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Normal families of meromorphic functions sharing one function

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Abstract

Suppose $p(z)$ is a holomorphic function, the multiplicity of its zeros is at most d, $P(z)$ is a nonconstant polynomial. Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, all of whose zeros and poles have multiplicity at least $\max\{\frac{k}{2}+d+1, k+d\}$. If for each pair of functions f and g in $\mathcal{F}$, $P(f)f^{(k)}$ and $P(g)g^{(k)}$ share a holomorphic function $p(z)$, then $\mathcal{F}$ is normal in D. It generalizes and extends the results of Jiang, Gao and Wu, Xu.

MSC: 30D35; 30D45

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1 Introduction and results

Let D be a domain in $\mathbb{C}$, let $\mathcal{F}$ be a family of meromorphic functions in D. $\mathcal{F}$ is said to be normal in D, in the sense of Montel, if for any sequence $\{f_n\} \in \mathcal{F}$ contains a subsequence $\{f_{nj}\}$ such that f_{nj} converges spherically locally uniformly in D to a meromorphic function or ∞ [1–3].

Let $a \in \mathbb{C} \cup \{\infty\}$, let f and g be two nonconstant meromorphic functions in D. If $f(z) - a$ and $g(z) - a$ have the same zeros (ignoring multiplicity), we say f and g share the value a in D.

In 1959, Hayman [1] proved that if f is a transcendental meromorphic function, then $f^n f'$ assumes every finite nonzero complex number infinitely often for any positive integer $n \geq 3$. He [4] conjectured that this remains valid for $n = 1$ and $n = 2$. Further, the case of $n = 2$ was confirmed by Mues [5] in 1979. The case $n = 1$ was considered and settled by Clunie [6].

In 1994, Yang and Yang [7] proposed a conjecture: If f is an entire function and $k \geq 2$, then $(ff^{(k)})^n - a(z)$ $(a(z) \not\equiv 0)$ has infinitely many zeros.

Zhang and Song [8] proved the following theorem.

Theorem A *Suppose that f is a transcendental meromorphic function, n, k are two positive integers, then when $n \geq 2$, $(ff^{(k)})^n - a(z)$ has infinitely many zeros, where $a(z) \not\equiv 0$ is a small function of f.*

In 2005, Wang [9] proved the following theorem.

Theorem B *Let f be a transcendental meromorphic function, let $c(z) \not\equiv 0$ be a small function of f, and let n, k be two positive integers. If $n \geq 3$, then $f^n f^{(k)} - c(z)$ has infinitely many zeros.*

In the case of $ff^{(k)}$, Yang and Yang [7] proposed a conjecture: If f is transcendental, then $ff^{(k)}$ assumes every finite nonzero complex number infinitely often. In 2006, Wang [10] proved that this conjecture holds when f has only zeros of multiplicity at least $k+1$ ($k \geq 2$).

In 2011, Meng and Hu [11] obtained the following theorem.

Theorem C *Take a positive integer k and a nonzero complex number a. Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D \in \mathbb{C}$ such that each $f \in \mathcal{F}$ has only zeros of multiplicity at least $k+1$. For each pair $(f,g) \in \mathcal{F}$, if $ff^{(k)}$ and $gg^{(k)}$ share a, then $\mathcal{F}$ is normal in D.*

In 2011, Jiang and Gao [12] obtained the following theorem.

Theorem D *Suppose that d (≥ 0) is an integer, $p(z)$ is an analytic function in D, and the multiplicity of its all zeros is at most d. Let $\mathcal{F}$ be a family of meromorphic functions in D, let n be a positive integer. If $n \geq 2d+2$ and for each pair of functions f and g in $\mathcal{F}$, $f^n f'$ and $g^n g'$ share $p(z)$ in D, then $\mathcal{F}$ is normal in D.*

In 2012, Wu and Xu [13] got the following theorem.

Theorem E *Let k be a positive integer, let $b \neq 0$ be a finite complex number, let P be a polynomial with either $\deg P \geq 3$ or $\deg P = 2$ and P having only one distinct zero, and let $\mathcal{F}$ be a family of meromorphic functions in D, all of whose zeros have multiplicity at least k. If for each pair of functions f and g in $\mathcal{F}$, $P(f)f^{(k)}$ and $P(g)g^{(k)}$ share b in D, then $\mathcal{F}$ is normal in D.*

It is natural to ask whether Theorem E can be improved by the idea of sharing a holomorphic function. In this paper, we study the problem and obtain the following theorems.

Theorem 1.1 *Suppose that $d \geq 0$ is an integer, $p(z) \not\equiv 0$ is a holomorphic function in D, and the multiplicity of its all zeros is at most d. Let $\mathcal{F}$ be a family of meromorphic functions in D, the multiplicity of all zeros and poles of $f \in \mathcal{F}$ is at least $\max\{\frac{k}{2}+d+1, k+d\}$. If for each pair of functions f and g in $\mathcal{F}$, $ff^{(k)}$ and $gg^{(k)}$ share $p(z)$ in D, then $\mathcal{F}$ is normal in D.*

Remark 1.1 Theorem 1.1 still holds when $p(z)$ is a nonzero finite constant.

Theorem 1.2 *Suppose that $d \geq 0$ is an integer, $p(z) \not\equiv 0$ is a holomorphic function in D, and the multiplicity of its all zeros is at most d. Let P be a nonconstant polynomial, $\mathcal{F}$ be a family of meromorphic functions in D, the multiplicity of all zeros and poles of $f \in \mathcal{F}$ is at least $\max\{\frac{k}{2}+d+1, k+d\}$. If for each pair of functions f and g in $\mathcal{F}$, $P(f)f^{(k)}$ and $P(g)g^{(k)}$ share $p(z)$ in D, then $\mathcal{F}$ is normal in D.*

2 Some lemmas

Lemma 2.1 (see [14]) *Let k be a positive integer, let $\mathcal{F}$ be a family of meromorphic functions in D such that each function $f \in \mathcal{F}$ has only zeros with multiplicities at least k, and suppose* *that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$, $f \in \mathcal{F}$. If $\mathcal{F}$ is not normal at $z_0 \in D$, then for each α, $0 \leq \alpha \leq k$, there exists a sequence of complex numbers $z_n \in D$, $z_n \to z_0$, a sequence of positive numbers $\rho_n \to 0$, and a sequence of functions $f_n \in \mathcal{F}$ such that*

$$g_n(\xi) = \frac{f_n(z_n + \rho_n \xi)}{\rho_n^{\alpha}} \to g(\xi)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least k, such that $g^{\sharp}(\xi) \leq g^{\sharp}(0) = kA + 1$. Moreover, $g(\xi)$ has order at most 2.

Lemma 2.2 (see [15]) *Let $f(z)$ be a meromorphic function and k be a positive integer. If $f^{(k)} \not\equiv 0$, then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f).$$

Lemma 2.3 (see [1]) *Let $f_1(z), f_2(z)$ be two meromorphic functions defined in $D = \{z : |z| < R\}$, then*

$$N(r, f_1 f_2) - N\left(r, \frac{1}{f_1 f_2}\right) = N(r, f_1) + N(r, f_2) - N\left(r, \frac{1}{f_1}\right) - N\left(r, \frac{1}{f_2}\right).$$

Lemma 2.4 (see [16]) *Let f be a transcendental meromorphic function, let $P_f(z)$, $Q_f(z)$ be two differential polynomials of f. If $f^n P_f = Q_f$ holds and the degree of Q_f is at most n, then $m(r, P_f) = S(r,f)$.*

Lemma 2.5 *Let d (≥ 0) be an integer, let k be a positive integer, and let $p(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0$ be a polynomial, where $a_d \neq 0, a_{d-1}, \ldots, a_0$ are constants. Suppose that f is a transcendental meromorphic function, all of whose zeros and poles have multiplicity at least 2, $p(z)$ is a small function of $f(z)$, then $ff^{(k)}(z) - p(z)$ has infinitely many zeros.*

Proof Let

$$\psi(z) = ff^{(k)} - p(z). \tag{2.1}$$

Suppose $ff^{(k)} - p(z)$ has only finitely many zeros, then $N(r, \frac{1}{\psi(z)}) = S(r,f)$. By (2.1), then

$$\left(\frac{\psi}{p}\right)' = \frac{f' f^{(k)}}{p} + \frac{ff^{(k+1)}}{p} + \left(\frac{1}{p}\right)' ff^{(k)}. \tag{2.2}$$

Let

$$\psi_1 = \frac{\psi}{p}.$$

Since the multiplicity of zeros of $f(z)$ is at least 2, we can get from (2.2) that

$$N\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{1}{(\psi_1)'}\right) + S(r,f). \tag{2.3}$$

By Lemma 2.2, we know that

$$N\left(r,\frac{1}{(\psi_1)'}\right)\le N\left(r,\frac{1}{\psi_1}\right)+\overline{N}(r,f)+S(r,f). \tag{2.4}$$

We can get from (2.2) that

$$\frac{ff^{(k)}}{p}\frac{(\psi_1)'}{\psi_1}-\frac{f'f^{(k)}}{p}-\frac{ff^{(k+1)}}{p}-\left(\frac{1}{p}\right)'ff^{(k)}=\frac{(\psi_1)'}{\psi_1},$$

i.e.,

$$f\left(\frac{f^{(k)}}{p}\frac{(\psi_1)'}{\psi_1}-\frac{f'f^{(k)}}{fp}-\frac{f^{(k+1)}}{p}-\left(\frac{1}{p}\right)'f^{(k)}\right)=\frac{(\psi_1)'}{\psi_1}. \tag{2.5}$$

Let

$$fH=\frac{(\psi_1)'}{\psi_1}, \tag{2.6}$$

where $H=\frac{f^{(k)}}{p}\frac{(\psi_1)'}{\psi_1}-\frac{f'f^{(k)}}{fp}-\frac{f^{(k+1)}}{p}-(\frac{1}{p})'f^{(k)}$. By Lemma 2.4, we get $m(r,H)=S(r,f)$.

From (2.5) and Lemma 2.3, we obtain that

$$\begin{aligned}m\left(r,\frac{1}{f}\right)&\le m(r,H)+m\left(r,\frac{\psi_1}{(\psi_1)'}\right)\\&\le N\left(r,\frac{(\psi_1)'}{\psi_1}\right)-N\left(r,\frac{\psi_1}{(\psi_1)'}\right)+m\left(r,\frac{(\psi_1)'}{\psi_1}\right)+S(r,f)\\&\le N\big(r,(\psi_1)'\big)+N\left(r,\frac{1}{\psi_1}\right)-N\left(r,\frac{1}{(\psi_1)'}\right)-N(r,\psi_1)+S(r,f)\\&\le\overline{N}(r,f)+N\left(r,\frac{1}{\psi}\right)-N\left(r,\frac{1}{(\psi_1)'}\right)+S(r,f).\end{aligned} \tag{2.7}$$

We can get from (2.6) that

$$ff^{(k)}\left(\frac{(\psi_1)'}{p\psi_1}-\frac{f'}{fp}-\frac{f^{(k+1)}}{pf^{(k)}}-\left(\frac{1}{p}\right)'\right)=\frac{(\psi_1)'}{\psi_1}. \tag{2.8}$$

Let

$$ff^{(k)}G=\frac{(\psi_1)'}{\psi_1}, \tag{2.9}$$

where $G=\frac{(\psi_1)'}{p\psi_1}-\frac{f'}{fp}-\frac{f^{(k+1)}}{pf^{(k)}}-(\frac{1}{p})'$. By Lemma 2.4, then $m(r,G)=S(r,f)$.

By (2.9), we have that

$$\begin{aligned}m\big(r,f^{(k)}\big)&\le m\left(r,\frac{(\psi_1)'}{\psi_1}\right)+m\left(r,\frac{1}{f}\right)+m\left(r,\frac{1}{G}\right)\\&\le m\left(r,\frac{1}{f}\right)+N(r,G)-N\left(r,\frac{1}{G}\right)+S(r,f).\end{aligned} \tag{2.10}$$

Since $\frac{(\psi_1)'}{\psi_1}$ has only simple poles, and by (2.9) we know that the poles of f are impossible G's. Hence the poles of G are only possible from the zeros and poles of $p(z)$ or the zeros of ψ_1, f and $f^{(k)}$.

Hence by (2.8) and (2.9), we obtain that

$$\begin{aligned} N(r,G) &\le \overline{N}\left(r,\frac{1}{\psi_1}\right) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}}\right) + N\left(r,\frac{1}{p}\right) \\ &\le \overline{N}\left(r,\frac{1}{\psi}\right) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}}\right) + S(r,f). \end{aligned} \tag{2.11}$$

Since $\frac{(\psi_1)'}{\psi_1}$ has only simple poles, so by (2.9) we know that

$$N\left(r,\frac{1}{G}\right) \ge N(r,f) + N\left(r,f^{(k)}\right) - \overline{N}(r,f). \tag{2.12}$$

Combining (2.7) and (2.10)-(2.12), we have

$$\begin{aligned} m\left(r,f^{(k)}\right) \le{}& \left\{\overline{N}\left(r,\frac{1}{\psi}\right) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}}\right)\right\} - \left\{N(r,f) + N\left(r,f^{(k)}\right) - \overline{N}(r,f)\right\} \\ &+ \left\{\overline{N}(r,f) + N\left(r,\frac{1}{\psi}\right) - N\left(r,\frac{1}{(\psi_1)'}\right)\right\} + S(r,f). \end{aligned}$$

Hence

$$\begin{aligned} T\left(r,f^{(k)}\right) \le{}& \overline{N}\left(r,\frac{1}{\psi}\right) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}}\right) - N(r,f) + \overline{N}(r,f) \\ &+ \overline{N}(r,f) + N\left(r,\frac{1}{\psi}\right) - N\left(r,\frac{1}{(\psi_1)'}\right) + S(r,f). \end{aligned}$$

Since the multiplicity of the zeros and poles of $f(z)$ is at least 2, by an elementary calculation and combing with Lemma 2.2, (2.3) and (2.4), the above inequality yields

$$\begin{aligned} T\left(r,f^{(k)}\right) &\le N\left(r,\frac{1}{f^{(k)}}\right) + 2N\left(r,\frac{1}{\psi}\right) + S(r,f) \\ &\le N\left(r,\frac{1}{f}\right) + k\overline{N}(r,f) + 2N\left(r,\frac{1}{\psi}\right) + S(r,f) \\ &\le (k+1)\overline{N}(r,f) + 3N\left(r,\frac{1}{\psi}\right) + S(r,f). \end{aligned} \tag{2.13}$$

Since the multiplicity of the poles of $f(z)$ is at least 2, we can get from (2.13) that

$$\begin{aligned} T\left(r,f^{(k)}\right) &\le \left(1-\frac{1}{k+2}\right)N\left(r,f^{(k)}\right) + 3N\left(r,\frac{1}{\psi}\right) + S(r,f) \\ &\le \left(1-\frac{1}{k+2}\right)N\left(r,f^{(k)}\right) + S(r,f). \end{aligned}$$

This implies $T(r,f^{(k)}) = S(r,f)$, then $f^{(k)}$ is a rational function, thus f is a rational function which contradicts with f is transcendental. Hence $ff^{(k)}(z) - p(z)$ has infinitely many zeros. □

Remark 2.1 When $p(z)$ is a nonzero finite constant or a small function of $f(z)$, similarly we can get the same conclusion.

Lemma 2.6 *Let d (≥ 0) be an integer, let k be a positive integer, and let $p(z) = a_d z^d + a_{d-1}z^{d-1} + \cdots + a_1 z + a_0$ be a polynomial, where $a_d \neq 0, a_{d-1}, \ldots, a_0$ are constants. If $f(z)$ is a nonconstant polynomial, all of whose zeros and poles have multiplicity at least $k+d$, then $ff^{(k)}(z) - p(z)$ has at least two distinct zeros, and $ff^{(k)}(z) - p(z) \not\equiv 0$.*

Proof We discuss the following two cases.

Case 1. If $ff^{(k)} - p(z) \neq 0$, then $ff^{(k)} - p(z) \equiv C$, where C is a nonzero constant. So $ff^{(k)} \equiv p(z) + C$. Since the multiplicity of all the zeros of f is at least $k+d$, thus $\deg(ff^{(k)}) \geq k + 2d$, which contradicts with $\deg(p(z)) = d$.

Case 2. If $ff^{(k)} - p(z)$ has only one zero z_0, we assume $ff^{(k)} - p(z) \equiv A(z-z_0)^l$, where A is a nonzero constant, l is a positive integer.

We discuss the following two cases.

(i) If $l \leq d+1$, then $ff^{(k)} \equiv p(z) + A(z-z_0)^l$. Since $\deg(ff^{(k)}) \geq k+2d$, the degree of the right of the equation is at most $d+1$, which is smaller than the degree of the left of the equation. We get a contradiction.

(ii) If $l > d+1$, then $ff^{(k)} \equiv p(z) + A(z-z_0)^l$. So $(ff^{(k)})^{(d)} \equiv a_d + Al \cdots (l-d+1)(z-z_0)^{l-d}$. Since $a_d \neq 0$, so $(ff^{(k)})^{(d)}$ has only simple zeros, which contradicts with the multiplicity of all the zeros of f is at least $k+d$.

By Case 1 and Case 2, $ff^{(k)} - p(z)$ has at least two distinct zeros.

If $ff^{(k)} - p(z) \equiv 0$, then similar to the proof of Case 1, we get a contradiction. Hence $ff^{(k)} - p(z) \not\equiv 0$. □

Lemma 2.7 *Let d (≥ 0) be an integer, let k be a positive integer, and let $p(z) = a_d z^d + a_{d-1}z^{d-1} + \cdots + a_1 z + a_0$ be a polynomial, where $a_d \neq 0, a_{d-1}, \ldots, a_0$ are constants. If $f(z)$ is a nonconstant rational function and not a polynomial, and the multiplicity of whose zeros and poles is at least $\frac{k}{2} + d + 1$, then $ff^{(k)}(z) - p(z)$ has at least two distinct zeros, and $ff^{(k)}(z) - p(z) \not\equiv 0$.*

Proof Since $f(z)$ is a nonconstant rational function and not a polynomial, then obviously $ff^{(k)}(z) - p(z) \not\equiv 0$. Let

$$f(z) = B\frac{(z-\alpha_1)^{m_1}(z-\alpha_2)^{m_2}\cdots(z-\alpha_s)^{m_s}}{(z-\beta_1)^{n_1}(z-\beta_2)^{n_2}\cdots(z-\beta_t)^{n_t}}, \tag{2.14}$$

where B is a nonzero constant. Since the multiplicity of the zeros and poles of f is at least $\frac{k}{2} + d + 1$, we have $m_i \geq \frac{k}{2} + d + 1$ $(i = 1, 2, \ldots, s)$, $n_j \geq \frac{k}{2} + d + 1$ $(j = 1, 2, \ldots, t)$. For simplicity, we denote

$$m_1 + m_2 + \cdots + m_s = m \geq \left(\frac{k}{2} + d + 1\right)s, \qquad n_1 + n_2 + \cdots + n_t = n \geq \left(\frac{k}{2} + d + 1\right)t.$$

By (2.19), we get

$$f^{(k)}(z) = B\frac{(z-\alpha_1)^{m_1-k}(z-\alpha_2)^{m_2-k}\cdots(z-\alpha_s)^{m_s-k}g(z)}{(z-\beta_1)^{n_1+k}(z-\beta_2)^{n_2+k}\cdots(z-\beta_t)^{n_t+k}}, \tag{2.15}$$

where $g(z) = (m-n)(m-n-1)\cdots(m-n-k+1)z^{k(s+t-1)} + \cdots + c_1 z + c_0$ is a polynomial, c_i $(i = 0,1)$ are constants and $\deg(g(z)) \le k(s+t-1)$. Thus (2.14) together with (2.15) implies

$$ff^{(k)}(z) = B\frac{(z-\alpha_1)^{2m_1-k}(z-\alpha_2)^{2m_2-k}\cdots(z-\alpha_s)^{2m_s-k}g(z)}{(z-\beta_1)^{2n_1+k}(z-\beta_2)^{2n_2+k}\cdots(z-\beta_t)^{2n_t+k}}. \tag{2.16}$$

By (2.16), we obtain

$$\bigl(ff^{(k)}(z)\bigr)^{(d+1)} = B\frac{(z-\alpha_1)^{2m_1-k-d-1}(z-\alpha_2)^{2m_2-k-d-1}\cdots(z-\alpha_s)^{2m_s-k-d-1}g_1(z)}{(z-\beta_1)^{2n_1+k+d+1}(z-\beta_2)^{2n_2+k+d+1}\cdots(z-\beta_t)^{2n_t+k+d+1}}, \tag{2.17}$$

where $\deg(g_1(z)) \le (k+d+1)(s+t-1)$.

Next, we discuss the following two cases.

Case 1. If $ff^{(k)} - p(z)$ has only one zero z_0, then let

$$ff^{(k)}(z) - p(z) = C\frac{(z-z_0)^l}{(z-\beta_1)^{2n_1+k}(z-\beta_2)^{2n_2+k}\cdots(z-\beta_t)^{2n_t+k}}. \tag{2.18}$$

Subcase 1.1. When $d \ge l$.

Combining (2.16) and (2.18), we get $d+2n+kt = \deg(g(z)) + 2m - ks \le k(s+t-1) + 2m - ks$. That is, $2(m-n) \ge k+d$, and then $m > n$.

Differentiating both sides of (2.18), we have

$$\begin{aligned}\bigl(ff^{(k)}(z)\bigr)^{(d+1)} &- p^{(d+1)}(z)\\ &= C\frac{g_2(z)}{(z-\beta_1)^{2n_1+k+d+1}(z-\beta_2)^{2n_2+k+d+1}\cdots(z-\beta_t)^{2n_t+k+d+1}},\end{aligned} \tag{2.19}$$

where $\deg(g_2(z)) \le t(d+1) + l - d - 1$.

By (2.17) and (2.19), we know $2m - (k+d+1)s \le \deg(g_2(z)) \le t(d+1) + l - d - 1$. Thus $2m - (k+d+1)s - t(d+1) \le l - d - 1$. Since $2m - (k+d+1)s - t(d+1) \ge 2m - (k+d+1)\frac{m}{k/2+d+1} - (d+1)\frac{n}{k/2+d+1} > 0$, then $0 < l - d - 1$, which contradicts with $d \ge l$.

Subcase 1.2. When $d < l$.

Differentiating both sides of (2.18), we have

$$\begin{aligned}\bigl(ff^{(k)}(z)\bigr)^{(d+1)} &- p^{(d+1)}(z)\\ &= D\frac{(z-z_0)^{l-d-1}g_3(z)}{(z-\beta_1)^{2n_1+k+d+1}(z-\beta_2)^{2n_2+k+d+1}\cdots(z-\beta_t)^{2n_t+k+d+1}},\end{aligned} \tag{2.20}$$

where $g_3(z) = (l-2n-kt)\cdots(l-2n-kt-d)z^{t(d+1)} + \cdots + d_1 z + d_0$, where d_i $(i = 0,1)$ are constants and $\deg(g_3(z)) \le t(d+1)$.

Differentiating both sides of (2.18) step by step for d times, we have z_0 is a zero of $(ff^{(k)}(z))^{(d)} - p^{(d)}(z)$, as $p^{(d)}(z) = a_d \ne 0$ and the multiplicity of all the zeros of $f(z)$ is at least $\frac{k}{2} + d + 1$, thus $\alpha_i \ne z_0$ $(i = 1,2,\ldots,s)$. When $p(z)$ is a constant, from (2.18) we can also get $\alpha_i \ne z_0$ $(i = 1,2,\ldots,s)$.

Here, we discuss three subcases as follows.

Subcase 1.2.1. When $l < 2n + kt + d$.

Combining (2.16) and (2.18), we get $d+2n+kt = \deg(g(z)) + 2m - ks \le k(s+t-1) + 2m - ks$. That is, $2(m-n) \ge k+d$, and then $m > n$.

Since $\alpha_i \neq z_0$ ($i = 1, 2, \ldots, s$), by (2.17) and (2.20), we have $t(d+1) \geq \deg(g_3(z)) \geq 2m - (k + d + 1)s$. Thus $2m \leq (k + d + 1)s + t(d + 1) \leq (k + d + 1)\frac{m}{k/2+d+1} + (d + 1)\frac{n}{k/2+d+1} < 2m$, which is impossible.

Subcase 1.2.2. When $l = 2n + kt + d$.

If $m > n$, by a similar discussion to Subcase 1.2.1, we can get a contradiction. Thus $m \leq n$. Since $\alpha_i \neq z_0$ ($i = 1, 2, \ldots, s$), by (2.17) and (2.20), we have $l - d - 1 \leq \deg(g_1(z)) \leq (k + d + 1)(s + t - 1)$, since $l = 2n + kt + d$, thus $2n + kt + d - d - 1 \leq (k + d + 1)(s + t - 1)$. Then $2n \leq (k + d + 1)s + (d + 1)t - (k + d) < (k + d + 1)\frac{m}{k/2+d+1} + (d + 1)\frac{n}{k/2+d+1} \leq 2n$, which is impossible.

Subcase 1.2.3. When $l > 2n + kt + d$.

By (2.16) and (2.18), we get $l = \deg(g(z)) + 2m - ks \leq k(s + t - 1) + 2m - ks = 2m + kt - k$. If $m > n$, by a similar discussion to Subcase 1.2.1, we get a contradiction. Thus $m \leq n$.

Case 2. If $ff^{(k)} - p(z)$ has no zero. Then $l = 0$ in (2.18), by a similar discussion to Subcase 1.1, we get a contradiction.

By Case 1 and Case 2, we get $ff^{(k)} - p(z)$ has at least two distinct zeros. □

3 Proof of Theorem 1.1

For any point z_0 in D, either $p(z_0) = 0$ or $p(z_0) \neq 0$.

Case 1. When $p(z_0) = 0$. We may assume $z_0 = 0$. Then $p(z) = a_d z^d + a_{d+1} z^{d+1} + \cdots = z^d h(z)$, where $a_d (\neq 0), a_{d+1}, \ldots$ are constants, $d \geq 1$, $h(z_0) \neq 0$, without loss of generality, let $h(z_0) = a_d$, where $h(z)$ is a holomorphic function.

Let $\mathcal{F}_1 = \{F_j | F_j = \frac{f_j}{z^{d/2}}, f_j \in \mathcal{F}\}$. If $\mathcal{F}_1$ is not normal at 0, then by Lemma 2.1, there exists a sequence of complex numbers $z_j \to 0$, a sequence of positive numbers $\rho_j \to 0$ and a sequence of functions $F_j \in \mathcal{F}_1$ such that $G_j(\xi) = \rho_j^{-\frac{k}{2}} F_j(z_j + \rho_j \xi) \to G(\xi)$ spherically locally uniformly in $\mathbb{C}$, where $G(\xi)$ is a nonconstant meromorphic function in $\mathbb{C}$, and the multiplicity of the zeros and poles of $G(\xi)$ is at least $\max\{\frac{k}{2} + d + 1, k + d\}$. Here, we discuss two cases as follows.

Case 1.1. There exists a subsequence of $\frac{z_j}{\rho_j}$, we may denote it as $\frac{z_j}{\rho_j}$ such that $\frac{z_j}{\rho_j} \to c$, c is a finite complex number. Then

$$\phi_j(\xi) = \frac{f_j(\rho_j \xi)}{\rho_j^{\frac{d+k}{2}}} = \frac{(\rho_j \xi)^{\frac{d}{2}} F_j(z_j + \rho_j(\xi - \frac{z_j}{\rho_j}))}{\rho_j^{\frac{d}{2}} \rho_j^{\frac{k}{2}}} \to \xi^{\frac{d}{2}} G(\xi - c) = H(\xi)$$

spherically locally uniformly in $\mathbb{C}$, so

$$\phi_j(\xi)\phi_j^{(k)}(\xi) - \frac{p(\rho_j \xi)}{\rho_j^d} = \frac{f_j(\rho_j \xi) f_j^{(k)}(\rho_j \xi) - p(\rho_j \xi)}{\rho_j^d} \to H(\xi)H^{(k)}(\xi) - a_d \xi^d$$

spherically locally uniformly in $\mathbb{C}$.

Since $\forall f \in \mathcal{F}$, the multiplicity of whose zeros and poles is at least $\max\{\frac{k}{2} + d + 1, k + d\}$, then the multiplicity of all zeros and poles of H is at least $\max\{\frac{k}{2} + d + 1, k + d\}$, by Lemmas 2.5-2.7, we get $H(\xi)H^{(k)}(\xi) - a_d \xi^d \not\equiv 0$, and $H(\xi)H^{(k)}(\xi) - a_d \xi^d$ has at least two distinct zeros.

Suppose ξ_0, ξ_0^* are two distinct zeros of $H(\xi)H^{(k)}(\xi) - a_d \xi^d$. We may choose a proper $\sigma > 0$ such that $D(\xi_0, \sigma) \cap D(\xi_0^*, \sigma) = \emptyset$, where $D(\xi_0, \sigma) = \{\xi || \xi - \xi_0| < \sigma\}$, $D(\xi_0^*, \sigma) = \{\xi || \xi - \xi_0^*| < \sigma\}$.

By Hurwitz's theorem, there exists a subsequence of $f_j(\rho_j\xi)f_j^{(k)}(\rho_j\xi) - p(\rho_j\xi)$, we may still denote it as $f_j(\rho_j\xi)f_j^{(k)}(\rho_j\xi) - p(\rho_j\xi)$, then there exist points $\xi_j \in D(\xi_0,\sigma)$ and points $\xi_j^* \in D(\xi_0^*,\sigma)$ such that for sufficiently large j, $f_j(\rho_j\xi_j)f_j^{(k)}(\rho_j\xi_j) - p(\rho_j\xi_j) = 0$, $f_j(\rho_j\xi_j^*)f_j^{(k)}(\rho_j\xi_j^*) - p(\rho_j\xi_j^*) = 0$.

Since $f_jf_j^{(k)}$ and $g_jg_j^{(k)}$ share $p(z)$ in D, it follows that for any positive integer m, $f_m(\rho_j\xi_j)f_m^{(k)}(\rho_j\xi_j) - p(\rho_j\xi_j) = 0$, $f_m(\rho_j\xi_j^*)f_m^{(k)}(\rho_j\xi_j^*) - p(\rho_j\xi_j^*) = 0$.

Fix m, let $j \to \infty$ and note $\rho_j\xi_j \to 0$, $\rho_j\xi_j^* \to 0$, we obtain $f_m(0)f_m^{(k)}(0) - p(0) = 0$.

Since the zeros of $f_m(0)f_m^{(k)}(0) - p(0)$ have no accumulation points, in fact when j is large enough, we have $\rho_j\xi_j = \rho_j\xi_j^* = 0$. Thus, when j is large enough, $\xi_0 = \xi_0^* = 0$, which contradicts with $D(\xi_0,\sigma) \cap D(\xi_0^*,\sigma) = \emptyset$. Thus, $\mathcal{F}_1$ is normal at 0.

Case 1.2. There exists a subsequence of $\frac{z_j}{\rho_j}$, we may denote it as $\frac{z_j}{\rho_j}$ such that $\frac{z_j}{\rho_j} \to \infty$. Then

$$\begin{aligned} f_j(z_j+\rho_j\xi)f_j^{(k)}(z_j+\rho_j\xi) &= (z_j+\rho_j\xi)^{\frac{d}{2}}F_j(z_j+\rho_j\xi)\Bigg[(z_j+\rho_j\xi)^{\frac{d}{2}}\big(F_j(z_j+\rho_j\xi)\big)^{(k)} \\ &\quad + \sum_{i=1}^{k} c_i(z_j+\rho_j\xi)^{\frac{d}{2}-i}\big(F_j(z_j+\rho_j\xi)\big)^{(k-i)}\Bigg] \\ &= (z_j+\rho_j\xi)^d G_j(\xi)G_j^{(k)}(\xi) + \sum_{i=1}^{k} c_i(z_j+\rho_j\xi)^{d-i}\rho_j^i G_j(\xi)G_j^{(k-i)}(\xi), \end{aligned}$$

where $c_i = \frac{d}{2}(\frac{d}{2}-1)\cdots(\frac{d}{2}-i+1)C_{d/2}^i$ when $\frac{d}{2} \geq i$, and $c_i = 0$ when $\frac{d}{2} < i$.

Thus, we have

$$\begin{aligned} &\frac{a_d f_j(z_j+\rho_j\xi)f_j^{(k)}(z_j+\rho_j\xi)}{p(z_j+\rho_j\xi)} - a_d \\ &= \left(G_j(\xi)G_j^{(k)}(\xi) + \sum_{i=1}^{k} c_i \frac{G_j(\xi)G_j^{(k-i)}(\xi)}{(\frac{z_j}{\rho_j}+\xi)^i}\right)\frac{a_d}{h(z_j+\rho_j\xi)} - a_d \\ &\to G(\xi)G^{(k)}(\xi) - a_d, \end{aligned}$$

spherically locally uniformly in $\mathbb{C} - \{\xi | G(\xi) = \infty\}$.

Since the multiplicity of all zeros and poles of G is at least $\max\{\frac{k}{2}+d+1, k+d\}$ and by Lemmas 2.5-2.7, we have $G(\xi)G^{(k)}(\xi) - a_d \not\equiv 0$, and $G(\xi)G^{(k)}(\xi) - a_d$ has at least two distinct zeros.

Suppose ξ_1, ξ_1^* are two distinct zeros of $G(\xi)G^{(k)}(\xi) - a_d$. We may choose a proper $\delta > 0$ such that $D(\xi_1,\delta) \cap D(\xi_1^*,\delta) = \emptyset$, where $D(\xi_1,\delta) = \{\xi || \xi - \xi_1| < \sigma\}$, $D(\xi_1^*,\delta) = \{\xi || \xi - \xi_1^*| < \delta\}$.

By Hurwitz's theorem, there exists a subsequence of $a_d f_j(z_j+\rho_j\xi)f_j^{(k)}(z_j+\rho_j\xi) - a_d p(z_j+\rho_j\xi)$, we may still denote it as $a_d f_j(z_j+\rho_j\xi)f_j^{(k)}(z_j+\rho_j\xi) - a_d p(z_j+\rho_j\xi)$, then there exist points $\xi_j \in D(\xi_1,\delta)$ and points $\xi_j^* \in D(\xi_1^*,\delta)$ such that for sufficiently large j, $a_d f_j(z_j+\rho_j\xi)f_j^{(k)}(z_j+\rho_j\xi) - a_d p(z_j+\rho_j\xi) = 0$, $a_d f_j(z_j+\rho_j\xi)f_j^{(k)}(z_j+\rho_j\xi) - a_d p(z_j+\rho_j\xi) = 0$.

Similar to the proof of Case 1.1, we get a contradiction. Then $\mathcal{F}_1$ is normal at 0.

By Case 1.1 and Case 1.2, we know $\mathcal{F}_1$ is normal at 0. Hence there exists $\triangle_\rho = \{z : |z| < \rho\}$ and a subsequence of F_{jk} of F_j such that F_{jk} converges spherically locally uniformly to a meromorphic function $F(z)$ or ∞ $(k \to \infty)$ in $\triangle_\rho$.

Here, we discuss the following two cases.

Case i. When k is large enough, $f_{jk} \neq 0$. Then $F(0) = \infty$. Thus, for $\forall$ constant $R > 0$, $\exists \sigma \in (0, \rho)$, we have $|F(z)| > R$ when $z \in \triangle_\rho$. Thus, for sufficiently large k, $|F_{jk}(z)| > \frac{R}{2}$, $\frac{1}{f_{jk}}$ is a holomorphic function in $\triangle_\rho$. Hence when $|z| = \frac{\sigma}{2}$,

$$\left|\frac{1}{f_{jk}}\right| = \left|\frac{1}{F_{jk}z^{d/2}}\right| \leq \frac{2^{d/2+1}}{R\sigma^{d/2}}.$$

By the maximum principle and Montel's theorem, $\mathcal{F}$ is normal at $z = 0$.

Case ii. There exists a subsequence of f_{jk}, we may still denote it as f_{jk}, such that $f_{jk}(0) = 0$. Since $\forall f \in \mathcal{F}$, the multiplicity of whose zeros is at least $\max\{\frac{k}{2} + d + 1, k + d\}$, then $F(0) = 0$. Thus, there exists $0 < r < \rho$ such that $F(z)$ is holomorphic in $\triangle_r = \{z : |z| < r\}$ and has a unique zero $z = 0$ in $\triangle_r$. Then F_{jk} converges spherically locally uniformly to a holomorphic function $F(z)$ in $\triangle_r$, f_{jk} converges spherically locally uniformly to a holomorphic function $z^{\frac{d}{2}}F(z)$ in $\triangle_r$. Hence $\mathcal{F}$ is normal at $z = 0$.

By Case i and Case ii, we obtain $\mathcal{F}$ is normal at $z = 0$.

Case 2. When $p(z_0) \neq 0$.

Suppose that $\mathcal{F}$ is not normal at z_0. Then by Lemma 2.1, there exists a sequence of complex numbers $z_t \to z_0$, a sequence of positive numbers $\rho_t \to 0$ and a sequence of functions $f_t \in \mathcal{F}$ such that $g_t(\xi) = \rho_t^{-\frac{k}{2}} f_t(z_t + \rho_t\xi) \to g(\xi)$ spherically locally uniformly in $\mathbb{C}$, where $g(\xi)$ is a nonconstant meromorphic function in $\mathbb{C}$, and the multiplicity of the zeros and poles of $g(\xi)$ is at least $\max\{\frac{k}{2} + d + 1, k + d\}$.

Hence by Lemmas 2.5-2.7, we have $g(\xi)g^{(k)}(\xi) - p(z_0) \not\equiv 0$, and $g(\xi)g^{(k)}(\xi) - p(z_0)$ has at least two distinct zeros. Similar to the proof of Case 1.1, we get a contradiction. Thus, $\mathcal{F}$ is normal at z_0.

Hence, $\mathcal{F}$ is normal in D as z_0 is arbitrary. The proof is complete.

4 Proof of Theorem 1.2

Because $P(z)$ has at least one zero, we may assume, with no loss of generality, that $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_q z^q$, where $q \geq 1$ is a positive integer and $a_q \neq 0$. Suppose that $\mathcal{F}$ is not normal in D. Then similar to the proof of Theorem 1.1, we can get a contradiction. Hence $\mathcal{F}$ is normal in D. The proof is complete.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
LQ and FH performed and drafted manuscript. All authors read and approved the final manuscript.

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