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Some companions of perturbed Ostrowski-type inequalities based on the quadratic kernel function with three sections and applications

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Abstract

In this paper, based on the quadratic kernel function with three sections, which was defined by Liu in 2009, we establish some companions of perturbed Ostrowski-type inequalities for the case when $f'' \in L^1[a,b]$, $f''' \in L^2[a,b]$ and $f'' \in L^2[a,b]$, respectively. The special cases of these results offer better estimation than the conventional trapezoidal formula and the midpoint formula. The results we get can apply to composite quadrature rules in numerical integration and probability density functions. The effectiveness of these applications is also illustrated through several specific examples due to better error estimates.

MSC: 26D15; 41A55; 41A80; 65C50

Keywords: perturbed Ostrowski type inequality; differentiable mapping; composite quadrature rule; probability density function

1 Introduction

In 1938, Ostrowski [1] established the following interesting integral inequality for differentiable mappings with bounded derivatives.

Theorem 1.1 *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) whose derivative is bounded on (a,b) and denote $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then for all $x \in [a,b]$ we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right](b-a)\|f'\|_\infty. \tag{1.1}$$

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

In [2], Guessab and Schmeisser proved the following companion of Ostrowski's inequality.

Theorem 1.2 *Let $f : [a,b] \to \mathbb{R}$ be satisfying the Lipschitz condition, i.e., $|f(t) - f(s)| \le M|t-s|$. Then for all $x \in [a, \frac{a+b}{2}]$ we have*

$$\left| \frac{f(x) + f(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{8} + 2\left(\frac{x - \frac{3a+b}{4}}{b-a} \right)^2 \right](b-a)M. \tag{1.2}$$

The constant $\frac{1}{8}$ *is sharp in the sense that it cannot be replaced by a smaller one. In* (1.2), *the point* $x = \frac{3a+b}{4}$ *gives the best estimator.*

Motivated by [2], Dragomir [3] proved some companions of Ostrowski's inequality for absolutely continuous functions. Recently, Alomari [4] studied the companion of Ostrowski inequality (1.2) for differentiable bounded mappings. In [5], Liu established some companions of an Ostrowski-type integral inequality for functions whose first derivatives are absolutely continuous and second derivatives belong to L^p ($1 \leq p \leq \infty$) spaces.

Theorem 1.3 *Let* $f : [a,b] \to \mathbb{R}$ *be such that* f' *is absolutely continuous on* $[a,b]$ *and* $f'' \in L^\infty[a,b]$. *Then for all* $x \in [a, \frac{a+b}{2}]$ *we have*

$$\begin{aligned} &\left|\frac{f(x)+f(a+b-x)}{2} - \left(x - \frac{3a+b}{4}\right)\frac{f'(x)-f'(a+b-x)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\quad \leq \left[\frac{1}{96} + \frac{1}{2}\frac{(x-\frac{3a+b}{4})^2}{(b-a)^2}\right](b-a)^2\|f''\|_\infty. \end{aligned} \tag{1.3}$$

The constant $\frac{1}{96}$ *is sharp in the sense that it cannot be replaced by a smaller one.*

For other related results, the reader may refer to [6–26] and the references therein.

The main aim of this paper is to establish some companions of perturbed Ostrowski-type inequalities for the case when $f'' \in L^1[a,b]$, $f''' \in L^2[a,b]$ and $f'' \in L^2[a,b]$, respectively. For our purpose, we will use the quadratic kernel function with three sections (see (2.1) below) which was defined by Liu in [5]. The special cases of the results we get offer better estimation than the conventional trapezoidal formula and the midpoint formula. These results can apply to composite quadrature rules in numerical integration and probability density functions. The effectiveness of these applications is also illustrated through several specific examples due to better error estimates.

2 Main results

To prove our main results, we need the following lemmas.

Lemma 2.1 [5] *Let* $f : [a,b] \to \mathbb{R}$ *be such that* f' *is absolutely continuous on* $[a,b]$. *Denote by* $K(x,t) : [a,b] \to \mathbb{R}$ *the kernel given by*

$$K(x,t) = \begin{cases} \frac{1}{2}(t-a)^2, & t \in [a,x], \\ \frac{1}{2}(t-\frac{a+b}{2})^2, & t \in (x, a+b-x], \\ \frac{1}{2}(t-b)^2, & t \in (a+b-x, b], \end{cases} \tag{2.1}$$

then the identity

$$\begin{aligned} &\frac{1}{b-a}\int_a^b K(x,t)f''(t)\,dt \\ &\quad = \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(x)+f(a+b-x)}{2} + \left(x - \frac{3a+b}{4}\right)\frac{f'(x)-f'(a+b-x)}{2} \end{aligned} \tag{2.2}$$

holds.

2.1 The case when $f'' \in L^1[a,b]$ and is bounded

Theorem 2.1 *Let $f : [a,b] \to \mathbb{R}$ be such that f' is absolutely continuous on $[a,b]$. If $f'' \in L^1[a,b]$ and $\gamma \le f''(t) \le \Gamma$, $\forall t \in [a,b]$, then for all $x \in [a, \frac{a+b}{2}]$ we have*

$$\begin{aligned}
&\left|\frac{f(x)+f(a+b-x)}{2} - \left(x - \frac{3a+b}{4}\right)\frac{f'(x)-f'(a+b-x)}{2}\right. \\
&\qquad \left. + \frac{f'(b)-f'(a)}{b-a}\left[\frac{1}{2}\left(x-\frac{3a+b}{4}\right)^2 + \frac{(b-a)^2}{96}\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\qquad \le (S-\gamma)\left[\frac{(b-a)^2}{48} + \frac{b-a}{4}\left|x - \frac{3a+b}{4}\right|\right]
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
&\left|\frac{f(x)+f(a+b-x)}{2} - \left(x - \frac{3a+b}{4}\right)\frac{f'(x)-f'(a+b-x)}{2}\right. \\
&\qquad \left. + \frac{f'(b)-f'(a)}{b-a}\left[\frac{1}{2}\left(x-\frac{3a+b}{4}\right)^2 + \frac{(b-a)^2}{96}\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\qquad \le (\Gamma - S)\left[\frac{(b-a)^2}{48} + \frac{b-a}{4}\left|x - \frac{3a+b}{4}\right|\right],
\end{aligned} \tag{2.4}$$

where $S = (f'(b)-f'(a))/(b-a)$.

Proof From (2.2) and the facts

$$\frac{1}{b-a}\int_a^b f''(t)\,dt = \frac{f'(b)-f'(a)}{b-a} \tag{2.5}$$

and

$$\frac{1}{b-a}\int_a^b K(x,t)\,dt = \frac{1}{2}\left(x - \frac{3a+b}{4}\right)^2 + \frac{(b-a)^2}{96}, \tag{2.6}$$

it follows that

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b K(x,t)f''(t)\,dt - \frac{1}{(b-a)^2}\int_a^b K(x,t)\,dt\int_a^b f''(t)\,dt \\
&\quad = \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(x)+f(a+b-x)}{2} + \left(x-\frac{3a+b}{4}\right)\frac{f'(x)-f'(a+b-x)}{2} \\
&\qquad - \frac{f'(b)-f'(a)}{b-a}\left[\frac{1}{2}\left(x-\frac{3a+b}{4}\right)^2 + \frac{(b-a)^2}{96}\right].
\end{aligned} \tag{2.7}$$

We denote

$$R_n(x) = \frac{1}{b-a}\int_a^b K(x,t)f''(t)\,dt - \frac{1}{(b-a)^2}\int_a^b K(x,t)\,dt\int_a^b f''(t)\,dt. \tag{2.8}$$

If $C \in \mathbb{R}$ is an arbitrary constant, then we have

$$R_n(x) = \frac{1}{b-a}\int_a^b \big(f''(t) - C\big)\left[K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)\,ds\right]dt. \tag{2.9}$$

Furthermore, we have

$$\left|R_n(x)\right| \le \frac{1}{b-a} \max_{t\in[a,b]} \left|K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)\,ds\right| \int_a^b \left|f''(t) - C\right|dt. \tag{2.10}$$

To compute

$$\begin{aligned}
&\max_{t\in[a,b]} \left|K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)\,ds\right| \\
&\quad = \max\left\{\left|\frac{1}{2}(x-a)^2 - \left[\frac{1}{2}\left(x - \frac{3a+b}{4}\right)^2 + \frac{(b-a)^2}{96}\right]\right|,\right. \\
&\qquad \left.\left|\frac{1}{2}\left(\frac{a+b}{2} - x\right)^2 - \left[\frac{1}{2}\left(x - \frac{3a+b}{4}\right)^2 + \frac{(b-a)^2}{96}\right]\right|, \frac{1}{2}\left(x - \frac{3a+b}{4}\right)^2 + \frac{(b-a)^2}{96}\right\} \\
&\quad = \max\left\{\frac{b-a}{24}|6x-5a-b|, \frac{b-a}{12}|3x-2a-b|, \frac{1}{2}\left(x - \frac{3a+b}{4}\right)^2 + \frac{(b-a)^2}{96}\right\},
\end{aligned} \tag{2.11}$$

we denote

$$\begin{aligned}
&y_1 = \frac{b-a}{24}|6x-5a-b|, \qquad y_2 = \frac{b-a}{12}|3x-2a-b|, \\
&y_3 = \frac{1}{2}\left(x - \frac{3a+b}{4}\right)^2 + \frac{(b-a)^2}{96}.
\end{aligned}$$

If we choose $y_1 = 0$, then we get $x_1 = \frac{5a+b}{6}$. If we choose $y_2 = 0$, then we get $x_2 = \frac{2a+b}{3}$. A direct computation gives that

$$\begin{cases} y_2 \ge \max\{y_1, y_3\}, & x \in [a, \frac{3a+b}{4}], \\ y_1 > \max\{y_2, y_3\}, & x \in (\frac{3a+b}{4}, \frac{a+b}{2}]. \end{cases} \tag{2.12}$$

Therefore, we get

$$\begin{aligned}
&\max_{t\in[a,b]} \left|K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)\,ds\right| \\
&\quad = \max\{y_1, y_2\} = \frac{(b-a)^2}{48} + \frac{b-a}{4}\left|x - \frac{3a+b}{4}\right|.
\end{aligned} \tag{2.13}$$

We also have

$$\int_a^b \left|f''(t) - \gamma\right|dt = (S-\gamma)(b-a) \tag{2.14}$$

and

$$\int_a^b \left|f''(t) - \Gamma\right|dt = (\Gamma - S)(b-a). \tag{2.15}$$

Therefore, we obtain (2.3) and (2.4) by using (2.7)-(2.10), (2.1)-(2.15) and choosing $C = \gamma$ and $C = \Gamma$ in (2.10), respectively. □

Corollary 2.1 *Under the assumptions of Theorem* 2.1, *choose*

(1) $x = \frac{3a+b}{4}$, *we have*

$$\left|\frac{f(\frac{3a+b}{4}) + f(\frac{a+3b}{4})}{2} + \frac{f'(b) - f'(a)}{b-a}\frac{(b-a)^2}{96} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{48}(S-\gamma)(b-a)^2, \tag{2.16}$$

$$\left|\frac{f(\frac{3a+b}{4}) + f(\frac{a+3b}{4})}{2} + \frac{f'(b) - f'(a)}{b-a}\frac{(b-a)^2}{96} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{48}(\Gamma - S)(b-a)^2. \tag{2.17}$$

(2) $x = a$, *we have*

$$\left|\frac{f(a) + f(b)}{2} - \frac{f'(b) - f'(a)}{b-a}\frac{(b-a)^2}{12} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{12}(S-\gamma)(b-a)^2,$$

$$\left|\frac{f(a) + f(b)}{2} - \frac{f'(b) - f'(a)}{b-a}\frac{(b-a)^2}{12} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{12}(\Gamma - S)(b-a)^2.$$

(3) $x = \frac{a+b}{2}$, *we have*

$$\left|f\left(\frac{a+b}{2}\right) + \frac{f'(b) - f'(a)}{b-a}\frac{(b-a)^2}{24} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{12}(S-\gamma)(b-a)^2,$$

$$\left|f\left(\frac{a+b}{2}\right) + \frac{f'(b) - f'(a)}{b-a}\frac{(b-a)^2}{24} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{12}(\Gamma - S)(b-a)^3.$$

Corollary 2.2 *Let f be as in Theorem* 2.1. *Additionally, if f is symmetric about $x = \frac{a+b}{2}$, then for all $x \in [a, \frac{a+b}{2}]$ we have*

$$\begin{aligned}&\left|f(x) - \left(x - \frac{3a+b}{4}\right)f'(x) + \frac{f'(b) - f'(a)}{b-a}\left[\frac{1}{2}\left(x - \frac{3a+b}{4}\right)^2 + \frac{(b-a)^2}{96}\right]\right.\\&\quad\left. - \frac{1}{b-a}\int_a^b f(t)\,dt\right|\\&\quad\le (S-\gamma)\left[\frac{(b-a)^2}{48} + \frac{b-a}{4}\left|x - \frac{3a+b}{4}\right|\right]\end{aligned}$$

and

$$\begin{aligned}&\left|f(x) - \left(x - \frac{3a+b}{4}\right)f'(x) + \frac{f'(b) - f'(a)}{b-a}\left[\frac{1}{2}\left(x - \frac{3a+b}{4}\right)^2 + \frac{(b-a)^2}{96}\right]\right.\\&\quad\left. - \frac{1}{b-a}\int_a^b f(t)\,dt\right|\\&\quad\le (\Gamma - S)\left[\frac{(b-a)^2}{48} + \frac{b-a}{4}\left|x - \frac{3a+b}{4}\right|\right].\end{aligned}$$

2.2 The case when $f''' \in L^2[a,b]$

Theorem 2.2 *Let* $f:[a,b]\to\mathbb{R}$ *be a thrice continuously differentiable mapping in* (a,b) *with* $f''' \in L^2[a,b]$. *Then for all* $x\in[a,\frac{a+b}{2}]$ *we have*

$$\begin{aligned}&\left|\frac{f(x)+f(a+b-x)}{2}-\left(x-\frac{3a+b}{4}\right)\frac{f'(x)-f'(a+b-x)}{2}\right.\\&\qquad\left.+\frac{f'(b)-f'(a)}{b-a}\left[\frac{1}{2}\left(x-\frac{3a+b}{4}\right)^2+\frac{(b-a)^2}{96}\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\&\qquad\le\frac{1}{\pi}\|f'''\|_2\left\{\frac{1}{320}(a+b-2x)^5+\frac{1}{10}(x-a)^5\right.\\&\qquad\left.-(b-a)\left[\frac{1}{2}\left(x-\frac{3a+b}{4}\right)^2+\frac{(b-a)^2}{96}\right]^2\right\}^{1/2}.\end{aligned}\tag{2.18}$$

Proof Let $R_n(x)$ be defined by (2.8). From (2.7), we get

$$\begin{aligned}R_n(x)&=\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(x)+f(a+b-x)}{2}+\left(x-\frac{3a+b}{4}\right)\frac{f'(x)-f'(a+b-x)}{2}\\&\quad-\frac{f'(b)-f'(a)}{b-a}\left[\frac{1}{2}\left(x-\frac{3a+b}{4}\right)^2+\frac{(b-a)^2}{96}\right].\end{aligned}\tag{2.19}$$

If we choose $C=f''((a+b)/2)$ in (2.9) and use the Cauchy inequality, then we get

$$\begin{aligned}&|R_n(x)|\\&\quad\le\frac{1}{b-a}\int_a^b\left|f''(t)-f''\left(\frac{a+b}{2}\right)\right|\left|K(x,t)-\frac{1}{b-a}\int_a^b K(x,s)\,ds\right|dt\\&\quad\le\frac{1}{b-a}\left[\int_a^b\left(f''(t)-f''\left(\frac{a+b}{2}\right)\right)^2dt\right]^{1/2}\\&\quad\quad\times\left[\int_a^b\left(K(x,t)-\frac{1}{b-a}\int_a^b K(x,s)\,ds\right)^2dt\right]^{1/2}.\end{aligned}\tag{2.20}$$

We can use the Diaz-Metcalf inequality (see [19, p.83] or [25, p.424]) to get

$$\int_a^b\left(f''(t)-f''\left(\frac{a+b}{2}\right)\right)^2dt\le\frac{(b-a)^2}{\pi^2}\|f'''\|_2^2.$$

We also have

$$\begin{aligned}&\int_a^b\left(K(x,t)-\frac{1}{b-a}\int_a^b K(x,s)\,ds\right)^2dt\\&\quad=\int_a^b K(x,t)^2\,dt-(b-a)\left[\frac{1}{2}\left(x-\frac{3a+b}{4}\right)^2+\frac{(b-a)^2}{96}\right]^2\\&\quad=\frac{1}{320}(a+b-2x)^5+\frac{1}{10}(x-a)^5-(b-a)\left[\frac{1}{2}\left(x-\frac{3a+b}{4}\right)^2+\frac{(b-a)^2}{96}\right]^2.\end{aligned}\tag{2.21}$$

Therefore, using the above relations, we obtain (2.18). □

Corollary 2.3 *Under the assumptions of Theorem* 2.2, *choose*

(1) $x = \frac{3a+b}{4}$, *we have*

$$\left|\frac{f(\frac{3a+b}{4})+f(\frac{a+3b}{4})}{2} + \frac{f'(b)-f'(a)}{b-a}\frac{(b-a)^2}{96} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{(b-a)^{5/2}}{48\pi\sqrt{5}}\|f'''\|_2. \tag{2.22}$$

(2) $x = a$, *we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{f'(b)-f'(a)}{b-a}\frac{(b-a)^2}{12} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{(b-a)^{5/2}}{12\pi\sqrt{5}}\|f'''\|_2.$$

(3) $x = \frac{a+b}{2}$, *we have*

$$\left|f\left(\frac{a+b}{2}\right) + \frac{f'(b)-f'(a)}{b-a}\frac{(b-a)^2}{24} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{(b-a)^{5/2}}{12\pi\sqrt{5}}\|f'''\|_2.$$

Corollary 2.4 *Let f be as in Theorem* 2.1. *Additionally, if f is symmetric about $x = \frac{a+b}{2}$, i.e., $f(a+b-x) = f(x)$, then for all $x \in [a, \frac{a+b}{2}]$ we have*

$$\begin{aligned}&\left|f(x) - \left(x - \frac{3a+b}{4}\right)f'(x) + \frac{f'(b)-f'(a)}{b-a}\left[\frac{1}{2}\left(x-\frac{3a+b}{4}\right)^2 + \frac{(b-a)^2}{96}\right]\right.\\&\left.\quad - \frac{1}{b-a}\int_a^b f(t)\,dt\right|\\&\quad\le \frac{1}{\pi}\|f'''\|_2\left\{\frac{1}{320}(a+b-2x)^5 + \frac{1}{10}(x-a)^5\right.\\&\quad\left.- (b-a)\left[\frac{1}{2}\left(x-\frac{3a+b}{4}\right)^2 + \frac{(b-a)^2}{96}\right]^2\right\}^{1/2}.\end{aligned}$$

2.3 The case when $f'' \in L^2[a,b]$

Theorem 2.3 *Let $f : [a,b] \to \mathbb{R}$ be such that f' is absolutely continuous on $[a,b]$ with $f'' \in L^2[a,b]$. Then for all $x \in [a, \frac{a+b}{2}]$ we have*

$$\begin{aligned}&\left|\frac{f(x)+f(a+b-x)}{2} - \left(x-\frac{3a+b}{4}\right)\frac{f'(x)-f'(a+b-x)}{2}\right.\\&\left.\quad + \frac{f'(b)-f'(a)}{b-a}\left[\frac{1}{2}\left(x-\frac{3a+b}{4}\right)^2 + \frac{(b-a)^2}{96}\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right|\\&\quad\le \frac{\sqrt{\sigma(f'')}}{b-a}\left\{\frac{1}{320}(a+b-2x)^5 + \frac{1}{10}(x-a)^5\right.\\&\quad\left.- (b-a)\left[\frac{1}{2}\left(x-\frac{3a+b}{4}\right)^2 + \frac{(b-a)^2}{96}\right]^2\right\}^{1/2},\end{aligned} \tag{2.23}$$

where $\sigma(f'')$ is defined by

$$\sigma(f'') = \|f''\|_2^2 - \frac{(f'(b)-f'(a))^2}{b-a} = \|f''\|_2^2 - S^2(b-a) \tag{2.24}$$

and S is defined in Theorem 2.1.

Proof Let $R_n(x)$ be defined by (2.8). If we choose $C = \frac{1}{b-a}\int_a^b f''(s)\,ds$ in (2.9) and use the Cauchy inequality and (2.21), then we get

$$\begin{aligned}
&\left|R_n(x)\right| \\
&\quad\le \frac{1}{b-a}\int_a^b \left|f''(t) - \frac{1}{b-a}\int_a^b f''(s)\,ds\right|\left|K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)\,ds\right|dt \\
&\quad\le \frac{1}{b-a}\left[\int_a^b \left(f''(t) - \frac{1}{b-a}\int_a^b f''(s)\,ds\right)^2 dt\right]^{1/2} \\
&\qquad\times \left[\int_a^b \left(K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)\,ds\right)^2 dt\right]^{1/2} \\
&\quad\le \frac{\sqrt{\sigma(f'')}}{b-a}\left\{\frac{1}{320}(a+b-2x)^5 + \frac{1}{10}(x-a)^5\right. \\
&\qquad\left. - (b-a)\left[\frac{1}{2}\left(x - \frac{3a+b}{4}\right)^2 + \frac{(b-a)^2}{96}\right]^2\right\}^{1/2}. \qquad \square
\end{aligned}$$

Corollary 2.5 *Under the assumptions of Theorem* 2.3, *choose*

(1) $x = \frac{3a+b}{4}$, *we have*

$$\begin{aligned}
&\left|\frac{f(\frac{3a+b}{4}) + f(\frac{a+3b}{4})}{2} + \frac{f'(b) - f'(a)}{b-a}\frac{(b-a)^2}{96} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\quad\le \frac{(b-a)^{3/2}}{48\sqrt{5}}\sqrt{\sigma(f'')}.
\end{aligned} \tag{2.25}$$

(2) $x = a$, *we have*

$$\left|\frac{f(a) + f(b)}{2} - \frac{f'(b) - f'(a)}{b-a}\frac{(b-a)^2}{12} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{(b-a)^{3/2}}{12\sqrt{5}}\sqrt{\sigma(f'')}.$$

(3) $x = \frac{a+b}{2}$, *we have*

$$\left|f\left(\frac{a+b}{2}\right) + \frac{f'(b) - f'(a)}{b-a}\frac{(b-a)^2}{24} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{(b-a)^{3/2}}{12\sqrt{5}}\sqrt{\sigma(f'')}.$$

Corollary 2.6 *Let f be as in Theorem* 2.1. *Additionally, if f is symmetric about $x = \frac{a+b}{2}$, then for all $x \in [a, \frac{a+b}{2}]$ we have*

$$\begin{aligned}
&\left|f(x) - \left(x - \frac{3a+b}{4}\right)f'(x) + \frac{f'(b) - f'(a)}{b-a}\left[\frac{1}{2}\left(x - \frac{3a+b}{4}\right)^2 + \frac{(b-a)^2}{96}\right]\right. \\
&\qquad\left. - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\quad\le \frac{\sqrt{\sigma(f'')}}{b-a}\left\{\frac{1}{320}(a+b-2x)^5 + \frac{1}{10}(x-a)^5\right. \\
&\qquad\left. - (b-a)\left[\frac{1}{2}\left(x - \frac{3a+b}{4}\right)^2 + \frac{(b-a)^2}{96}\right]^2\right\}^{1/2}.
\end{aligned}$$

3 Application to composite quadrature rules

Let $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be a partition of the interval $[a,b]$ and $h_i = x_{i+1} - x_i$ $(i = 0,1,2,\ldots,n-1)$.

Consider the perturbed composite quadrature rules

$$Q_n^1(I_n,f) = \frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right)\right]h_i + \sum_{i=0}^{n-1}\frac{f'(x_{i+1}) - f'(x_i)}{96}h_i^2 \tag{3.1}$$

and

$$Q_n^2(I_n,f) = \frac{1}{2}\sum_{i=0}^{n-1}\left[f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right)\right]h_i + \frac{\Gamma + \gamma}{192}\sum_{i=0}^{n-1}h_i^3. \tag{3.2}$$

The following results hold.

Theorem 3.1 *Let $f : [a,b] \to \mathbb{R}$ be such that f' is absolutely continuous on $[a,b]$. If $f'' \in L^1[a,b]$ and $\gamma \leq f''(t) \leq \Gamma$, $\forall t \in [a,b]$, then for all $x \in [a, \frac{a+b}{2}]$ we have*

$$\int_a^b f(t)\,dt = Q_n^1(I_n,f) + R_n^1(I_n,f),$$

where $Q_n^1(I_n,f)$ is defined by formula (3.1), *and the remainder $R_n^1(I_n,f)$ satisfies the estimate*

$$\left|R_n^1(I_n,f)\right| \leq \frac{1}{48}(S - \gamma)\sum_{i=0}^{n-1}h_i^3 \tag{3.3}$$

and

$$\left|R_n^1(I_n,f)\right| \leq \frac{1}{48}(\Gamma - S)\sum_{i=0}^{n-1}h_i^3. \tag{3.4}$$

Proof Applying inequality (2.1) and (2.1) to the intervals $[x_i, x_{i+1}]$, we get

$$\left|\int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{1}{2}\left[f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right)\right]h_i - \frac{f'(x_{i+1}) - f'(x_i)}{96}h_i^2\right| \leq \frac{1}{48}(S - \gamma)h_i^3$$

and

$$\left|\int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{1}{2}\left[f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right)\right]h_i - \frac{f'(x_{i+1}) - f'(x_i)}{96}h_i^2\right| \leq \frac{1}{48}(\Gamma - S)h_i^3$$

for $i = 0,1,2,\ldots,n-1$. Now summing over i from 0 to $n-1$ and using the triangle inequality, we get (3.3) and (3.4). □

Theorem 3.2 *Let $f : [a,b] \to \mathbb{R}$ be a thrice continuously differentiable mapping in (a,b) with $f''' \in L^2[a,b]$. Then for all $x \in [a, \frac{a+b}{2}]$ we have*

$$\int_a^b f(t)\,dt = Q_n^1(I_n,f) + R_n^1(I_n,f),$$

where $Q_n^1(I_n,f)$ is defined by formula (3.1), *and the remainder $R_n^1(I_n,f)$ satisfies the estimate*

$$\left|R_n^1(I_n,f)\right| \le \frac{\|f'''\|_2}{48\pi\sqrt{5}} \sum_{i=0}^{n-1} h_i^{7/2}. \tag{3.5}$$

Proof Applying inequality (2.3) to the intervals $[x_i, x_{i+1}]$, we get

$$\begin{aligned}&\left|\int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{1}{2}\left[f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right)\right]h_i - \frac{f'(x_{i+1}) - f'(x_i)}{96}h_i^2\right| \\ &\quad \le \frac{h_i^{7/2}}{48\pi\sqrt{5}}\|f'''\|_2\end{aligned}$$

for $i = 0, 1, 2, \ldots, n-1$. Now summing over i from 0 to $n-1$ and using the triangle inequality, we get (3.5). □

Theorem 3.3 *Let $f : [a,b] \to \mathbb{R}$ be such that f' is absolutely continuous on $[a,b]$ with $f'' \in L^2[a,b]$. Then for all $x \in [a, \frac{a+b}{2}]$ we have*

$$\int_a^b f(t)\,dt = Q_n^1(I_n,f) + R_n^1(I_n,f),$$

where $Q_n^1(I_n,f)$ is defined by formula (3.1), *and the remainder $R_n^1(I_n,f)$ satisfies the estimate*

$$\left|R_n^1(I_n,f)\right| \le \frac{\sqrt{\sigma(f'')}}{48\sqrt{5}} \sum_{i=0}^{n-1} h_i^{5/2}. \tag{3.6}$$

Proof Applying inequality (2.5) to the intervals $[x_i, x_{i+1}]$, we get

$$\begin{aligned}&\left|\int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{1}{2}\left[f\left(\frac{3x_i + x_{i+1}}{4}\right) + f\left(\frac{x_i + 3x_{i+1}}{4}\right)\right]h_i - \frac{f'(x_{i+1}) - f'(x_i)}{96}h_i^2\right| \\ &\quad \le \frac{h_i^{5/2}}{48\sqrt{5}}\sqrt{\sigma(f'')}\end{aligned}$$

for $i = 0, 1, 2, \ldots, n-1$. Now summing over i from 0 to $n-1$ and using the triangle inequality, we get (3.6). □

To illustrate the effectiveness of the perturbed composite quadrature rules (3.1) and (3.2), we compute the approximate values of several specific examples using these two rules and the composite trapezoidal formula

$$T_n(I_n,f) = \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right],$$

Table 1 Numerical results

$f(x)$	n	$[a,b]$	$\int_a^b f(x)\,dx$	T_n	Error of T_n	Q_n^1 and Q_n^2	Error of Q_n^1 and Q_n^2
$\cos x - x$	20	$[0, \frac{\pi}{2}]$	–0.233701	–0.234215	5.14E-4	–0.233636	6.5E-5
$e^{2x}\cos(e^x)$	20	[0, 1]	–1.176887	–1.181466	4.579E-3	–1.176316	5.71E-4
$\frac{1}{x^4+4x^2+3}$	10	[0, 1]	0.241549	0.241393	1.56E-4	0.241569	2E-5
$\tan x + x$	20	$[0, \frac{\pi}{4}]$	0.654999	0.655127	1.28E-4	0.654983	7E-6
$\ln(x^2+1)$	20	[–1, 1]	0.527887	0.529554	1.667E-3	0.527679	2.08E-4

respectively, and then we compare their errors. We get Table 1, from which the power of these two rules in numerical integration is demonstrated due to better error estimates.

4 Application to probability density functions

Now, let X be a random variable taking values in the finite interval $[a,b]$, with the probability density function $f:[a,b]\to[0,1]$ and with the cumulative distribution function

$$F(x) = \Pr(X \le x) = \int_a^x f(t)\,dt.$$

The following results hold.

Theorem 4.1 *With the assumptions of Theorem* 2.1, *we have*

$$\begin{aligned}
&\left|\frac{1}{2}\bigl[F(x)+F(a+b-x)\bigr] - \left(x-\frac{3a+b}{4}\right)\frac{f(x)-f(a+b-x)}{2}\right. \\
&\qquad \left. + \frac{f(b)-f(a)}{b-a}\left[\frac{1}{2}\left(x-\frac{3a+b}{4}\right)^2 + \frac{(b-a)^2}{96}\right] - \frac{b-E(X)}{b-a}\right| \\
&\qquad \le (S-\gamma)\left[\frac{(b-a)^2}{48} + \frac{b-a}{4}\left|x-\frac{3a+b}{4}\right|\right]
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
&\left|\frac{1}{2}\bigl[F(x)+F(a+b-x)\bigr] - \left(x-\frac{3a+b}{4}\right)\frac{f(x)-f(a+b-x)}{2}\right. \\
&\qquad \left. + \frac{f(b)-f(a)}{b-a}\left[\frac{1}{2}\left(x-\frac{3a+b}{4}\right)^2 + \frac{(b-a)^2}{96}\right] - \frac{b-E(X)}{b-a}\right| \\
&\qquad \le (\Gamma-S)\left[\frac{(b-a)^2}{48} + \frac{b-a}{4}\left|x-\frac{3a+b}{4}\right|\right]
\end{aligned} \tag{4.2}$$

for all $x \in [a, \frac{a+b}{2}]$, *where* $E(X)$ *is the expectation of* X.

Proof By (2.3) and (2.4) on choosing $f = F$ and taking into account

$$E(X) = \int_a^b t\,dF(t) = b - \int_a^b F(t)\,dt,$$

we obtain (4.1) and (4.2). □

Corollary 4.1 *Under the assumptions of Theorem* 4.1 *with* $x = \frac{3a+b}{4}$, *we have*

$$\begin{aligned}&\left|\frac{1}{2}\left[F\left(\frac{3a+b}{4}\right)+F\left(\frac{a+3b}{4}\right)\right]+\frac{b-a}{96}\left[f(b)-f(a)\right]-\frac{b-E(x)}{b-a}\right|\\&\quad\leq\frac{1}{48}(S-\gamma)(b-a)^2\end{aligned}\tag{4.3}$$

and

$$\begin{aligned}&\left|\frac{1}{2}\left[F\left(\frac{3a+b}{4}\right)+F\left(\frac{a+3b}{4}\right)\right]+\frac{b-a}{96}\left[f(b)-f(a)\right]-\frac{b-E(x)}{b-a}\right|\\&\quad\leq\frac{1}{48}(\Gamma-S)(b-a)^2.\end{aligned}\tag{4.4}$$

Theorem 4.2 *With the assumptions of Theorem* 2.2, *we have*

$$\begin{aligned}&\left|\frac{1}{2}\left[F(x)+F(a+b-x)\right]-\left(x-\frac{3a+b}{4}\right)\frac{f(x)-f(a+b-x)}{2}\right.\\&\qquad\left.+\frac{f(b)-f(a)}{b-a}\left[\frac{1}{2}\left(x-\frac{3a+b}{4}\right)^2+\frac{(b-a)^2}{96}\right]-\frac{b-E(X)}{b-a}\right|\\&\quad\leq\frac{1}{\pi}\left\|f'''\right\|_2\left\{\frac{1}{320}(a+b-2x)^5+\frac{1}{10}(x-a)^5\right.\\&\qquad\left.-(b-a)\left[\frac{1}{2}\left(x-\frac{3a+b}{4}\right)^2+\frac{(b-a)^2}{96}\right]^2\right\}^{1/2}\end{aligned}\tag{4.5}$$

for all $x\in[a,\frac{a+b}{2}]$, *where* $E(X)$ *is the expectation of* X.

Proof By (2.18) on choosing $f = F$ and taking into account

$$E(X)=\int_a^b t\,dF(t)=b-\int_a^b F(t)\,dt,$$

we obtain (4.5). □

Corollary 4.2 *Under the assumptions of Theorem* 4.2 *with* $x = \frac{3a+b}{4}$, *we have*

$$\begin{aligned}&\left|\frac{1}{2}\left[F\left(\frac{3a+b}{4}\right)+F\left(\frac{a+3b}{4}\right)\right]+\frac{b-a}{96}\left[f(b)-f(a)\right]-\frac{b-E(x)}{b-a}\right|\\&\quad\leq\frac{(b-a)^{5/2}}{48\pi\sqrt{5}}\left\|f'''\right\|_2.\end{aligned}\tag{4.6}$$

Theorem 4.3 *With the assumptions of Theorem* 2.3, *we have*

$$\begin{aligned}&\left|\frac{1}{2}\left[F(x)+F(a+b-x)\right]-\left(x-\frac{3a+b}{4}\right)\frac{f(x)-f(a+b-x)}{2}\right.\\&\qquad\left.+\frac{f(b)-f(a)}{b-a}\left[\frac{1}{2}\left(x-\frac{3a+b}{4}\right)^2+\frac{(b-a)^2}{96}\right]-\frac{b-E(X)}{b-a}\right|\\&\quad\leq\frac{\sqrt{\sigma(f'')}}{b-a}\left\{\frac{1}{320}(a+b-2x)^5+\frac{1}{10}(x-a)^5\right.\end{aligned}$$

$$- (b-a)\left[\frac{1}{2}\left(x - \frac{3a+b}{4}\right)^2 + \frac{(b-a)^2}{96}\right]^2\Bigg\}^{1/2} \tag{4.7}$$

for all $x \in [a, \frac{a+b}{2}]$, *where* $E(X)$ *is the expectation of* X.

Proof By (2.23) on choosing $f = F$ and taking into account

$$E(X) = \int_a^b t\,dF(t) = b - \int_a^b F(t)\,dt,$$

we obtain (4.7). □

Corollary 4.3 *Under the assumptions of Theorem* 4.3 *with* $x = \frac{3a+b}{4}$, *we have*

$$\left|\frac{1}{2}\left[F\left(\frac{3a+b}{4}\right) + F\left(\frac{a+3b}{4}\right)\right] + \frac{b-a}{96}\big[f(b) - f(a)\big] - \frac{b-E(x)}{b-a}\right| \\ \leq \frac{(b-a)^{3/2}}{48\sqrt{5}}\sqrt{\sigma\left(f''\right)}. \tag{4.8}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed to the writing of the present article and they read and approved the final manuscript.

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