divergence-free property of u and b, we deduce

$$\begin{aligned}
M &= \int u\cdot\nabla\omega\cdot\Delta\omega\,\mathrm{d}x = \int u_i\,\partial_i\omega_j\,\partial^2_{kk}\omega_j\,\mathrm{d}x = -\int \partial_k u_i\,\partial_i\omega_j\,\partial_k\omega_j\,\mathrm{d}x \\
&= -\int \partial_x u_i\,\partial_i\omega_j\,\partial_x\omega_j\,\mathrm{d}x - \int \partial_y u_i\,\partial_i\omega_j\,\partial_y\omega_j\,\mathrm{d}x - \int \partial_z u_i\,\partial_i\omega_j\,\partial_z\omega_j\,\mathrm{d}x \\
&= M_1 + M_2 + M_3.
\end{aligned}$$

We estimate each term as follows:

$$\begin{aligned}
M_1 &= -\int \partial_x u_i\,\partial_i\omega_j\,\partial_x\omega_j\,\mathrm{d}x \le C\|u_x\|^{\frac12}\|\nabla\omega\|^{\frac12}\|\nabla\omega\|^{\frac12}\|\nabla\omega_x\|^{\frac12}\|\nabla\omega_y\|^{\frac12}\|\partial_x u_z\|^{\frac12} \\
&\le \frac{1}{62}\|\nabla\omega_x\|^2 + \frac{1}{40}\|\nabla\omega_y\|^2 + C\|u_x\|\|\omega_x\|\|\nabla\omega\|^2 \\
&\le \frac{1}{62}\|\nabla\omega_x\|^2 + \frac{1}{40}\|\nabla\omega_y\|^2 + C\bigl(\|u_x\|^2 + \|\omega_x\|^2\bigr)\|\nabla\omega\|^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
M_2 &= -\int \partial_y u_i\,\partial_i\omega_j\,\partial_y\omega_j\,\mathrm{d}x \\
&\le C\|u_y\|^{\frac12}\|\nabla\omega\|^{\frac12}\|\nabla\omega\|^{\frac12}\|\nabla\omega_x\|^{\frac12}\|\nabla\omega_y\|^{\frac12}\|\partial_y u_z\|^{\frac12} \\
&\le \frac{1}{62}\|\nabla\omega_x\|^2 + \frac{1}{40}\|\nabla\omega_y\|^2 + C\bigl(\|u_y\|^2 + \|\omega_y\|^2\bigr)\|\nabla\omega\|^2.
\end{aligned}$$

Now, we turn to bound M_3,

$$\begin{aligned}
M_3 &= -\int \partial_z u_i\,\partial_i\omega_j\,\partial_z\omega_j\,\mathrm{d}x \\
&= -\int \partial_z u_1\,\partial_x\omega_j\,\partial_z\omega_j\,\mathrm{d}x - \int \partial_z u_2\,\partial_y\omega_j\,\partial_z\omega_j\,\mathrm{d}x - \int \partial_z u_3\,\partial_z\omega_j\,\partial_z\omega_j\,\mathrm{d}x \\
&= M_{31} + M_{32} + M_{33}.
\end{aligned}$$

Similarly, we can deduce that

$$\begin{aligned}
M_{31} &= -\int \partial_z u_1\,\partial_x\omega_j\,\partial_z\omega_j\,\mathrm{d}x \\
&\le C\|\omega\|^{\frac12}\|\omega_x\|^{\frac12}\|\nabla\omega\|^{\frac12}\|\omega_x\|^{\frac12}\|\nabla\omega_x\|^{\frac12}\|\nabla\omega_y\|^{\frac12} \\
&\le \frac{1}{62}\|\nabla\omega_x\|^2 + \frac{1}{40}\|\nabla\omega_y\|^2 + C\bigl(\|\omega\|^2 + \|\omega_x\|^2\bigr)\|\nabla\omega\|^2, \\
M_{32} &= -\int \partial_z u_2\,\partial_y\omega_j\,\partial_z\omega_j\,\mathrm{d}x \\
&\le C\|\omega\|^{\frac12}\|\omega_y\|^{\frac12}\|\nabla\omega\|^{\frac12}\|\omega_x\|^{\frac12}\|\nabla\omega_y\|^{\frac12}\|\nabla\omega_y\|^{\frac12} \\
&\le \frac{1}{40}\|\nabla\omega_y\|^2 + C\bigl(\|\omega\|^2 + \|\omega_x\|^2\bigr)\|\nabla\omega\|^2,
\end{aligned}$$

$$
\begin{aligned}
M_{33} &= -\int \partial_z u_3\, \partial_z \omega_j\, \partial_z \omega_j \,\mathrm{d}x = \int (\partial_x u_1 + \partial_y u_2)\, \partial_z \omega_j\, \partial_z \omega_j \,\mathrm{d}x \\
&\le C\|u_x\|^{\frac{1}{2}} \|\nabla\omega\|^{\frac{1}{2}} \|\nabla\omega\|^{\frac{1}{2}} \|\partial_x u_z\|^{\frac{1}{2}} \|\nabla\omega_x\|^{\frac{1}{2}} \|\nabla\omega_y\|^{\frac{1}{2}} \\
&\quad + C\|u_y\|^{\frac{1}{2}} \|\nabla\omega\|^{\frac{1}{2}} \|\nabla\omega\|^{\frac{1}{2}} \|\partial_y u_z\|^{\frac{1}{2}} \|\nabla\omega_x\|^{\frac{1}{2}} \|\nabla\omega_y\|^{\frac{1}{2}} \\
&\le \frac{1}{62}\|\nabla\omega_x\|^2 + \frac{1}{40}\|\nabla\omega_y\|^2 + C\big(\|u_x\|^2 + \|u_y\|^2 + \|\omega_x\|^2 + \|\omega_y\|^2\big)\|\nabla\omega\|^2.
\end{aligned}
$$

As for N, integrating by parts, we deduce that

$$
\begin{aligned}
N &= \int u \cdot \nabla j \cdot \Delta j \,\mathrm{d}x = \int u_i\, \partial_i j_j\, \partial_{kk}^2 j_j \,\mathrm{d}x = -\int \partial_k u_i\, \partial_i j_j\, \partial_k j_j \,\mathrm{d}x \\
&= -\int \partial_x u_i\, \partial_i j_j\, \partial_x j_j \,\mathrm{d}x - \int \partial_y u_i\, \partial_i j_j\, \partial_y j_j \,\mathrm{d}x - \int \partial_z u_i\, \partial_i j_j\, \partial_z j_j \,\mathrm{d}x \\
&= N_1 + N_2 + N_3.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
N_1 &= -\int \partial_x u_i\, \partial_i j_j\, \partial_x j_j \,\mathrm{d}x \\
&\le C\|u_x\|^{\frac{1}{2}} \|\nabla j\|^{\frac{1}{2}} \|j_x\|^{\frac{1}{2}} \|\omega_x\|^{\frac{1}{2}} \|\nabla j_x\|^{\frac{1}{2}} \|\nabla j_y\|^{\frac{1}{2}} \\
&\le \frac{1}{50}\|\nabla j_x\|^2 + \frac{1}{48}\|\nabla j_y\|^2 + C\big(\|u_x\|^2 + \|\omega_x\|^2\big)\|\nabla j\|^2, \\
N_2 &= -\int \partial_y u_i\, \partial_i j_j\, \partial_y j_j \,\mathrm{d}x \\
&\le C\|u_y\|^{\frac{1}{2}} \|\nabla j\|^{\frac{1}{2}} \|j_y\|^{\frac{1}{2}} \|\omega_y\|^{\frac{1}{2}} \|\nabla j_x\|^{\frac{1}{2}} \|\nabla j_y\|^{\frac{1}{2}} \\
&\le \frac{1}{50}\|\nabla j_x\|^2 + \frac{1}{48}\|\nabla j_y\|^2 + C\big(\|u_y\|^2 + \|\omega_y\|^2\big)\|\nabla j\|^2.
\end{aligned}
$$

As for N_3, we obtain

$$
\begin{aligned}
N_3 &= -\int \partial_z u_i\, \partial_i j_j\, \partial_z j_j \,\mathrm{d}x \\
&= -\int \partial_z u_1\, \partial_x j_j\, \partial_z j_j \,\mathrm{d}x - \int \partial_z u_2\, \partial_y j_j\, \partial_z j_j \,\mathrm{d}x - \int \partial_z u_3\, \partial_z j_j\, \partial_z j_j \,\mathrm{d}x \\
&= N_{31} + N_{32} + N_{33}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
N_{31} &= -\int \partial_z u_1\, \partial_x j_j\, \partial_z j_j \,\mathrm{d}x \\
&\le C\|\omega\|^{\frac{1}{2}} \|j_x\|^{\frac{1}{2}} \|\nabla j\|^{\frac{1}{2}} \|\omega_x\|^{\frac{1}{2}} \|\nabla j_x\|^{\frac{1}{2}} \|\nabla j_y\|^{\frac{1}{2}} \\
&\le \frac{1}{50}\|\nabla j_x\|^2 + \frac{1}{48}\|\nabla j_y\|^2 + C\big(\|\omega\|^2 + \|\omega_x\|^2\big)\|\nabla j\|^2,
\end{aligned}
$$

$$
\begin{aligned}
N_{32} &= -\int \partial_z u_2\, \partial_y j_j\, \partial_z j_j\, \mathrm{d}x \\
&\le C\|\omega\|^{\frac{1}{2}}\|j_y\|^{\frac{1}{2}}\|\nabla j\|^{\frac{1}{2}}\|\omega_x\|^{\frac{1}{2}}\|\nabla j_y\|^{\frac{1}{2}}\|\nabla j_y\|^{\frac{1}{2}} \\
&\le \frac{1}{48}\|\nabla j_y\|^2 + C\big(\|\omega\|^2 + \|\omega_x\|^2\big)\|\nabla j\|^2, \\
N_{33} &= -\int \partial_z u_3\, \partial_z j_j\, \partial_z j_j\, \mathrm{d}x \\
&\le C\|u_x\|^{\frac{1}{2}}\|\nabla j\|^{\frac{1}{2}}\|\nabla j\|^{\frac{1}{2}}\|\nabla j_x\|^{\frac{1}{2}}\|\nabla j_y\|^{\frac{1}{2}}\|\omega_x\|^{\frac{1}{2}} \\
&\quad + C\|u_y\|^{\frac{1}{2}}\|\nabla j\|^{\frac{1}{2}}\|\nabla j\|^{\frac{1}{2}}\|\nabla j_x\|^{\frac{1}{2}}\|\nabla j_y\|^{\frac{1}{2}}\|\omega_y\|^{\frac{1}{2}} \\
&\le \frac{1}{50}\|\nabla j_x\|^2 + \frac{1}{48}\|\nabla j_y\|^2 + C\big(\|u_x\|^2 + \|u_y\|^2 + \|\omega_x\|^2 + \|\omega_y\|^2\big)\|\nabla j\|^2.
\end{aligned}
$$

We now turn to bound P,

$$
\begin{aligned}
P &= -\int \omega\cdot\nabla\omega\cdot\Delta\omega\, \mathrm{d}x = -\int \omega_i\, \partial_i u_j\, \partial_{kk}^2\omega_j\, \mathrm{d}x \\
&= \int \partial_k\omega_i\, \partial_i u_j\, \partial_k\omega_j\, \mathrm{d}x + \int \omega_i\, \partial_k\, \partial_i u_j\, \partial_k\omega_j\, \mathrm{d}x = P_1 + P_2.
\end{aligned}
$$

For P_1, we have

$$
\begin{aligned}
P_1 &= \int \partial_k\omega_i\, \partial_i u_j\, \partial_k\omega_j\, \mathrm{d}x \\
&= \int \partial_x\omega_i\, \partial_i u_j\, \partial_x\omega_j\, \mathrm{d}x + \int \partial_y\omega_i\, \partial_i u_j\, \partial_y\omega_j\, \mathrm{d}x + \int \partial_z\omega_i\, \partial_i u_j\, \partial_z\omega_j\, \mathrm{d}x \\
&= P_{11} + P_{12} + P_{13}, \\
P_{11} &= \int \partial_x\omega_i\, \partial_i u_j\, \partial_x\omega_j\, \mathrm{d}x \\
&\le C\|\omega_x\|^{\frac{1}{2}}\|\omega\|^{\frac{1}{2}}\|\omega_x\|^{\frac{1}{2}}\|\omega_x\|^{\frac{1}{2}}\|\nabla\omega_x\|^{\frac{1}{2}}\|\nabla\omega_x\|^{\frac{1}{2}} \\
&\le \frac{1}{62}\|\nabla\omega_x\|^2 + C\big(\|\omega\|^2 + \|\omega_x\|^2\big)\|\nabla\omega\|^2, \\
P_{12} &= \int \partial_y\omega_i\, \partial_i u_j\, \partial_y\omega_j\, \mathrm{d}x \\
&\le C\|\omega_y\|^{\frac{1}{2}}\|\omega\|^{\frac{1}{2}}\|\omega_y\|^{\frac{1}{2}}\|\omega_x\|^{\frac{1}{2}}\|\nabla\omega_y\|^{\frac{1}{2}}\|\nabla\omega_y\|^{\frac{1}{2}} \\
&\le \frac{1}{40}\|\nabla\omega_y\|^2 + C\big(\|\omega\|^2 + \|\omega_x\|^2\big)\|\nabla\omega\|^2.
\end{aligned}
$$

For P_{13}, we see that

$$
\begin{aligned}
P_{13} &= \int \partial_z\omega_i\, \partial_i u_j\, \partial_z\omega_j\, \mathrm{d}x \\
&= \int \partial_z\omega_1\, \partial_x u_j\, \partial_z\omega_j\, \mathrm{d}x + \int \partial_z\omega_2\, \partial_y u_j\, \partial_z\omega_j\, \mathrm{d}x + \int \partial_z\omega_3\, \partial_z u_j\, \partial_z\omega_j\, \mathrm{d}x \\
&= P_{13}^1 + P_{13}^2 + P_{13}^3.
\end{aligned}
$$

Thus, we can bound P_{13} as follows:

$$\begin{aligned}
P_{13}^1 &= \int \partial_z\omega_1\,\partial_x u_j\,\partial_z\omega_j\,\mathrm{d}x \le C\|\nabla\omega\|^{\frac12}\|u_x\|^{\frac12}\|\nabla\omega\|^{\frac12}\|\nabla\omega_x\|^{\frac12}\|\nabla\omega_y\|^{\frac12}\|\omega_x\|^{\frac12}\\
&\le \frac{1}{62}\|\nabla\omega_x\|^2 + \frac{1}{40}\|\nabla\omega_y\|^2 + C\big(\|u_x\|^2 + \|\omega_x\|^2\big)\|\nabla\omega\|^2,\\
P_{13}^2 &= \int \partial_z\omega_2\,\partial_y u_j\,\partial_z\omega_j\,\mathrm{d}x \le C\|\nabla\omega\|^{\frac12}\|u_y\|^{\frac12}\|\nabla\omega\|^{\frac12}\|\nabla\omega_x\|^{\frac12}\|\nabla\omega_y\|^{\frac12}\|\omega_y\|^{\frac12}\\
&\le \frac{1}{62}\|\nabla\omega_x\|^2 + \frac{1}{40}\|\nabla\omega_y\|^2 + C\big(\|u_y\|^2 + \|\omega_y\|^2\big)\|\nabla\omega\|^2,\\
P_{13}^2 &= \int \partial_z\omega_3\,\partial_z u_j\,\partial_z\omega_j\,\mathrm{d}x\\
&= \int \partial_z(\partial_x u_2 - \partial_y u_1)\,\partial_z u_j\,\partial_z\omega_j\,\mathrm{d}x\\
&\le C\|\omega_x\|^{\frac12}\|\omega\|^{\frac12}\|\nabla\omega\|^{\frac12}\|\nabla\omega_x\|^{\frac12}\|\omega_x\|^{\frac12}\|\nabla\omega_y\|^{\frac12}\\
&\quad + C\|\omega_y\|^{\frac12}\|\omega\|^{\frac12}\|\nabla\omega\|^{\frac12}\|\nabla\omega_y\|^{\frac12}\|\omega_y\|^{\frac12}\|\nabla\omega_x\|^{\frac12}\\
&\le \frac{1}{62}\|\nabla\omega_x\|^2 + \frac{1}{40}\|\nabla\omega_y\|^2 + C\big(\|\omega\|^2 + \|\omega_x\|^2 + \|\omega_y\|^2\big)\|\nabla\omega\|^2.
\end{aligned}$$

Now, we turn to P_2,

$$\begin{aligned}
P_2 &= \int \omega_i\,\partial_k\,\partial_i u_j\,\partial_k\omega_j\,\mathrm{d}x\\
&= \int \omega_1\,\partial_k\,\partial_x u_j\,\partial_k\omega_j\,\mathrm{d}x + \int \omega_2\,\partial_k\,\partial_y u_j\,\partial_k\omega_j\,\mathrm{d}x + \int \omega_3\,\partial_k\,\partial_z u_j\,\partial_k\omega_j\,\mathrm{d}x\\
&= P_{21} + P_{22} + P_{23}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
P_{21} &= \int \omega_1\,\partial_k\,\partial_x u_j\,\partial_k\omega_j\,\mathrm{d}x\\
&\le C\|\omega\|^{\frac12}\|\omega_x\|^{\frac12}\|\nabla\omega\|^{\frac12}\|\omega_x\|^{\frac12}\|\nabla\omega_y\|^{\frac12}\|\nabla\omega_x\|^{\frac12}\\
&\le \frac{1}{62}\|\nabla\omega_x\|^2 + \frac{1}{40}\|\nabla\omega_y\|^2 + C\big(\|\omega\|^2 + \|\omega_x\|^2\big)\|\nabla\omega\|^2,\\
P_{22} &= \int \omega_2\,\partial_k\,\partial_y u_j\,\partial_k\omega_j\,\mathrm{d}x\\
&\le C\|\omega\|^{\frac12}\|\omega_y\|^{\frac12}\|\nabla\omega\|^{\frac12}\|\omega_x\|^{\frac12}\|\nabla\omega_y\|^{\frac12}\|\nabla\omega_y\|^{\frac12}\\
&\le \frac{1}{40}\|\nabla\omega_y\|^2 + C\big(\|\omega\|^2 + \|\omega_x\|^2\big)\|\nabla\omega\|^2,\\
P_{23} &= \int \omega_3\,\partial_k\,\partial_z u_j\,\partial_k\omega_j\,\mathrm{d}x\\
&\le C\|u_x\|^{\frac12}\|\nabla\omega\|^{\frac12}\|\nabla\omega\|^{\frac12}\|\nabla\omega_x\|^{\frac12}\|\nabla\omega_y\|^{\frac12}\|\omega_x\|^{\frac12}\\
&\quad + C\|u_y\|^{\frac12}\|\nabla\omega\|^{\frac12}\|\nabla\omega\|^{\frac12}\|\nabla\omega_x\|^{\frac12}\|\nabla\omega_y\|^{\frac12}\|\omega_y\|^{\frac12}\\
&\le \frac{1}{62}\|\nabla\omega_x\|^2 + \frac{1}{40}\|\nabla\omega_y\|^2 + C\big(\|u_x\|^2 + \|u_y\|^2 + \|\omega_x\|^2 + \|\omega_y\|^2\big)\|\nabla\omega\|^2.
\end{aligned}$$

To bound Q, we see that

$$\begin{aligned}
Q &= \int j\cdot\nabla b\cdot\Delta\omega\,\mathrm{d}x = \int j_i\,\partial_i b_j\,\partial^2_{kk}\omega_j\,\mathrm{d}x \\
&= -\int \partial_k j_i\,\partial_i b_j\,\partial_k\omega_j\,\mathrm{d}x - \int j_i\,\partial_k\,\partial_i b_j\,\partial_k\omega_j\,\mathrm{d}x = Q_1 + Q_2, \\
Q_1 &= -\int \partial_x j_i\,\partial_i b_j\,\partial_x\omega_j\,\mathrm{d}x - \int \partial_y j_i\,\partial_i b_j\,\partial_y\omega_j\,\mathrm{d}x - \int \partial_z j_i\,\partial_i b_j\,\partial_z\omega_j\,\mathrm{d}x \\
&= Q_{11} + Q_{12} + Q_{13}, \\
Q_{11} &= -\int \partial_x j_i\,\partial_i b_j\,\partial_x\omega_j\,\mathrm{d}x \le C\|j_x\|^{\frac12}\|j\|^{\frac12}\|\omega_x\|^{\frac12}\|\nabla j_x\|^{\frac12}\|\nabla j\|^{\frac12}\|\nabla\omega_x\|^{\frac12} \\
&\le \frac{1}{62}\|\nabla\omega_x\|^2 + \frac{1}{50}\|\nabla j_x\|^2 + C\big(\|j\|^2 + \|j_x\|^2\big)\big(\|\nabla\omega\|^2 + \|\nabla j\|^2\big), \\
Q_{12} &= -\int \partial_y j_i\,\partial_i b_j\,\partial_y\omega_j\,\mathrm{d}x \le C\|j_y\|^{\frac12}\|j\|^{\frac12}\|\omega_x\|^{\frac12}\|\nabla j_y\|^{\frac12}\|j_x\|^{\frac12}\|\nabla\omega_x\|^{\frac12} \\
&\le \frac{1}{62}\|\nabla\omega_x\|^2 + \frac{1}{48}\|\nabla j_y\|^2 + C\big(\|j\|^2 + \|j_y\|^2\big)\big(\|\nabla\omega\|^2 + \|\nabla j\|^2\big).
\end{aligned}$$

As for Q_{13}, we see that

$$\begin{aligned}
Q_{13} &= -\int \partial_z j_i\,\partial_i b_j\,\partial_z\omega_j\,\mathrm{d}x \\
&= -\int \partial_z j_1\,\partial_x b_j\,\partial_z\omega_j\,\mathrm{d}x - \int \partial_z j_2\,\partial_y b_j\,\partial_z\omega_j\,\mathrm{d}x - \int \partial_z j_3\,\partial_z b_j\,\partial_z\omega_j\,\mathrm{d}x \\
&= Q^1_{13} + Q^2_{13} + Q^3_{13}.
\end{aligned}$$

Thus, we can deduce that

$$\begin{aligned}
Q^1_{13} &= -\int \partial_z j_1\,\partial_x b_j\,\partial_z\omega_j\,\mathrm{d}x \\
&\le C\|\nabla j\|^{\frac12}\|b_x\|^{\frac12}\|\nabla\omega\|^{\frac12}\|\nabla j_x\|^{\frac12}\|j_x\|^{\frac12}\|\nabla\omega_y\|^{\frac12} \\
&\le \frac{1}{40}\|\nabla\omega_y\|^2 + \frac{1}{50}\|\nabla j_x\|^2 + C\big(\|b_x\|^2 + \|j_x\|^2\big)\big(\|\nabla\omega\|^2 + \|\nabla j\|^2\big), \\
Q^2_{13} &= -\int \partial_z j_2\,\partial_y b_j\,\partial_z\omega_j\,\mathrm{d}x \\
&\le C\|\nabla j\|^{\frac12}\|b_x\|^{\frac12}\|\nabla\omega\|^{\frac12}\|\nabla j_x\|^{\frac12}\|j_y\|^{\frac12}\|\nabla\omega_y\|^{\frac12} \\
&\le \frac{1}{40}\|\nabla\omega_y\|^2 + \frac{1}{50}\|\nabla j_x\|^2 + C\big(\|b_y\|^2 + \|j_y\|^2\big)\big(\|\nabla\omega\|^2 + \|\nabla j\|^2\big), \\
Q^3_{13} &= -\int \partial_z j_3\,\partial_z b_j\,\partial_z\omega_j\,\mathrm{d}x = -\int \partial_z(\partial_x b_2 - \partial_y b_1)\,\partial_z b_j\,\partial_z\omega_j\,\mathrm{d}x \\
&\le C\|j_x\|^{\frac12}\|j\|^{\frac12}\|\nabla\omega\|^{\frac12}\|\nabla j_x\|^{\frac12}\|j_y\|^{\frac12}\|\nabla\omega_x\|^{\frac12} \\
&\quad + C\|j_y\|^{\frac12}\|j\|^{\frac12}\|\nabla\omega\|^{\frac12}\|\nabla j_y\|^{\frac12}\|j_y\|^{\frac12}\|\nabla\omega_x\|^{\frac12} \\
&\le \frac{1}{62}\|\nabla\omega_x\|^2 + \frac{1}{50}\|\nabla j_x\|^2 + \frac{1}{48}\|\nabla j_y\|^2 + C\big(\|j\|^2 + \|j_x\|^2 + \|j_y\|^2\big)\big(\|\nabla\omega\|^2 + \|\nabla j\|^2\big).
\end{aligned}$$

Now, we turn to Q_2,

$$\begin{aligned}Q_2 &= -\int b\cdot\nabla j\cdot\Delta\omega\,\mathrm{d}x = -\int j_i\,\partial_k\,\partial_i b_j\,\partial_k\omega_j\,\mathrm{d}x\\ &= -\int j_1\,\partial_k\,\partial_x b_j\,\partial_k\omega_j\,\mathrm{d}x - \int j_2\,\partial_k\,\partial_y b_j\,\partial_k\omega_j\,\mathrm{d}x - \int j_3\,\partial_k\,\partial_z b_j\,\partial_k\omega_j\,\mathrm{d}x\\ &= Q_{21}+Q_{22}+Q_{23}.\end{aligned}$$

We deduce that

$$\begin{aligned}Q_{21} &= -\int j_1\,\partial_k\,\partial_x b_j\,\partial_k\omega_j\,\mathrm{d}x\\ &\le C\|j\|^{\frac12}\|j_x\|^{\frac12}\|\nabla\omega\|^{\frac12}\|\nabla\omega_x\|^{\frac12}\|j_y\|^{\frac12}\|\nabla j_x\|^{\frac12}\\ &\le \frac{1}{62}\|\nabla\omega_x\|^2+\frac{1}{50}\|\nabla j_x\|^2+C\big(\|j\|^2+\|j_x\|^2\big)\big(\|\nabla\omega\|^2+\|\nabla j\|^2\big),\\ Q_{22} &= -\int j_2\,\partial_k\,\partial_y b_j\,\partial_k\omega_j\,\mathrm{d}x\\ &\le C\|j\|^{\frac12}\|j_y\|^{\frac12}\|\nabla\omega\|^{\frac12}\|\nabla\omega_x\|^{\frac12}\|j_y\|^{\frac12}\|\nabla j_y\|^{\frac12}\\ &\le \frac{1}{62}\|\nabla\omega_x\|^2+\frac{1}{48}\|\nabla j_y\|^2+C\big(\|j\|^2+\|j_y\|^2\big)\big(\|\nabla\omega\|^2+\|\nabla j\|^2\big),\\ Q_{23} &= -\int j_3\,\partial_k\,\partial_z b_j\,\partial_k\omega_j\,\mathrm{d}x\\ &\le C\|b_x\|^{\frac12}\|\nabla j\|^{\frac12}\|\nabla\omega\|^{\frac12}\|\nabla\omega_x\|^{\frac12}\|\nabla j_y\|^{\frac12}\|j_x\|^{\frac12}\\ &\quad + C\|b_y\|^{\frac12}\|\nabla j\|^{\frac12}\|\nabla\omega\|^{\frac12}\|\nabla\omega_x\|^{\frac12}\|\nabla j_y\|^{\frac12}\|j_y\|^{\frac12}\\ &\le \frac{1}{62}\|\nabla\omega_x\|^2+\frac{1}{48}\|\nabla j_y\|^2+C\big(\|b_x\|^2+\|j_x\|^2+\|b_y\|^2+\|j_y\|^2\big)\big(\|\nabla\omega\|^2+\|\nabla j\|^2\big).\end{aligned}$$

Here we start to estimate R as follows:

$$\begin{aligned}R &= -\int b\cdot\nabla j\cdot\Delta\omega\,\mathrm{d}x - \int b\cdot\nabla\omega\cdot\Delta j\,\mathrm{d}x\\ &= -\int b_i\,\partial_i j_j\,\partial^2_{kk}\omega_j\,\mathrm{d}x - \int b_i\,\partial_i\omega_j\,\partial^2_{kk}j_j\,\mathrm{d}x\\ &= \int \partial_k b_i\,\partial_i j_j\,\partial_k\omega_j\,\mathrm{d}x + \int \partial_k b_i\,\partial_i\omega_j\,\partial_k j_j\,\mathrm{d}x = R_1+R_2.\end{aligned}$$

For R_1, we have

$$\begin{aligned}R_1 &= \int \partial_k b_i\,\partial_i j_j\,\partial_k\omega_j\,\mathrm{d}x\\ &= \int \partial_x b_i\,\partial_i j_j\,\partial_x\omega_j\,\mathrm{d}x + \int \partial_y b_i\,\partial_i j_j\,\partial_y\omega_j\,\mathrm{d}x + \int \partial_z b_i\,\partial_i j_j\,\partial_z\omega_j\,\mathrm{d}x\\ &= R_{11}+R_{12}+R_{13}.\end{aligned}$$

We deduce that

$$\begin{aligned}
R_{11} &= \int \partial_x b_i \, \partial_i j_j \, \partial_x \omega_j \, \mathrm{d}x \\
&\le C\|b_x\|^{\frac{1}{2}} \|\nabla j\|^{\frac{1}{2}} \|\omega_x\|^{\frac{1}{2}} \|\nabla j_x\|^{\frac{1}{2}} \|\nabla \omega_x\|^{\frac{1}{2}} \|j_x\|^{\frac{1}{2}} \\
&\le \frac{1}{62}\|\nabla \omega_x\|^2 + \frac{1}{50}\|\nabla j_x\|^2 + C\big(\|b_x\|^2 + \|\omega_x\|^2\big)\|\nabla j\|^2, \\
R_{12} &= \int \partial_y b_i \, \partial_i j_j \, \partial_y \omega_j \, \mathrm{d}x \\
&\le C\|b_y\|^{\frac{1}{2}} \|\nabla j\|^{\frac{1}{2}} \|\omega_y\|^{\frac{1}{2}} \|\nabla j_x\|^{\frac{1}{2}} \|\nabla \omega_y\|^{\frac{1}{2}} \|j_y\|^{\frac{1}{2}} \\
&\le \frac{1}{50}\|\nabla j_x\|^2 + \frac{1}{40}\|\nabla \omega_y\|^2 + C\big(\|b_y\|^2 + \|\omega_y\|^2\big)\|\nabla j\|^2.
\end{aligned}$$

For R_{13}, we have

$$\begin{aligned}
R_{13} &= \int \partial_z b_i \, \partial_i j_j \, \partial_z \omega_j \, \mathrm{d}x \\
&= \int \partial_z b_1 \, \partial_x j_j \, \partial_z \omega_j \, \mathrm{d}x + \int \partial_z b_2 \, \partial_y j_j \, \partial_z \omega_j \, \mathrm{d}x + \int \partial_z b_3 \, \partial_z j_j \, \partial_z \omega_j \, \mathrm{d}x = R_{13}^1 + R_{13}^2 + R_{13}^3.
\end{aligned}$$

Similarly,

$$\begin{aligned}
R_{13}^1 &= \int \partial_z b_1 \, \partial_x j_j \, \partial_z \omega_j \, \mathrm{d}x \\
&\le C\|j\|^{\frac{1}{2}} \|j_x\|^{\frac{1}{2}} \|\nabla \omega\|^{\frac{1}{2}} \|\nabla \omega_x\|^{\frac{1}{2}} \|\nabla j_x\|^{\frac{1}{2}} \|j_y\|^{\frac{1}{2}} \\
&\le \frac{1}{62}\|\nabla \omega_x\|^2 + \frac{1}{50}\|\nabla j_x\|^2 + C\big(\|j\|^2 + \|j_x\|^2\big)\big(\|\nabla \omega\|^2 + \|\nabla j\|^2\big), \\
R_{13}^2 &= \int \partial_z b_2 \, \partial_y j_j \, \partial_z \omega_j \, \mathrm{d}x \\
&\le C\|j\|^{\frac{1}{2}} \|j_y\|^{\frac{1}{2}} \|\nabla \omega\|^{\frac{1}{2}} \|\nabla \omega_x\|^{\frac{1}{2}} \|j_y\|^{\frac{1}{2}} \|\nabla j_y\|^{\frac{1}{2}} \\
&\le \frac{1}{62}\|\nabla \omega_x\|^2 + \frac{1}{48}\|\nabla j_y\|^2 + C\big(\|j\|^2 + \|j_y\|^2\big)\big(\|\nabla \omega\|^2 + \|\nabla j\|^2\big), \\
R_{13}^3 &= \int \partial_z b_3 \, \partial_z j_j \, \partial_z \omega_j \, \mathrm{d}x \\
&= -\int (\partial_x b_1 + \partial_y b_2) \, \partial_z j_j \, \partial_z \omega_j \, \mathrm{d}x \\
&\le C\|b_x\|^{\frac{1}{2}} \|\nabla j\|^{\frac{1}{2}} \|\nabla \omega\|^{\frac{1}{2}} \|\nabla \omega_x\|^{\frac{1}{2}} \|\nabla j_y\|^{\frac{1}{2}} \|j_x\|^{\frac{1}{2}} \\
&\quad + C\|b_y\|^{\frac{1}{2}} \|\nabla j\|^{\frac{1}{2}} \|\nabla \omega\|^{\frac{1}{2}} \|\nabla \omega_x\|^{\frac{1}{2}} \|\nabla j_y\|^{\frac{1}{2}} \|j_y\|^{\frac{1}{2}} \\
&\le \frac{1}{62}\|\nabla \omega_x\|^2 + \frac{1}{48}\|\nabla j_y\|^2 + C\big(\|j\|^2 + \|j_y\|^2\big)\big(\|\nabla \omega\|^2 + \|\nabla j\|^2\big).
\end{aligned}$$

Now, we turn to R_2,

$$\begin{aligned}
R_2 &= \int \partial_k b_i \, \partial_i \omega_j \, \partial_k j_j \, \mathrm{d}x \\
&= \int \partial_x b_i \, \partial_i \omega_j \, \partial_x j_j \, \mathrm{d}x + \int \partial_y b_i \, \partial_i \omega_j \, \partial_y j_j \, \mathrm{d}x + \int \partial_z b_i \, \partial_i \omega_j \, \partial_z j_j \, \mathrm{d}x = R_{21} + R_{22} + R_{23}.
\end{aligned}$$

Thus, similarly, we deduce that

$$\begin{aligned}
R_{21} &= \int \partial_x b_i \, \partial_i \omega_j \, \partial_x j_j \, \mathrm{d}x \le C\|b_x\|^{\frac{1}{2}} \|\nabla\omega\|^{\frac{1}{2}} \|j_x\|^{\frac{1}{2}} \|\nabla\omega_x\|^{\frac{1}{2}} \|\nabla j_x\|^{\frac{1}{2}} \|j_x\|^{\frac{1}{2}} \\
&\le \frac{1}{62}\|\nabla\omega_x\|^2 + \frac{1}{50}\|\nabla j_x\|^2 + C\big(\|b_x\|^2 + \|j_x\|^2\big)\big(\|\nabla\omega\|^2 + \|\nabla j\|^2\big), \\
R_{22} &= \int \partial_y b_i \, \partial_i \omega_j \, \partial_y j_j \, \mathrm{d}x \le C\|b_y\|^{\frac{1}{2}} \|\nabla\omega\|^{\frac{1}{2}} \|j_y\|^{\frac{1}{2}} \|\nabla\omega_x\|^{\frac{1}{2}} \|\nabla j_y\|^{\frac{1}{2}} \|j_y\|^{\frac{1}{2}} \\
&\le \frac{1}{62}\|\nabla\omega_x\|^2 + \frac{1}{48}\|\nabla j_y\|^2 + C\big(\|b_y\|^2 + \|j_y\|^2\big)\big(\|\nabla\omega\|^2 + \|\nabla j\|^2\big).
\end{aligned}$$

For R_{23}, we have

$$\begin{aligned}
R_{23} &= \int \partial_z b_i \, \partial_i \omega_j \, \partial_z j_j \, \mathrm{d}x \\
&= \int \partial_z b_1 \, \partial_x \omega_j \, \partial_z j_j \, \mathrm{d}x + \int \partial_z b_2 \, \partial_y \omega_j \, \partial_z j_j \, \mathrm{d}x + \int \partial_z b_3 \, \partial_z \omega_j \, \partial_z j_j \, \mathrm{d}x \\
&= R_{23}^1 + R_{23}^2 + R_{23}^3.
\end{aligned}$$

Similarly,

$$\begin{aligned}
R_{23}^1 &= \int \partial_z b_1 \, \partial_x \omega_j \, \partial_z j_j \, \mathrm{d}x \le C\|j\|^{\frac{1}{2}} \|\omega_x\|^{\frac{1}{2}} \|\nabla j\|^{\frac{1}{2}} \|\nabla\omega_x\|^{\frac{1}{2}} \|\nabla j_x\|^{\frac{1}{2}} \|j_y\|^{\frac{1}{2}} \\
&\le \frac{1}{62}\|\nabla\omega_x\|^2 + \frac{1}{50}\|\nabla j_x\|^2 + C\big(\|j\|^2 + \|\omega_x\|^2\big)\|\nabla j\|^2, \\
R_{23}^2 &= \int \partial_z b_2 \, \partial_y \omega_j \, \partial_z j_j \, \mathrm{d}x \le C\|j\|^{\frac{1}{2}} \|\omega_x\|^{\frac{1}{2}} \|\nabla j\|^{\frac{1}{2}} \|\nabla j_x\|^{\frac{1}{2}} \|j_y\|^{\frac{1}{2}} \|\nabla\omega_y\|^{\frac{1}{2}} \\
&\le \frac{1}{50}\|\nabla j_x\|^2 + \frac{1}{40}\|\nabla\omega_y\|^2 + C\big(\|j\|^2 + \|j_y\|^2\big)\big(\|\nabla\omega\|^2 + \|\nabla j\|^2\big), \\
R_{23}^3 &= \int \partial_z b_3 \, \partial_z \omega_j \, \partial_z j_j \, \mathrm{d}x \\
&= -\int (\partial_x b_1 + \partial_y b_2) \, \partial_z \omega_j \, \partial_z j_j \, \mathrm{d}x \\
&\le C\|b_x\|^{\frac{1}{2}} \|\nabla\omega\|^{\frac{1}{2}} \|\nabla j\|^{\frac{1}{2}} \|\nabla\omega_x\|^{\frac{1}{2}} \|\nabla j_y\|^{\frac{1}{2}} \|j_x\|^{\frac{1}{2}} \\
&\quad + C\|b_y\|^{\frac{1}{2}} \|\nabla\omega\|^{\frac{1}{2}} \|\nabla j\|^{\frac{1}{2}} \|\nabla\omega_x\|^{\frac{1}{2}} \|\nabla j_y\|^{\frac{1}{2}} \|j_y\|^{\frac{1}{2}} \\
&\le \frac{1}{62}\|\nabla\omega_x\|^2 + \frac{1}{48}\|\nabla j_y\|^2 + C\big(\|b_x\|^2 + \|b_y\|^2 + \|j_x\|^2 + \|j_y\|^2\big)\big(\|\nabla\omega\|^2 + \|\nabla j\|^2\big).
\end{aligned}$$

Finally, for the last term S, we have

$$\begin{aligned}
S &= -\int \varepsilon_{ijk}(\partial_j b_l \, \partial_l u_k - \partial_j u_l \, \partial_l b_k)\Delta j_i \, \mathrm{d}x \\
&= \int \varepsilon_{ijk} \, \partial_m(\partial_j b_l \, \partial_l u_k - \partial_j u_l \, \partial_l b_k) \, \partial_m j_i \, \mathrm{d}x \\
&= S_1 + S_2.
\end{aligned}$$

We first consider S_1,

$$\begin{aligned}
S_1 &= \int \varepsilon_{ijk}\,\partial_m(\partial_j b_l\,\partial_l u_k)\,\partial_m j_i\,\mathrm{d}x \\
&= \int \varepsilon_{ijk}\,\partial_x(\partial_j b_l\,\partial_l u_k)\,\partial_x j_i\,\mathrm{d}x + \int \varepsilon_{ijk}\,\partial_y(\partial_j b_l\,\partial_l u_k)\,\partial_y j_i\,\mathrm{d}x + \int \varepsilon_{ijk}\,\partial_z(\partial_j b_l\,\partial_l u_k)\,\partial_z j_i\,\mathrm{d}x \\
&= S_{11} + S_{12} + S_{13}.
\end{aligned}$$

We deduce that

$$\begin{aligned}
S_{11} &= \int \varepsilon_{ijk}\,\partial_x(\partial_j b_l\,\partial_l u_k)\,\partial_x j_i\,\mathrm{d}x = \int \varepsilon_{ijk}\,\partial_x\,\partial_j b_l\,\partial_l u_k\,\partial_x j_i\,\mathrm{d}x + \int \varepsilon_{ijk}\,\partial_j b_l\,\partial_x\,\partial_l u_k\,\partial_x j_i\,\mathrm{d}x \\
&\le C\|j_x\|^{\frac{1}{2}}\|\omega\|^{\frac{1}{2}}\|j_x\|^{\frac{1}{2}}\|\omega_x\|^{\frac{1}{2}}\|\nabla j_x\|^{\frac{1}{2}}\|\nabla j_y\|^{\frac{1}{2}} \\
&\quad + C\|j\|^{\frac{1}{2}}\|\omega_x\|^{\frac{1}{2}}\|j_x\|^{\frac{1}{2}}\|j_x\|^{\frac{1}{2}}\|\nabla\omega_x\|^{\frac{1}{2}}\|\nabla j_x\|^{\frac{1}{2}} \\
&\le \frac{1}{62}\|\nabla\omega_x\|^2 + \frac{1}{50}\|\nabla j_x\|^2 + \frac{1}{48}\|\nabla j_y\|^2 + C\big(\|j_x\|^2 + \|\omega\|^2 + \|j\|^2\big)\big(\|\nabla\omega\|^2 + \|\nabla j\|^2\big), \\
S_{12} &= \int \varepsilon_{ijk}\,\partial_y(\partial_j b_l\,\partial_l u_k)\,\partial_y j_i\,\mathrm{d}x = \int \varepsilon_{ijk}\,\partial_y\,\partial_j b_l\,\partial_l u_k\,\partial_y j_i\,\mathrm{d}x + \int \varepsilon_{ijk}\,\partial_j b_l\,\partial_y\,\partial_l u_k\,\partial_y j_i\,\mathrm{d}x \\
&\le C\|j_y\|^{\frac{1}{2}}\|\omega\|^{\frac{1}{2}}\|j_y\|^{\frac{1}{2}}\|\nabla j_x\|^{\frac{1}{2}}\|\nabla j_y\|^{\frac{1}{2}}\|\nabla\omega\|^{\frac{1}{2}} \\
&\quad + C\|j\|^{\frac{1}{2}}\|\omega_y\|^{\frac{1}{2}}\|j_y\|^{\frac{1}{2}}\|j_x\|^{\frac{1}{2}}\|\nabla\omega_y\|^{\frac{1}{2}}\|\nabla j_x\|^{\frac{1}{2}} \\
&\le \frac{1}{40}\|\nabla\omega_y\|^2 + \frac{1}{50}\|\nabla j_x\|^2 + \frac{1}{48}\|\nabla j_y\|^2 \\
&\quad + C\big(\|\omega\|^2 + \|j_y\|^2 + \|j\|^2 + \|j_x\|^2\big)\big(\|\nabla\omega\|^2 + \|\nabla j\|^2\big).
\end{aligned}$$

As for S_{13}, we see that

$$\begin{aligned}
S_{13} &= \int \varepsilon_{ijk}\,\partial_z(\partial_j b_l\,\partial_l u_k)\,\partial_z j_i\,\mathrm{d}x \\
&= \int \varepsilon_{ijk}\,\partial_z\,\partial_j b_l\,\partial_l u_k\,\partial_z j_i\,\mathrm{d}x + \int \varepsilon_{ijk}\,\partial_j b_l\,\partial_z\,\partial_l u_k\,\partial_z j_i\,\mathrm{d}x \\
&= \int \varepsilon_{ijk}\,\partial_z\,\partial_x b_l\,\partial_l u_k\,\partial_z j_i\,\mathrm{d}x + \int \varepsilon_{ijk}\,\partial_z\,\partial_y b_l\,\partial_l u_k\,\partial_z j_i\,\mathrm{d}x + \int \varepsilon_{ijk}\,\partial_z\,\partial_z b_l\,\partial_l u_k\,\partial_z j_i\,\mathrm{d}x \\
&\quad + \int \varepsilon_{ijk}\,\partial_x b_l\,\partial_z\,\partial_l u_k\,\partial_z j_i\,\mathrm{d}x + \int \varepsilon_{ijk}\,\partial_y b_l\,\partial_z\,\partial_l u_k\,\partial_z j_i\,\mathrm{d}x + \int \varepsilon_{ijk}\,\partial_z b_l\,\partial_z\,\partial_l u_k\,\partial_z j_i\,\mathrm{d}x \\
&= S_{13}^1 + S_{13}^2 + S_{13}^3 + S_{13}^4 + S_{13}^5 + S_{13}^6.
\end{aligned}$$

We deduce each term step by step as follows:

$$\begin{aligned}
S_{13}^1 &= \int \varepsilon_{ijk}\,\partial_z\,\partial_x b_l\,\partial_l u_k\,\partial_z j_i\,\mathrm{d}x \\
&\le C\|j_x\|^{\frac{1}{2}}\|\omega\|^{\frac{1}{2}}\|\nabla j\|^{\frac{1}{2}}\|\nabla j_y\|^{\frac{1}{2}}\|\omega_x\|^{\frac{1}{2}}\|\nabla j_x\|^{\frac{1}{2}} \\
&\le \frac{1}{50}\|\nabla j_x\|^2 + \frac{1}{48}\|\nabla j_y\|^2 + C\big(\|\omega\|^2 + \|j_x\|^2\big)\big(\|\nabla\omega\|^2 + \|\nabla j\|^2\big),
\end{aligned}$$

$$\begin{aligned}
S_{13}^2 &= \int \varepsilon_{ijk}\,\partial_z\,\partial_y b_l\,\partial_l u_k\,\partial_z j_i\,\mathrm{d}x \\
&\le C\|j_y\|^{\frac12}\|\omega\|^{\frac12}\|\nabla j\|^{\frac12}\|\nabla j_x\|^{\frac12}\|\omega_y\|^{\frac12}\|\nabla j_y\|^{\frac12} \\
&\le \frac{1}{50}\|\nabla j_x\|^2 + \frac{1}{48}\|\nabla j_y\|^2 + C\bigl(\|\omega\|^2 + \|j_y\|^2\bigr)\bigl(\|\nabla\omega\|^2 + \|\nabla j\|^2\bigr).
\end{aligned}$$

For S_{13}^3, we have

$$\begin{aligned}
S_{13}^3 &= \int \varepsilon_{ijk}\,\partial_z\,\partial_z b_l\,\partial_l u_k\,\partial_z j_i\,\mathrm{d}x \\
&= \int \varepsilon_{ijk}\,\partial_z\,\partial_z b_1\,\partial_x u_k\,\partial_z j_i\,\mathrm{d}x + \int \varepsilon_{ijk}\,\partial_z\,\partial_z b_2\,\partial_y u_k\,\partial_z j_i\,\mathrm{d}x + \int \varepsilon_{ijk}\,\partial_z\,\partial_z b_3\,\partial_z u_k\,\partial_z j_i\,\mathrm{d}x \\
&= S_{13}^{31} + S_{13}^{32} + S_{13}^{33}.
\end{aligned}$$

Thus, we deduce that

$$\begin{aligned}
S_{13}^{31} &= \int \varepsilon_{ijk}\,\partial_z\,\partial_z b_1\,\partial_x u_k\,\partial_z j_i\,\mathrm{d}x \\
&\le C\|\nabla j\|^{\frac12}\|u_x\|^{\frac12}\|\nabla j\|^{\frac12}\|\nabla j_x\|^{\frac12}\|\nabla j_y\|^{\frac12}\|\omega_x\|^{\frac12} \\
&\le \frac{1}{50}\|\nabla j_x\|^2 + \frac{1}{48}\|\nabla j_y\|^2 + C\bigl(\|u_x\|^2 + \|\omega_x\|^2\bigr)\|\nabla j\|^2, \\
S_{13}^{32} &= \int \varepsilon_{ijk}\,\partial_z\,\partial_z b_2\,\partial_y u_k\,\partial_z j_i\,\mathrm{d}x \\
&\le C\|\nabla j\|^{\frac12}\|u_y\|^{\frac12}\|\nabla j\|^{\frac12}\|\nabla j_x\|^{\frac12}\|\nabla j_y\|^{\frac12}\|\omega_x\|^{\frac12} \\
&\le \frac{1}{50}\|\nabla j_x\|^2 + \frac{1}{48}\|\nabla j_y\|^2 + C\bigl(\|u_y\|^2 + \|\omega_y\|^2\bigr)\|\nabla j\|^2, \\
S_{13}^{33} &= \int \varepsilon_{ijk}\,\partial_z\,\partial_z b_3\,\partial_z u_k\,\partial_z j_i\,\mathrm{d}x \\
&= -\int \varepsilon_{ijk}\,\partial_z(\partial_x b_1 + \partial_y b_2)\,\partial_z u_k\,\partial_z j_i\,\mathrm{d}x \\
&\le C\|j_x\|^{\frac12}\|\omega\|^{\frac12}\|\nabla j\|^{\frac12}\|\nabla j_x\|^{\frac12}\|\nabla j_y\|^{\frac12}\|\nabla\omega\|^{\frac12} \\
&\quad + C\|j_y\|^{\frac12}\|\omega\|^{\frac12}\|\nabla j\|^{\frac12}\|\nabla j_x\|^{\frac12}\|\nabla j_y\|^{\frac12}\|\nabla\omega\|^{\frac12} \\
&\le \frac{1}{50}\|\nabla j_x\|^2 + \frac{1}{48}\|\nabla j_y\|^2 + C\bigl(\|\omega\|^2 + \|j_x\|^2 + \|j_y\|^2\bigr)\bigl(\|\nabla\omega\|^2 + \|\nabla j\|^2\bigr), \\
S_{13}^4 &= \int \varepsilon_{ijk}\,\partial_x b_l\,\partial_z\,\partial_l u_k\,\partial_z j_i\,\mathrm{d}x \\
&\le C\|b_x\|^{\frac12}\|\nabla\omega\|^{\frac12}\|\nabla j\|^{\frac12}\|\nabla\omega_x\|^{\frac12}\|\nabla j_y\|^{\frac12}\|j_x\|^{\frac12} \\
&\le \frac{1}{62}\|\nabla\omega_x\|^2 + \frac{1}{48}\|\nabla j_y\|^2 + C\bigl(\|b_x\|^2 + \|j_x\|^2\bigr)\bigl(\|\nabla\omega\|^2 + \|\nabla j\|^2\bigr), \\
S_{13}^5 &= \int \varepsilon_{ijk}\,\partial_y b_l\,\partial_z\,\partial_l u_k\,\partial_z j_i\,\mathrm{d}x \\
&\le C\|b_y\|^{\frac12}\|\nabla\omega\|^{\frac12}\|\nabla j\|^{\frac12}\|\nabla\omega_x\|^{\frac12}\|\nabla j_y\|^{\frac12}\|j_y\|^{\frac12} \\
&\le \frac{1}{62}\|\nabla\omega_x\|^2 + \frac{1}{48}\|\nabla j_y\|^2 + C\bigl(\|b_y\|^2 + \|j_y\|^2\bigr)\bigl(\|\nabla\omega\|^2 + \|\nabla j\|^2\bigr).
\end{aligned}$$

For the last term S_{13}^{6}, we see that

$$\begin{aligned} S_{13}^{6} &= \int \varepsilon_{ijk}\,\partial_z b_l\,\partial_z\,\partial_l u_k\,\partial_z j_i\,\mathrm{d}x \\ &= \int \varepsilon_{ijk}\,\partial_z b_1\,\partial_z\,\partial_x u_k\,\partial_z j_i\,\mathrm{d}x + \int \varepsilon_{ijk}\,\partial_z b_2\,\partial_z\,\partial_y u_k\,\partial_z j_i\,\mathrm{d}x + \int \varepsilon_{ijk}\,\partial_z b_3\,\partial_z\,\partial_z u_k\,\partial_z j_i\,\mathrm{d}x \\ &= S_{13}^{61} + S_{13}^{62} + S_{13}^{63}. \end{aligned}$$

Then,

$$\begin{aligned} S_{13}^{61} &= \int \varepsilon_{ijk}\,\partial_z b_1\,\partial_z\,\partial_x u_k\,\partial_z j_i\,\mathrm{d}x \le C\|j\|^{\frac{1}{2}}\|\omega_x\|^{\frac{1}{2}}\|\nabla j\|^{\frac{1}{2}}\|\nabla j_x\|^{\frac{1}{2}}\|j_y\|^{\frac{1}{2}}\|\nabla\omega_x\|^{\frac{1}{2}} \\ &\le \frac{1}{62}\|\nabla\omega_x\|^2 + \frac{1}{50}\|\nabla j_x\|^2 + C\bigl(\|j\|^2 + \|\omega_x\|^2\bigr)\|\nabla j\|^2, \\ S_{13}^{62} &= \int \varepsilon_{ijk}\,\partial_z b_2\,\partial_z\,\partial_y u_k\,\partial_z j_i\,\mathrm{d}x \le C\|j\|^{\frac{1}{2}}\|\omega_y\|^{\frac{1}{2}}\|\nabla j\|^{\frac{1}{2}}\|\nabla j_x\|^{\frac{1}{2}}\|j_y\|^{\frac{1}{2}}\|\nabla\omega_y\|^{\frac{1}{2}} \\ &\le \frac{1}{40}\|\nabla\omega_y\|^2 + \frac{1}{50}\|\nabla j_x\|^2 + C\bigl(\|j\|^2 + \|\omega_y\|^2\bigr)\|\nabla j\|^2, \\ S_{13}^{63} &= \int \varepsilon_{ijk}\,\partial_z b_3\,\partial_z\,\partial_z u_k\,\partial_z j_i\,\mathrm{d}x \\ &= -\int \varepsilon_{ijk}(\partial_x b_1 + \partial_y b_2)\,\partial_z\,\partial_z u_k\,\partial_z j_i\,\mathrm{d}x \\ &\le C\|b_x\|^{\frac{1}{2}}\|\nabla\omega\|^{\frac{1}{2}}\|\nabla j\|^{\frac{1}{2}}\|\nabla\omega_x\|^{\frac{1}{2}}\|\nabla j_y\|^{\frac{1}{2}}\|j_y\|^{\frac{1}{2}} \\ &\quad + C\|b_y\|^{\frac{1}{2}}\|\nabla\omega\|^{\frac{1}{2}}\|\nabla j\|^{\frac{1}{2}}\|\nabla\omega_x\|^{\frac{1}{2}}\|\nabla j_y\|^{\frac{1}{2}}\|j_y\|^{\frac{1}{2}} \\ &\le \frac{1}{62}\|\nabla\omega_x\|^2 + \frac{1}{48}\|\nabla j_y\|^2 + C\bigl(\|b_x\|^2 + \|b_y\|^2 + \|j_y\|^2\bigr)\|\nabla j\|^2. \end{aligned}$$

As for S_2, deduced by similar methods, we can obtain the following inequality (for simplicity we omit the details here):

$$\begin{aligned} S_2 &\le \frac{1}{62}\|\nabla\omega_x\|^2 + \frac{1}{40}\|\nabla\omega_y\|^2 + \frac{1}{50}\|\nabla j_x\|^2 \\ &\quad + \frac{1}{48}\|\nabla j_y\|^2 + \bigl(\|\omega_x\|^2 + \|\omega\|^2 + \|j\|^2 + \|\omega_y\|^2 \\ &\quad + \|b_x\|^2 + \|j_x\|^2 + \|b_y\|^2 + \|j_y\|^2 + \|u_x\|^2 + \|u_y\|^2 + \|\omega_y\|^2\bigr)\bigl(\|\nabla\omega\|^2 + \|\nabla j\|^2\bigr). \end{aligned}$$

Then, substituting all the above estimates into (11), we finally deduce that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigl(\|\nabla\omega\|^2 + \|\nabla j\|^2\bigr) + \frac{1}{2}\bigl(\|\nabla\omega_x\|^2 + \|\nabla\omega_y\|^2 + \|\nabla j_x\|^2 + \|\nabla j_y\|^2\bigr) \\ &\quad \le C\bigl(\|\omega_x\|^2 + \|\omega\|^2 + \|j\|^2 + \|\omega_y\|^2 + \|b_x\|^2 + \|j_x\|^2 + \|b_y\|^2 + \|j_y\|^2 \\ &\qquad + \|u_x\|^2 + \|u_y\|^2 + \|\omega_y\|^2\bigr)\bigl(\|\nabla\omega\|^2 + \|\nabla j\|^2\bigr). \end{aligned}$$

Then we complete the proof of Lemma 2.3 by Gronwall's lemma. □

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by using the method of vanishing viscosity. To this end, we consider the following regularized problem:

$$u_t^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon = -\nabla p^\varepsilon + u_{xx}^\varepsilon + u_{yy}^\varepsilon + \varepsilon u_{zz}^\varepsilon + b^\varepsilon \cdot \nabla b^\varepsilon, \tag{12}$$

$$b_t^\varepsilon + u^\varepsilon \cdot \nabla b^\varepsilon = b_{xx}^\varepsilon + b_{yy}^\varepsilon + \varepsilon b_{zz}^\varepsilon + b^\varepsilon \cdot \nabla u^\varepsilon, \tag{13}$$

$$\operatorname{div} u^\varepsilon = 0, \qquad \operatorname{div} b^\varepsilon = 0, \tag{14}$$

with smooth initial data

$$u^\varepsilon(0,x) = \psi_\varepsilon * u_0, \qquad b^\varepsilon(0,x) = \psi_\varepsilon * b_0, \tag{15}$$

where $\psi_\varepsilon(x,y) = \varepsilon^{-2}\psi(x/\varepsilon, y/\varepsilon)$ is the standard mollifier satisfying

$$\psi \geq 0, \qquad \psi \in C_0^\infty(\mathbb{R}^3) \quad \text{and} \quad \int \psi \,\mathrm{d}x = 1.$$

Now, an application of the classical result shows that for any $T > 0$, there exists a unique global smooth solution $(u^\varepsilon, b^\varepsilon)$ of (12)-(15) on $\mathbb{R}^3 \times (0,T)$ satisfying the global bounds stated in Lemma 2.2 and 2.3, which are uniform in ε. So, by standard compactness arguments, we can extract a subsequence $(u^{\varepsilon_j}, b^{\varepsilon_j})$ and pass to the limit as $j \to \infty$ to get that the limit function (u,b) is indeed a global smooth solution of the problem (1)-(4). The proof of Theorem 1.1 is therefore completed.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
MS carried out the main work and drafted the manuscript. JW participated in completing the proof of Lemma 2.2. All authors read and approved the final manuscript.

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References
1. Wang, F, Wang, K: Global existence of 3D MHD equations with mixed partial dissipation and magnetic diffusion. Nonlinear Anal., Real World Appl. **14**, 525-536 (2013)
2. Cabannes, H, Theoretical Magnetofluiddynamics. Academic Press, New York (1970)
3. Cao, C, Wu, J: Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion. Adv. Math. **226**, 1803-1822 (2011)

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