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On almost sure limiting behavior of a dependent random sequence

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Abstract
We study some sufficient conditions for the almost certain convergence of averages of arbitrarily dependent random variables by certain summability methods. As corollaries, we generalized some known results.

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Keywords: dependent random variable; summability; dominated random sequence

1 Introduction

In reference [1], Chow and Teicher gave a limit theorem of almost certain summability of i.i.d. random variables as follows.

Theorem (Chow *et al.*, 1971) *Let* $a(x)$, $x > 0$ *be a positive non-increasing function and* $a_n = a(n)$, $A_n = \sum_{k=1}^n a_k$, $b_n = A_n/a_n$, *where*

(1) $A_n \to \infty$;

(2) $0 < \liminf_n \frac{b_n}{n} a(\log b_n) \le \limsup_n \frac{b_n}{n} a(\log b_n) < \infty$;

(3) $xa(\log^+ x)$ *is non-decreasing for* $x > 0$, *then i.i.d.* $\{X, X_n\}$ *are* a_n *summable, i.e.,*

$$T_n = A_n^{-1} \sum_{k=1}^n a_k X_k - C_n \to 0 \quad a.c.$$

for some choice of centering constants C_n, *if and only if*

$$E|X|a\big(\log^+ |X|\big) < \infty.$$

Motivated by Chow and Teicher's idea, in this paper we consider the problem of arbitrarily dependent random variables and their limiting behavior from a new perspective.

Throughout this paper, let $\mathbb{N}$ denote the set of positive integers, $\{X, X_n, \mathcal{F}_n, n \in \mathbb{N}\}$ be a stochastic sequence defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, *i.e.*, the sequence of σ-fields $\{\mathcal{F}_n, n \in \mathbb{N}\}$ in $\mathcal{F}$ is increasing in n, and $\{\mathcal{F}_n\}$ are adapted to random variables $\{X_n\}$, $\mathcal{F}_0$ denotes the trivial σ field $\{\Phi, \Omega\}$ and $\mathbf{1}_{[\cdot]}$ the indicator function.

We begin by introducing some terminology and lemmas.

Definition 1 (Adler *et al.*, 1987 [2]) Let $\{X_n, n \in \mathbb{N}\}$ be a sequence of random variables, and it is said to be stochastically dominated by a random variable X (we write $\{X_n, n \in \mathbb{N}\} \prec X$)

if there exists a constant $C > 0$, for almost every $\omega \in \Omega$, such that

$$\sup_{n\in\mathbb{N}} \mathbb{P}\{|X_n| > t\} \le C\mathbb{P}\{|X| > t\} \quad \text{for all } t > 0.$$

Lemma 1 (Chow *et al.*, 1978 [3]) *Let $\{X_n, \mathcal{F}_n, n \in \mathbb{N}\}$ be an L_p $(1 \le p \le 2)$ martingale difference sequence, if $\sum_{n=1}^{\infty} E(|X_n|^p|\mathcal{F}_{n-1}) < \infty$, then $\sum_{n=1}^{\infty} X_n$ a.c. converges.*

Lemma 2 *Let $\{X, X_n, n \in \mathbb{N}\}$ be a sequence of random variables. If $\{X_n\} \prec X$, then for all $t > 0$,*

$$\mathbb{E}|X_n|^2\mathbf{1}_{[|X_n|\le t]} \le C\big[t^2\mathbb{P}\big(|X| > t\big) + \mathbb{E}X^2\mathbf{1}_{[|X|\le t]}\big].$$

Proof By the integral equality

$$2\int_0^t s\mathbb{P}\big(|X_n| > s\big)\,ds = t^2\mathbb{P}\big(|X_n| > t\big) + \mathbb{E}|X_n|^2\mathbf{1}_{[|X_n|\le t]},$$

it follows that

$$\begin{aligned}\mathbb{E}|X_n|^2\mathbf{1}_{[|X_n|\le t]} &\le 2\int_0^t s\mathbb{P}\big(|X_n| > s\big)\,ds \\ &\le 2C\int_0^t s\mathbb{P}\big(|X| > s\big)\,ds = C\big[t^2\mathbb{P}\big(|X| > t\big) + \mathbb{E}X^2\mathbf{1}_{[|X|\le t]}\big].\end{aligned}$$ □

2 Strong law of large numbers

In this section, we always assume that $a(x)$, $x > 0$ is a positive non-increasing function and $a_n = a(n)$, $A_n = \sum_{k=1}^n a_k$, $b_n = A_n/a_n$, where

(1) $A_n \to \infty$;
(2) $0 < \liminf_n \frac{b_n}{n} a(\log b_n) \le \limsup_n \frac{b_n}{n} a(\log b_n) < \infty$;
(3) $xa(\log^+ x)$ is non-decreasing for $x > 0$.

Theorem 1 *Let $\{X, X_n\}$ be a sequence of random variables with $\{X_n\} \prec X$. If $E|X|a(\log^+|X|) < \infty$, then*

$$\lim_n \frac{1}{A_n}\sum_{k=1}^n a_k\big[X_k - E(X_k\mathbf{1}_{[|X_k|\le b_k]}|\mathcal{F}_{k-1})\big] = 0, \quad \textit{a.c.} \tag{2.1}$$

Proof To prove (2.1) by applying the Kronecker lemma, it suffices to show that

$$\text{the series } \sum_{n=1}^{\infty} \frac{X_n - E(X_n\mathbf{1}_{[|X_n|\le b_n]}|\mathcal{F}_{n-1})}{b_n} \quad \text{converges a.c.}$$

Since $0 < a(x) \downarrow$, (1) guarantees that $b_n \uparrow \infty$. Choose m_0 such that $n \ge m_0$ implies

$$\alpha n \le b_n a(\log b_n) \le \beta n \tag{2.2}$$

whence $b_n \ge \alpha n[a(\log b_m)]^{-1}$ for $n \ge m \ge m_0$ entailing

$$\sum_{k=m}^{\infty} b_k^{-2} \le \frac{a^2(\log b_m)}{\alpha^2 m}. \tag{2.3}$$

Put $Y_n = X_n\mathbf{1}_{[|X_n|\le b_n]}, Z_n = X_n\mathbf{1}_{[|X_n|>b_n]}$, obviously, $X_n = Y_n + Z_n, n \in \mathbb{N}$. Note that $\{X_n\} \prec X$ and the condition $E|X|a(\log^+|X|) < \infty$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\mathbb{P}\bigl(|X_n| > b_n\bigr)\\
&\quad= \sum_{n=1}^{m_0-1}\mathbb{P}\bigl(|X_n| > b_n\bigr) + \sum_{n=m_0}^{\infty}\mathbb{P}\bigl(|X_n| > b_n\bigr)\\
&\quad\le m_0 - 1 + C\sum_{n=m_0}^{\infty}\mathbb{P}\bigl(|X| > b_n\bigr)\\
&\quad\le m_0 - 1 + C\sum_{n=m_0}^{\infty}\mathbb{P}\bigl(|X|a\bigl(\log|X|\bigr) \ge b_n a(\log b_n)\bigr)\\
&\quad\le m_0 - 1 + C\sum_{n=m_0}^{\infty}\mathbb{P}\bigl(|X|a\bigl(\log|X|\bigr) \ge \alpha n\bigr) < \infty,
\end{aligned}\tag{2.4}$$

which shows

$$\mathbb{P}(X_n \ne Z_n, \text{i.o.}) = 0. \tag{2.5}$$

Let $W_n = \frac{Y_n}{b_n} - E(\frac{Y_n}{b_n}|\mathcal{F}_{n-1})$, then $(W_n, \mathcal{F}_n, n \in \mathbb{N})$ is a martingale difference sequence. Since

$$\begin{aligned}
&\sum_{k=1}^{\infty}E\frac{Y_k^2}{b_k^2} = \sum_{k=1}^{\infty}\frac{EX_k^2\mathbf{1}_{[|X_k|\le b_k]}}{b_k^2}\\
&\quad\le C\sum_{k=1}^{\infty}\biggl[E\mathbf{1}_{[|X|>b_k]} + \frac{EX^2\mathbf{1}_{[|X|\le b_k]}}{b_k^2}\biggr] \quad \text{(by Lemma 2)}\\
&\quad= C\sum_{k=1}^{\infty}E\mathbf{1}_{[|X|>b_k]} + C\Biggl(\sum_{k=1}^{m_0-1} + \sum_{k=m_0}^{\infty}\Biggr)\frac{EX^2\mathbf{1}_{[|X|\le b_k]}}{b_k^2}\\
&\quad\le C\sum_{k=1}^{\infty}\mathbb{P}\bigl(|X| > b_k\bigr) + C(m_0 - 1) + C\sum_{k=m_0}^{\infty}b_k^{-2}\biggl(\int_{[|X_k|\le b_{m_0-1}]}X^2 + \sum_{i=m_0}^{k}\int_{[b_{i-1}<|X|\le b_i]}X^2\biggr)\\
&\quad\le \mathcal{O}(1) + C\sum_{i=m_0}^{\infty}\sum_{k=i}^{\infty}b_k^{-2}\int_{[b_{i-1}<|X|\le b_i]}X^2 \quad \bigl(\text{by (2.4)}\bigr)\\
&\quad\le \mathcal{O}(1) + \alpha^{-2}C\sum_{i=m_0}^{\infty}i^{-1}a^2(\log b_i)\int_{[b_{i-1}<|X|\le b_i]}X^2 \quad \bigl(\text{by (2.3)}\bigr)\\
&\quad\le \mathcal{O}(1) + \alpha^{-2}\beta C\sum_{i=m_0}^{\infty}a(\log b_i)\int_{[b_{i-1}<|X|\le b_i]}|X| \quad \bigl(\text{by (2.2)}\bigr)\\
&\quad\le \mathcal{O}(1) + \alpha^{-2}\beta C\sum_{i=m_0}^{\infty}\int_{[b_{i-1}<|X|\le b_i]}|X|a\bigl(\log|X|\bigr) < \infty.
\end{aligned}\tag{2.6}$$

Note that

$$E\left[\sum_{n=1}^{\infty} E\big(W_n^2|\mathcal{F}_{n-1}\big)\right] \le E\left[\sum_{n=1}^{\infty} E\left(\frac{Y_n^2}{b_n^2}\Big|\mathcal{F}_{n-1}\right)\right] = \sum_{n=1}^{\infty} E\frac{Y_n^2}{b_n^2} < \infty, \tag{2.7}$$

which implies that $\sum_{n=1}^{\infty} E(W_n^2|\mathcal{F}_{n-1}) < \infty$ a.c. Hence, by Lemma 1, we have $\sum_{n=1}^{\infty} W_n$ a.c. convergence.

Theorem 1 follows from (2.5) and (2.7). □

Theorem 1 also includes some particular cases of means, we can establish the following.

Corollary 1 *Let $\{X, X_n, n \in \mathbb{N}\}$ be a sequence of random variables with $\{X_n\} \prec X$. If for some $\varepsilon > 0$, $E\frac{|X|}{\log|X|\mathbf{1}_{[|X|>\varepsilon]}} < \infty$, then*

$$\lim_n \frac{1}{\log n} \sum_{k=1}^{n} \left[\frac{X_k - E(X_k \mathbf{1}_{[|X_k| \le k\log k]}|\mathcal{F}_{k-1})}{k}\right] = 0, \quad \textit{a.c.}$$

Corollary 2 *Let $\{X, X_n, n \in \mathbb{N}\}$ be a sequence of random variables with $\{X_n\} \prec X$ and for some $k \ge 2$,*

$$a_n = \big[n(\log n)\cdots(\log_{k-1} n)\big]^{-1},$$

where $\log_1 n = \log n$, $\log_k n = \log(\log_{k-1} n)$, $k \ge 2$, if for all large $C > 0$,

$$E\frac{|X|\mathbf{1}_{[|X|>C]}}{(\log|X|)\cdots(\log_k|X|)} < \infty,$$

then

$$\lim_n \frac{1}{A_n} \sum_{k=1}^{n} a_k\big[X_k - E(X_k \mathbf{1}_{[|X_k| \le b_k]}|\mathcal{F}_{k-1})\big] = 0, \quad \textit{a.c.}$$

Corollary 3 *Let $\{X, X_n, n \in \mathbb{N}\}$ be a sequence of random variables with $\{X_n\} \prec X$. Further, let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and $\mathcal{F}_{-n} = \{\phi, \Omega\}$, $n \ge 0$. If $E|X|a(\log^+|X|) < \infty$, then for any $m \ge 1$,*

$$\lim_n \frac{1}{A_n} \sum_{k=1}^{n} a_k\big[X_k - E(X_k \mathbf{1}_{[|X_k| \le b_k]}|\mathcal{F}_{k-m})\big] = 0, \quad \textit{a.c.} \tag{2.8}$$

Proof Since $\{X_{nm+l}, \mathcal{F}_{nm+l}, n \ge 1\}$ is an adapted stochastic sequence and $\{X_{nm+l}\} \prec X$, by Theorem 1, we have for $l = 0, 1, \ldots, m-1$ that

$$\sum_{n=1}^{\infty} \frac{X_{nm+l} - E(X_{nm+l}\mathbf{1}_{[|X_{nm+l}| \le b_{nm+l}]}|\mathcal{F}_{(n-1)m+l})}{b_{nm+l}} \quad \text{converges a.c.}$$

Therefore, we have

$$\begin{aligned}&\sum_{n=m}^{\infty}\frac{X_n - E(X_n\mathbf{1}_{[|X_n|\le b_n]}|\mathcal{F}_{n-m})}{b_n}\\&\quad=\sum_{l=0}^{m-1}\sum_{n=1}^{\infty}\frac{X_{nm+l} - E(X_{nm+l}\mathbf{1}_{[|X_{nm+l}|\le b_{nm+l}]}|\mathcal{F}_{(n-1)m+l})}{b_{nm+l}}\quad\text{converges a.c.}\end{aligned}$$

□

Corollary 4 *Let $\{X_n, n\in\mathbb{N}\}$ be a sequence of m-dependent random variables. Further, let $\mathcal{F}_n = \sigma(X_1,\dots,X_n)$ and $\mathcal{F}_{-n} = \{\phi,\Omega\}, n\ge 0$. If there exists a random variable X such that $\{X_n\}\prec X$ and $E|X|a(\log^+|X|)<\infty$, then*

$$\lim_n \frac{1}{A_n}\sum_{k=1}^{n} a_k\big[X_k - E(X_k\mathbf{1}_{[|X_k|\le b_k]})\big] = 0,\quad a.c.$$

Proof Note that $\{X_n, n\in\mathbb{N}\}$ is a sequence of m-dependent random variables, then $E(X_n|\mathcal{F}_{n-m}) = EX_n$, Corollary 4 follows directly from Corollary 3. □

Definition 2 (Stout, 1974) Let $\{X_n, n\in\mathbb{N}\}$ be a sequence of random variables, and let $\mathcal{F}_n^m = \sigma(X_n,\dots,X_m)$. We say that the sequence $\{X_n, n\in\mathbb{N}\}$ is *-mixing if there exists a positive integer M and a non-decreasing function $\varphi(n)$ defined on integers $n\ge M$ with $\lim_n\varphi(n) = 0$ such that for $n > M$, $A\in\mathcal{F}_0^m$ and $B\in\mathcal{F}_{m+n}^{\infty}$, the relation

$$\big|\mathbb{P}(A\cap B) - \mathbb{P}(A)P(B)\big| \le \varphi(n)\mathbb{P}(A)\mathbb{P}(B)$$

holds for any integer $m\ge 1$.

It has been proved (*cf.* [4]) that the *-mixing condition is equivalent to the condition

$$\big|\mathbb{P}\big(B|\mathcal{F}_0^m\big) - \mathbb{P}(B)\big| \le \varphi(n)\mathbb{P}(B),\quad \text{a.c.}$$

for $B\in\mathcal{F}_{m+n}^{\infty}$ and $m\ge 1$ implies

$$\big|E\big(X_{n+m}|\mathcal{F}_0^m\big) - EX_{n+m}\big| \le \varphi(n)E|X_{n+m}|,\quad \text{a.c.} \tag{2.9}$$

Theorem 2 *Let $\{X, X_n, n\in\mathbb{N}\}$ be a sequence of *-mixing random variables with $\{X_n\}\prec X$. Further, let $\mathcal{F}_n = \sigma(X_1,\dots,X_n)$ and $\mathcal{F}_{-n} = \{\phi,\Omega\}$, $n\ge 0$. If $\max\{E|X|, E|X|a(\log^+|X|)\}<\infty$, then*

$$\lim_n \frac{1}{A_n}\sum_{k=1}^{n} a_k[X_k - EX_k\mathbf{1}_{[|X_k|\le b_k]}] = 0,\quad a.c.$$

Proof By Corollary 3, we have, for each $m\ge 1$,

$$\lim_n \frac{1}{A_n}\sum_{k=1}^{n} a_k\big[X_k - E(X_k\mathbf{1}_{[|X_k|\le b_k]}|\mathcal{F}_{k-m})\big] = 0,\quad \text{a.c.}$$

Since $\{X_n, n \in \mathbb{N}\}$ is *-mixing, by (2.8) and (2.9), we obtain

$$
\begin{aligned}
&\left|\frac{1}{A_n}\sum_{k=1}^{n} a_k[X_k - EX_k\mathbf{1}_{[|X_k|\le b_k]}]\right| \\
&\quad\le \left|\frac{1}{A_n}\sum_{k=1}^{n} a_k\bigl[X_k - E(X_k\mathbf{1}_{[|X_k|\le b_k]}|\mathcal{F}_{k-m})\bigr]\right| \\
&\qquad + \frac{1}{A_n}\sum_{k=1}^{n} a_k\bigl|\bigl[E(X_k\mathbf{1}_{[|X_k|\le b_k]}|\mathcal{F}_{k-m}) - EX_k\mathbf{1}_{[|X_k|\le b_k]}\bigr]\bigr| \\
&\quad\le \left|\frac{1}{A_n}\sum_{k=1}^{n} a_k\bigl[X_k - E(X_k\mathbf{1}_{[|X_k|\le b_k]}|\mathcal{F}_{k-m})\bigr]\right| + \frac{\varphi(m)}{A_n}\sum_{k=1}^{n} a_k E|X_k|\mathbf{1}_{[|X_k|\le b_k]} \\
&\quad\le \left|\frac{1}{A_n}\sum_{k=1}^{n} a_k\bigl[X_k - E(X_k\mathbf{1}_{[|X_k|\le b_k]}|\mathcal{F}_{k-m})\bigr]\right| + \varphi(m)E|X| \to 0 \quad \text{a.c. (as } n\to\infty).
\end{aligned}
$$

Thus, using the Kroneker lemma, Theorem 2 follows. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
WZ and FA carried out the design of the study and performed the analysis, WZ drafted the manuscript. All authors read and approved the final manuscript.

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