ERROR BOUNDS FOR GENERAL MIXED QUASIVARIATIONAL INEQUALITIES

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Received 8 August 2004 and in revised form 19 December 2004

It is well known that mixed quasivariational inequalities are equivalent to the implicit fixed-point problems. We use this alternative equivalent formulation to suggest and consider some merit functions for general mixed quasivariational inequalities. We use these merit functions to obtain error bounds for the solution under some mild conditions. Some special cases are also discussed.

1. Introduction

Variational inequalities introduced by Stampacchia [25] in the early sixties have been generalized and extended in various directions using innovative techniques. A useful and significant generalization of variational inequalities is called the mixed quasivariational inequality involving the nonlinear bifunction which enables us to study the free, moving, unilateral, and equilibrium problems arising in elasticity, fluid flow through porous media, finance, economics, transportation, circuit, and structural analysis in a unified framework; see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]. As a result of interaction between different branches of mathematical and engineering sciences, we now have a variety of techniques including the projection methods and their variant forms, auxiliary principle, resolvent equations to suggest and analyze various iterative algorithms for solving variational inequalities, and related optimization problems. It is well known that the projection method and its variant forms cannot be extended for mixed quasivariational inequalities due to the presence of the bifunction. However, if the bifunction is a proper, convex, and lower semicontinuous function with respect to the first argument, then it has been shown (see [13]) that the mixed quasivariational inequalities are equivalent to the fixed-point problem. This alternative equivalent formulation has been used to suggest and analyze some iterative methods for solving mixed quasivariational inequalities. Using this alternative equivalence, we define the natural residue vector, which is also known as the merit function. In recent years, much attention has been given to construct and investigate some regularized and D-merit functions associated with classical variational inequalities. These merit functions play an important part in developing several iterative methods for solving variational inequalities

International Journal of Mathematics and Mathematical Sciences 2005:7 (2005) 1085–1100
DOI: 10.1155/IJMMS.2005.1085

and related optimization problems; see [4, 19, 20, 21, 22, 23, 24, 26]. On the other hand, there are no such merit functions for mixed quasivariational inequalities. In this paper, we consider and investigate some merit functions for mixed quasivariational inequalities and use these merit functions to obtain error bounds for the solution of mixed quasivariational inequalities. As special cases, we obtain some new and previously known results for variational inequalities and related problems. Thus the results obtained in this paper can be viewed as an extension and refinement of previously known results.

2. Formulations and basic facts

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let K be a closed convex set in H and $T, g : H \to H$ nonlinear operators. Let $\varphi(\cdot,\cdot) : H \times H \to R \cup \{+\infty\}$ be a continuous bifunction with respect to both arguments . We consider the problem of finding $u \in H : g(u) \in H$ such that

$$\langle Tu, g(v) - g(u) \rangle + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \quad \forall v \in H : g(v) \in H, \tag{2.1}$$

which is called the *general mixed quasivariational inequality* and has been studied extensively in recent years; see [14, 17].

If the bifunction $\varphi(\cdot,\cdot)$ is proper, convex, and lower semicontinuous with respect to the first argument, then problem (2.1) is equivalent to finding $u \in H : g(u) \in H$ such that

$$0 \in Tu + \partial\varphi(g(u), g(u)), \tag{2.2}$$

which is known as finding a zero-sum of two (more) maximal monotone operators and has been studied extensively in recent years.

For $g \equiv I$, the identity operator, problem (2.1) reduces to the following problem. Find $u \in H$ such that

$$\langle Tu, v - u \rangle + \varphi(v, u) - \varphi(u, u) \geq 0, \quad \forall v \in H. \tag{2.3}$$

Problem (2.3) is called the mixed quasivariational inequality; see [1, 2, 3, 8, 11, 13, 14, 15, 17].

If $\varphi(u, v) = \varphi(v)$, for all $v \in H$, then problem (2.1) is equivalent to finding $u \in H : g(u) \in H$ such that

$$\langle Tu, g(v) - g(u) \rangle + \varphi(g(v)) - \varphi(g(u)) \geq 0, \quad \forall v \in H : g(v) \in H, \tag{2.4}$$

which is called the general mixed variational inequality.

If the bifunction $\varphi(\cdot)$ is the indicator function of a closed and convex set K in H, that is,

$$\varphi(u) = \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{otherwise}, \end{cases} \tag{2.5}$$

then problem (2.4) is equivalent to finding $u \in H$, $g(u) \in K$ such that

$$\langle Tu, g(v) - g(u) \rangle \geq 0, \quad \forall g(v) \in K, \tag{2.6}$$

which is known as the general variational inequality introduced and studied by Noor [10] in 1988. It turned out that a wide class of nonsymmetric and odd-order free, moving and equilibrium problems arising in finance, economics, transportation, elasticity, telecommunication network, optimization, and operations research can be studied in the unified and general framework of problems (2.1)–(2.6). For $g \equiv I$, the identity operator, we obtained the corresponding classical variational inequality problems; see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26].

We also need the following well-known results and concepts.

Definition 2.1. The operator $T : H \to H$ is said to be

(a) strongly g-monotone if and only if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, g(u) - g(v)\rangle \geq \alpha \left\| g(u) - g(v)\right\|^2, \quad \forall u, v \in H; \tag{2.7}$$

(b) g-monotone if and only if

$$\langle Tu - Tv, g(u) - g(v)\rangle \geq 0, \quad \forall u, v \in H; \tag{2.8}$$

(c) g-Lipschitz continuous if there exists a constant $\beta > 0$ such that

$$\| Tu - Tv\| \leq \beta \left\| g(u) - g(v)\right\|, \quad \forall u, v \in H; \tag{2.9}$$

(d) hemicontinuous if for all $u, v \in H$, the mapping $t \in [0, t]$ implies that

$$\langle T(u + t(v - u)), v\rangle \tag{2.10}$$

is continuous.

From (a) and (c), we have $\alpha \leq \beta$. For $g = I$, the identity operator, Definition 2.1 reduces to the well-known definition of strongly monotone and Lipschitz continuity of T.

Remark 2.2. We would like to point out that if the operator T is strongly monotone with a constant $\alpha > 0$, then

$$\alpha \|u - v\|^2 \leq \langle Tu - Tv, u - v\rangle \leq \|Tu - Tv\| \, \|u - v\| \tag{2.11}$$

implies that

$$\|Tu - Tv\| \geq \alpha \|u - v\|, \quad \forall u, v \in H. \tag{2.12}$$

In this case, we say that the operator T is strong nonexpansion with a constant $\alpha > 0$. Note that the strong monotonicity implies expansioncivity, but not conversely. It is clear that if the operator T is strongly g-monotone and g is strongly nonexpansion, then

$$\langle Tu - Tv, g(u) - g(v)\rangle \geq \alpha \left\| g(u) - g(v)\right\|^2 \geq \alpha \|u - v\|^2, \quad \forall u, v \in H. \tag{2.13}$$

Definition 2.3. The bifunction $\varphi(\cdot, \cdot)$ is said to be *skew-symmetric* if and only if

$$\varphi(u, u) - \varphi(u, v) - \varphi(v, u) + \varphi(v, v) \geq 0, \quad \forall u, v \in H. \tag{2.14}$$

Clearly, if the skew-symmetric bifunction $\varphi(\cdot,\cdot)$ is linear in both arguments, then $\varphi(u,u) \geq 0$, for all $u \in H$. In fact,

$$\varphi(u,u) - \varphi(u,v) - \varphi(v,u) + \varphi(v,v) = \varphi(u-v,u-v) \geq 0, \quad \forall u,v \in H. \tag{2.15}$$

Definition 2.4 (see [1]). Let A be a maximal monotone operator, then the resolvent operator associated with A is defined as

$$J_A(u) = (I+\rho A)^{-1}(u), \quad \forall u \in H, \tag{2.16}$$

where $\rho > 0$ is a constant and I is the identity operator.

Remark 2.5. It is well known that the subdifferential $\partial\varphi(\cdot,\cdot)$ of a convex, proper, and lower semicontinuous function $\varphi(\cdot,\cdot): H \times H \to R \cup \{+\infty\}$ is a maximal monotone with respect to the first argument, its resolvent is defined by

$$J_{\varphi(u)} = \left(I+\rho\partial\varphi(\cdot,u)\right)^{-1} \equiv \left(I+\rho\partial\varphi(u)\right)^{-1}, \tag{2.17}$$

where $\partial\varphi(u) \equiv \partial\varphi(\cdot,u)$, unless otherwise specified.

The resolvent operator $J_{\varphi(g(u))}$ has the following characterization.

Lemma 2.6. *For a given $u \in H : g(u) \in H$, $z \in H : g(z) \in H$ satisfies the inequality*

$$\langle g(u)-g(z), g(v)-g(u)\rangle + \rho\varphi(g(v),g(u)) - \rho\varphi(g(u),g(u)) \geq 0, \quad \forall v \in H : g(v) \in H, \tag{2.18}$$

if and only if

$$g(u) = J_{\varphi(g(u))}g(z), \tag{2.19}$$

where $J_{\varphi(g(u))} = (I+\rho\partial\varphi(\cdot,g(u)))^{-1}$ is the resolvent operator and $\rho > 0$ is a constant.

Proof. Clearly,

$$\begin{aligned}(2.18) &\Longleftrightarrow g(z)-g(u) \in \rho\partial\varphi(g(u),g(u)) \\ &\Longleftrightarrow g(z) \in g(u) + \rho\partial\varphi(g(u),g(u)) \equiv \left(I+\rho\partial\varphi(\cdot,g(u))\right)(g(u)) \\ &\Longleftrightarrow g(u) = \left(I+\rho\partial\varphi(\cdot,g(u))\right)^{-1}g(z) \equiv J_{\varphi(g(u))}g(z),\end{aligned} \tag{2.20}$$

the required result. □

Lemma 2.7. *Let the operator T be g-monotone and hemicontinuous and let the operator g be convex. If the bifunction $\varphi(\cdot,\cdot)$ is convex in the first argument, then problem (2.1) is equivalent to finding $u \in H$ such that*

$$\langle Tv, g(v)-g(u)\rangle + \varphi(g(v),g(u)) - \varphi(g(u),g(u)) \geq 0, \quad \forall v \in H. \tag{2.21}$$

Proof. Let $u \in H$ be a solution of (2.1). Then

$$\langle Tu, g(v)-g(u)\rangle + \varphi(g(v),g(u)) - \varphi(g(u),g(u)) \geq 0, \quad \forall v \in H, \tag{2.22}$$

which implies, using the g-monotonicity of T,

$$\langle Tv, g(v) - g(u)\rangle + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \quad \forall v \in H. \tag{2.23}$$

Conversely, let $u \in H$ be such that (2.21) holds. For $t \in [0,1]$, $u, v \in H$, $v_t = u + t(v-u) \in H$. Taking $v = v_t$ in (2.21), we have

$$\begin{aligned} 0 &\leq t\langle Tv_t, g(v) - g(u)\rangle + \varphi(g(v_t), u) - \varphi(u,u) \\ &\leq t\langle Tv_t, g(v) - g(u)\rangle + t\{\varphi(g(v), g(u)) - \varphi(g(u), g(u))\}, \end{aligned} \tag{2.24}$$

since g is convex and $\varphi(\cdot,\cdot)$ is also convex with respect to the first argument.

Dividing the above inequality by t and letting $t \to 0$, we have

$$\langle Tu, g(v) - g(u)\rangle + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \quad \forall v \in H, \tag{2.25}$$

the required (2.1). □

Remark 2.8. Inequality of type (2.21) is called the *dual general mixed quasivariational inequality.* From Lemma 2.7, it is clear that the solution sets of both problems (2.1) and (2.21) are equivalent. Lemma 2.7 plays an important part in the approximation of the variational inequalities. Lemma 2.7 can be viewed as a natural generalization of Minty's lemma (see [9]).

We now study those conditions under which the mixed quasivariational inequality (2.1) has a unique solution, which is the main motivation of our next result.

Theorem 2.9. *Let T be a strongly g-monotone with constant $\alpha > 0$ and g-Lipschitz continuous operator with constant $\beta > 0$. Let g be an injective operator. If the bifunction $\varphi(\cdot,\cdot)$ is skew-symmetric and $0 < \rho < 2\alpha/\beta^2$, then the general mixed quasivariational inequality (2.1) has a unique solution.*

Proof. (a) Uniqueness. Let $u_1 \neq u_2 \in H$ be two solutions of (2.1). Then, we have

$$\langle Tu_1, g(v) - g(u_1)\rangle + \varphi(g(v), g(u_1)) - \varphi(g(u_1), g(u_1)) \geq 0, \quad \forall v \in H, \tag{2.26}$$

$$\langle Tu_2, g(v) - g(u_2)\rangle + \varphi(g(v), g(u_2)) - \varphi(g(u_2), g(u_2)) \geq 0, \quad \forall v \in H. \tag{2.27}$$

Taking $v = u_2$ in (2.26) and $v = u_1$ in (2.27), adding the resultant and using the skew-symmetry of the bifunction $\varphi(\cdot,\cdot)$, we have

$$\begin{aligned} \langle Tu_1 - Tu_2, g(u_1) - g(u_2)\rangle &\leq \varphi(g(u_1), g(u_2)) - \varphi(g(u_1), g(u_1)) - \varphi(g(u_2), g(u_2)) \\ &\quad + \varphi(g(u_2), g(u_1)) \\ &\leq 0. \end{aligned} \tag{2.28}$$

Since T is strongly g-monotone, there exists a constant $\alpha > 0$ such that

$$\alpha\|g(u_1) - g(u_2)\|^2 \leq \langle Tu_1 - Tu_2, g(u_1) - g(u_2)\rangle \leq 0, \tag{2.29}$$

which implies that $g(u_1) = g(u_2) \to u_1 = u_2$, the uniqueness of the solution of (2.1), since g is an injective operator.

(b) Existence. We now use the auxiliary principle technique to prove the existence of a solution of (2.1). For a given $u \in H$, we consider the problem of finding a unique $w \in H$ such that

$$\begin{aligned} &\langle g(w), g(v) - g(w)\rangle + \rho\varphi(g(v), g(w)) - \rho\varphi(g(w), g(w)) \\ &\quad \geq \langle g(u), g(v) - g(w)\rangle - \rho\langle Tu, g(v) - g(w)\rangle, \quad \forall v \in H, \end{aligned} \tag{2.30}$$

where $\rho > 0$ is a constant. The parameter ρ plays a crucial part in proving that the mapping defined by the relation (2.30) is a contraction and consequently has a fixed point satisfying the original problem.

The inequality of type (2.30) is called the auxiliary variational inequality associated with problem (2.1). It is clear that the relation (2.30) defines a mapping $u \to w$. It is enough to show that the mapping $u \to w$, defined by the relation (2.30), has a fixed point belonging to H satisfying the mixed quasivariational inequality (2.1). Let $w_1 \neq w_2$ be two solutions of (2.30) related to $u_1, u_2 \in H$, respectively. It is sufficient to show that for a well-chosen $\rho > 0$,

$$||w_1 - w_2|| \leq \theta||u_1 - u_2||, \tag{2.31}$$

with $0 < \theta < 1$, where θ is independent of u_1 and u_2. Taking $v = w_2$ (resp., w_1) in (2.30) related to u_1 (resp., u_2), adding the resultants, and using the skew-symmetry of the bifunction $\varphi(\cdot,\cdot)$, we have

$$\langle g(w_1) - g(w_2), g(w_1) - g(w_2)\rangle \leq \langle g(u_1) - g(u_2) - \rho(Tu_1 - Tu_2), g(w_1) - g(w_2)\rangle, \tag{2.32}$$

from which we have

$$\begin{aligned} ||g(w_1) - g(w_2)||^2 &\leq ||g(u_1) - g(u_2) - \rho(Tu_1 - Tu_2)||^2 \\ &\leq ||g(u_1) - g(u_2)||^2 - 2\rho\langle g(u_1) - g(u_2), Tu_1 - Tu_2\rangle \\ &\quad + \rho^2||Tu_1 - Tu_2||^2 \\ &\leq ||g(u_1) - g(u_2)||^2 - 2\rho\alpha||g(u_1) - g(u_2)||^2 + \rho^2\beta^2||g(u_1) - g(u_2)||^2 \\ &\leq (1 - 2\rho\alpha + \rho^2\beta^2)||g(u_1) - g(u_2)||^2, \end{aligned} \tag{2.33}$$

since T is both a strongly monotone and Lipschitz continuous operator with constants $\alpha > 0$ and $\beta > 0$, respectively. Since g is injective, it follows that

$$||w_1 - w_2|| \leq \theta||u_1 - u_2||, \tag{2.34}$$

where $\theta = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2} < 1$ for $0 < \rho < 2\alpha/\beta^2$, showing that the mapping defined by (2.30) has a fixed point belonging to H, which is the solution of (2.1), the required result. $\square$

We note that if the operator T is symmetric, positive, and the bifunction $\varphi(\cdot,\cdot)$ is convex in the first argument, then the solution of the auxiliary mixed quasivariational inequality (2.30) is equivalent to finding the minimum of the functional $I[w]$, where

$$\begin{aligned} I[w] = {} & \frac{1}{2}\langle g(w)-g(u),g(w)-g(u)\rangle + \rho\langle Tu,g(w)-g(u)\rangle \\ & + \rho\varphi(g(u),g(w)) - \rho\varphi(g(u),g(u)), \quad \forall u \in H, \end{aligned} \tag{2.35}$$

which is a differentiable functional associated with the inequality (2.30). This auxiliary functional can be used to construct a gap (merit) function whose stationary points solve the variational inequality (2.1). In fact, one can easily show that the mixed quasivariational inequality (2.1) is equivalent to the optimization problem. This approach is used to suggest and analyze some descent iterative methods for solving mixed quasivariational inequalities.

We also need the following condition.

Assumption 2.10. For all $u,v,w \in H$, the operator $J_{\varphi(u)}$ satisfies the condition

$$||J_{\varphi(u)}w - J_{\varphi(v)}w|| \le \nu\|u-v\|, \tag{2.36}$$

where $\nu > 0$ is a constant. It is shown in [12] that Assumption 2.10 is satisfied for some special cases.

Definition 2.11. A function $M : H \to R \cup \{+\infty\}$ is called a merit (gap) function for the mixed quasivariational inequalities (2.1) if and only if

(i) $M(u) \ge 0$, $\forall v \in H$;

(ii) $M(\overline{u}) = 0$ if and only if $\overline{u} \in H$ solves (2.1).

3. Main results

In this section, we consider some merit functions and obtain error bounds for the general mixed quasivariational inequalities (2.1) and related optimization problems. For this purpose, we need the following result, which can be proved by using Lemma 2.6.

Lemma 3.1. *The general mixed quasivariational inequality (2.1) has a solution $u \in H$ if and only if $u \in H$ satisfies the relation*

$$g(u) = J_{\varphi(g(u))}[g(u) - \rho Tu], \tag{3.1}$$

where $\rho > 0$ is a constant and $J_{\varphi(g(u))} = (I + \rho\partial\varphi(\cdot,g(u)))^{-1}$ is the resolvent operator.

Lemma 3.1 implies that the mixed quasivariational inequalities (2.1) are equivalent to the fixed-point problem (3.1). This alternative equivalent formulation plays an important part in suggesting and analyzing several iterative methods for solving variational inequalities. This fixed-point formulation has been used to suggest the following iterative method for problem (2.1).

We now consider the residue vector

$$R_\rho(u) \equiv R(u) := g(u) - J_{\varphi(g(u))}[g(u) - \rho Tu]. \tag{3.2}$$

It is clear from Lemma 3.1 that (2.1) has a solution $u \in H$ if and only if $u \in H$ is a root of the equation

$$R(u) = 0. \tag{3.3}$$

It is known that the normal residue vector $R(u)$ defined by the relation (3.2) is a merit function for the mixed quasivariational inequality (2.1). We use this merit function to derive the global error bounds for the solution of (2.1).

Theorem 3.2. *Let $\overline{u} \in H$ be a solution of (2.1) and let Assumption 2.10 hold. Let g be both strongly nonexpanding and Lipschitz continuous with constants $\sigma > 0$ and $\delta > 0$. If the operator T is both strongly g-monotone and g-Lipschitz continuous with constants $\alpha > 0$ and $\beta > 0$, respectively, then*

$$\frac{1}{k_1}||R(u)|| \leq \|u - \overline{u}\| \leq k_2||R(u)||, \quad \forall u \in H, \tag{3.4}$$

where k_1, k_2 are generic constants.

Proof. Let $\overline{u} \in H$ be solution of (2.1). Then

$$\langle T\overline{u}, g(v) - g(\overline{u})\rangle + \varphi(g(v), g(\overline{u})) - \varphi(g(\overline{u}), g(\overline{u})) \geq 0, \quad \forall v \in H. \tag{3.5}$$

Taking $g(v) = J_{\varphi(g(u))}[g(u) - \rho Tu]$ in (3.5), we have

$$\begin{aligned} &\langle T\overline{u}, J_{\varphi(g(u))}[g(u) - \rho Tu] - g(\overline{u})\rangle + \varphi(J_{\varphi(g(u))}[g(u) - \rho Tu], g(\overline{u})) \\ &\quad - \varphi(g(\overline{u}), g(\overline{u})) \geq 0. \end{aligned} \tag{3.6}$$

Letting $g(u) = J_{\varphi(g(u))}[g(u) - \rho Tu]$, $g(z) = g(u) - \rho Tu$, and $g(v) = g(\overline{u})$ in (2.18), we have

$$\begin{aligned} &\langle \rho Tu + J_{\varphi(g(u))}[g(u) - \rho Tu] - g(u), g(\overline{u}) - J_{\varphi(g(u))}[g(u) - \rho Tu]\rangle \\ &\quad + \rho\varphi(g(\overline{u})), J_{\varphi(g(u))}[g(u) - \rho Tu] \\ &\quad - \rho\varphi(J_{\varphi(g(u))}[g(u) - \rho Tu], J_{\varphi(g(u))}[g(u) - \rho Tu]) \geq 0. \end{aligned} \tag{3.7}$$

Adding (3.6), (3.7), and using the skew-symmetry of the bifunction $\varphi(\cdot,\cdot)$, we obtain

$$\left\langle T\overline{u} - Tu + \frac{1}{\rho}(g(u) - J_{\varphi(g(u))}[g(u) - \rho Tu]), J_{\varphi(g(u))}[g(u) - \rho Tu] - g(\overline{u})\right\rangle \geq 0. \tag{3.8}$$

Since T is a strongly g-monotone and g is nonexpanding, there exists a constant $\alpha > 0$ such that

$$\begin{aligned}
\alpha\|\overline{u}-u\|^2 &\le \|g(\overline{u})-g(u)\|^2 \le \langle T\overline{u}-Tu, g(\overline{u})-g(u)\rangle \\
&= \langle T\overline{u}-Tu, g(\overline{u})-J_{\varphi}[u-\rho Tu]\rangle + \langle T\overline{u}-Tu, J_{\varphi(g(u))}[g(u)-\rho Tu]-g(u)\rangle \\
&\le \frac{1}{\rho}\langle g(u)-J_{\varphi(g(u))}[g(u)-\rho Tu], J_{\varphi(g(u))}[g(u)-\rho Tu]-g(u)+g(u)-g(\overline{u})\rangle \\
&\quad + \langle T\overline{u}-Tu, J_{\varphi(g(u))}[g(u)-\rho Tu]-g(u)\rangle \\
&\le -\frac{1}{\rho}\|R(u)\|^2 + \frac{1}{\rho}\|R(u)\|\,\|g(u)-g(\overline{u})\| + \|T\overline{u}-Tu\|\,\|R(u)\| \\
&\le \frac{1}{\rho}(1+\beta)\|R(u)\|\,\|g(\overline{u})-g(u)\| \\
&\le \frac{\delta}{\rho}(1+\beta)\|R(u)\|\,\|\overline{u}-u\|,
\end{aligned} \tag{3.9}$$

which implies that

$$\|\overline{u}-u\| \le k_2\|R(u)\|, \tag{3.10}$$

the right-hand inequality in (3.4) with $k_2 = (\delta/\alpha\rho)(1+\beta)$, where $\delta > 0$ is the Lipschitz constant of g.

Now from Assumption 2.10 and g-Lipschitz continuity of T, we have

$$\begin{aligned}
\|R(u)\| &= \|g(u)-J_{\varphi(g(u))}[g(u)-\rho Tu]\| \\
&= \|g(u)-g(\overline{u})+J_{\varphi(g(\overline{u}))}[g(\overline{u})-\rho T\overline{u}]-J_{\varphi(g(u))}[g(u)-\rho Tu]\| \\
&\le \|g(u)-g(\overline{u})\| + \|J_{\varphi(g(\overline{u}))}[g(\overline{u})-\rho T\overline{u}]-J_{\varphi(g(\overline{u}))}[g(u)-\rho Tu]\| \\
&\quad + \|J_{\varphi(g(\overline{u}))}[g(u)-\rho Tu]-J_{\varphi(g(u))}[g(u)-\rho Tu]\| \\
&\le \|g(u)-g(\overline{u})\| + \nu\|g(u)-g(\overline{u})\| + \|g(u)-g(\overline{u})+\rho(Tu-T\overline{u})\| \\
&\le \{2+\nu+\rho\beta\}\|g(u)-g(\overline{u})\| = k_1\|u-\overline{u}\|,
\end{aligned} \tag{3.11}$$

from which we have

$$\frac{1}{k_1}\|R(u)\| \le \|u-\overline{u}\|, \tag{3.12}$$

the leftmost inequality in (3.4) with $k_1 = (2+\nu+\rho\beta)\delta$, where $\delta > 0$ is the Lipschitz constant of g. Combining (3.10) and (3.12), we obtain the required (3.4). □

Letting $u = 0$ in (3.4), we have

$$\frac{1}{k_1}\|R(0)\| \le \|\overline{u}\| \le k_2\|R(0)\|. \tag{3.13}$$

Combining (3.4) and (3.13), we obtain the relative error bounds for any point $u \in H$.

Theorem 3.3. *Assume that all the assumptions of Theorem 3.2 hold. If* $0 \neq \overline{u} \in H$ *is the unique solution of (2.1), then*

$$c_1\left\|\frac{R(u)}{R(0)}\right\| \leq \left\|\frac{u-\overline{u}}{\overline{u}}\right\| \leq c_2\left\|\frac{R(u)}{R(0)}\right\|. \tag{3.14}$$

We now consider another merit function associated with problem (2.1), which can be viewed as a regularized merit function. From (2.35), we have

$$\begin{aligned} M_\rho(u) := \max_{v\in H}\Big\{&\langle Tu, g(u)-g(v)\rangle - \varphi(g(v),g(v)) + \varphi(g(u),g(v)) \\ &- \frac{1}{2\rho}\|g(u)-g(v)\|^2\Big\}, \quad u \in H, \end{aligned} \tag{3.15}$$

which is called the regularized merit (gap) function associated with problem (2.1).

We note that if $\varphi(\cdot,\cdot) = \varphi(\cdot)$ is an indicator function of a closed convex set K in H, then the merit function (3.15) reduces to the known merit function for general variational inequalities (2.4), that is,

$$M_\rho(u) := \max_{v\in K}\Big\{\langle Tu, g(u)-g(v)\rangle - \frac{1}{2\rho}\|g(u)-g(v)\|^2\Big\}, \quad u \in K, \tag{3.16}$$

which is a natural extension of a regularized merit function of Fukushima [4]. Thus it is clear that the merit function $M_\rho(u)$ defined by (3.15) can be viewed as a natural generalization of the regularized merit function associated with the general variational inequalities (2.4).

We note that the function $M_\rho(u)$ can be written as

$$\begin{aligned} M_\rho(u) = &\langle Tu, g(u) - J_{\varphi(g(u))}[g(u)-\rho Tu]\rangle + \varphi(g(u), J_{\varphi(g(u))}[g(u)-\rho Tu]) \\ &- \varphi(J_{\varphi(g(u))}[g(u)-\rho Tu], J_{\varphi(g(u))}[g(u)-\rho Tu]) \\ &- \frac{1}{2\rho}\|g(u) - J_{\varphi(g(u))}[g(u)-\rho Tu]\|^2, \quad \forall u \in H, \end{aligned} \tag{3.17}$$

from which it follows that $M_\rho(u) \geq 0$, for all $u \in H$.

We now show that the function $M_\rho(u)$ defined by (3.15) is a merit function and this is the main motivation of our next result.

Theorem 3.4. *For all* $u \in H$,

$$M_\rho(u) \geq \frac{1}{2\rho}\|R(u)\|^2. \tag{3.18}$$

In particular, $M_\rho(u) = 0$ *if and only if* $u \in H$ *is a solution of (2.1).*

Proof. Setting $g(v) = g(u)$, $g(u) = J_{\varphi(g(u))}[g(u) - \rho Tu]$, and $g(z) = g(u) - \rho Tu$ in (2.18), we have

$$\begin{aligned}
&\Big\langle Tu - \frac{1}{\rho}\big(g(u) - J_{\varphi(g(u))}[g(u) - \rho Tu]\big), g(u) - J_{\varphi(g(u))}[g(u) - \rho Tu]\Big\rangle \\
&\quad + \varphi\big(g(u), J_{\varphi(g(u))}[g(u) - \rho Tu]\big) \\
&\quad - \varphi\big(J_{\varphi(g(u))}[g(u) - \rho Tu], J_{\varphi(g(u))}[g(u) - \rho Tu]\big) \geq 0,
\end{aligned} \tag{3.19}$$

which implies that

$$\begin{aligned}
&\langle Tu, R(u)\rangle - \varphi\big(J_{\varphi(g(u))}[g(u) - \rho Tu], J_{\varphi(g(u))}[g(u) - \rho Tu]\big) \\
&\quad + \varphi\big(g(u), J_{\varphi(g(u))}[g(u) - \rho Tu]\big) \geq \frac{1}{\rho}||R(u)||^2.
\end{aligned} \tag{3.20}$$

Combining (3.17) and (3.20), we have

$$\begin{aligned}
M_\rho(u) &= \langle Tu, R(u)\rangle - \varphi\big(J_{\varphi(g(u))}[g(u) - \rho Tu], J_{\varphi(g(u))}[g(u) - \rho Tu]\big) \\
&\quad + \varphi\big(g(u), J_{\varphi(g(u))}[g(u) - \rho Tu]\big) - \frac{1}{2\rho}||R(u)||^2 \\
&\geq \frac{1}{2\rho}||R(u)||^2 - \frac{1}{2\rho}||R(u)||^2 \\
&= \frac{1}{2\rho}||R(u)||^2,
\end{aligned} \tag{3.21}$$

the required result (3.18). Clearly, we have $M_\rho(u) \geq 0$, for all $u \in H$.

Now if $M_\rho(u) = 0$, then clearly $R(u) = 0$. Hence by Lemma 3.1, we see that $u \in H$ is a solution of (2.1). Conversely, if $u \in H$ is a solution of (2.1), then $g(u) = J_{\varphi(g(u))}[g(u) - \rho Tu]$ by Lemma 3.1. Consequently, from (3.15), we see that $M_\rho(u) = 0$, the required result. □

From Theorem 3.4, we see that the function $M_\rho(u)$ defined by (3.15) is a merit function for the mixed quasivariational inequalities (2.1). It is known that the regularized merit function is differentiable whenever T and the bifunction $\varphi(\cdot,\cdot)$ are differentiable. We now derive the error bounds without using the Lipschitz continuity of the T.

Theorem 3.5. *Let T be strongly monotone with a constant $\alpha > 0$ and let the bifunction $\varphi(\cdot,\cdot)$ be a skew-symmetric function. If g is strongly nonexpanding with a constant $\sigma > 0$, then*

$$\|u - \overline{u}\|^2 \leq \frac{(2\rho)}{(2\alpha\rho - 1)\sigma M_\rho(u)}, \quad \forall u \in H. \tag{3.22}$$

Proof. From (3.15) and the strong monotonicity of T, we have

$$\begin{aligned} M_\rho(u) &\geq \langle Tu, g(u)-g(\overline{u})\rangle + \varphi(g(u),g(\overline{u})) - \varphi(g(\overline{u}),g(\overline{u})) - \frac{1}{2\rho}\|g(u)-g(\overline{u})\|^2 \\ &\geq \langle T\overline{u}, g(u)-g(\overline{u})\rangle + \alpha\|g(u)-g(\overline{u})\|^2 + \varphi(g(u),g(\overline{u})) - \varphi(g(\overline{u}),g(\overline{u})) \\ &\quad - \frac{1}{2\rho}\|g(u)-g(\overline{u})\|^2. \end{aligned} \tag{3.23}$$

Taking $v = u$ in (3.5), we have

$$\langle T\overline{u}, g(u)-g(\overline{u})\rangle + \varphi(g(u),g(\overline{u})) - \varphi(g(\overline{u}),g(\overline{u})) \geq 0. \tag{3.24}$$

From (3.23), (3.24), and using the strong nonexpansion of g with constant $\sigma > 0$, we have

$$\begin{aligned} M_\rho(u) &\geq \alpha\|g(u)-g(\overline{u})\|^2 - \frac{1}{2\rho}\|g(u)-g(\overline{u})\|^2 \\ &= \left(\alpha - \frac{1}{2\rho}\right)\sigma\|g(u)-g(\overline{u})\|^2, \end{aligned} \tag{3.25}$$

from which the result (3.22) follows. □

We consider another merit function associated with mixed quasivariational inequalities (2.1), which can be viewed as a difference of two regularized merit functions. Such a type of merit functions was introduced and studied by many authors for solving variational inequalities and complementarity problems; see [4, 20, 23]. Here we define the D-merit function by a formal difference of the regularized merit function defined by (3.15). To this end, we consider the function

$$\begin{aligned} D_{\rho,\mu}(u) = \max_{v\in H}\Big\{ &\langle Tu, g(u)-g(v)\rangle + \varphi(g(u),g(v)) - \varphi(g(v),g(v)) \\ &+ \frac{1}{2\rho}\|g(u)-g(v)\|^2 - \frac{1}{2\rho}\|g(u)-g(v)\|^2\Big\}, \quad \forall v \in H, \end{aligned} \tag{3.26}$$

which is called the D-merit function associated with the mixed quasivariational inequalities (2.1). The D-merit function defined by (3.26) can be written as

$$\begin{aligned} D_{\rho,\mu}(u) &= \langle Tu, J_{\varphi(g(u))}[g(u)-\mu Tu] - J_{\varphi(g(u))}[g(u)-\rho Tu]\rangle \\ &\quad + \varphi(J_{\varphi(g(u))}[g(u)-\mu Tu], g(u)) - \varphi(J_{\varphi(g(u))}[g(u)-\rho Tu], g(u)) \\ &\quad + \frac{1}{2\mu}\|g(u) - J_{\varphi(g(u))}[g(u)-\mu Tu]\|^2 - \frac{1}{2\rho}\|g(u) - J_{\varphi(g(u))}[g(u)-\rho Tu]\|^2 \\ &= \langle Tu, R_\rho(u) - R_\mu(u)\rangle + \varphi(J_{\varphi(g(u))}[g(u)-\mu Tu], g(u)) \\ &\quad - \varphi(J_{\varphi(g(u))}[g(u)-\rho Tu], g(u)) + \frac{1}{2\mu}\|R_\mu(u)\|^2 \\ &\quad - \frac{1}{2\rho}\|R_\rho(u)\|^2, \quad u \in H,\ \rho > \mu > 0. \end{aligned} \tag{3.27}$$

It is clear that the $D_{\rho,\mu}(u)$ is finite everywhere. We now show that the function $D_{\rho,\mu}(u)$ defined by (3.26) is indeed a merit function for the mixed quasivariational inequalities (2.1) and this is the motivation of our next result.

Theorem 3.6. *For all $u \in H$, $\rho > \mu > 0$,*

$$(\rho-\mu)||R_\rho(u)||^2 \geq 2\rho\mu D_{\rho,\mu}(u) \geq (\rho-\mu)||R_\mu(u)||^2. \tag{3.28}$$

In particular, $D_{\rho,\mu}(u) = 0$ if and only if $u \in H$ solves problem (2.1).

Proof. Taking $g(v) = J_{\varphi(g(u))}[g(u)-\mu Tu]$, $g(u) = J_{\varphi(g(u))}[g(u)-\rho Tu]$, and $g(z) = g(u) - \rho Tu$ in (2.18), we have

$$\begin{aligned}
&\langle J_{\varphi(g(u))}[g(u)-\rho Tu] - g(u) + \rho Tu, J_{\varphi(g(u))}[g(u)-\mu Tu] - J_{\varphi(g(u))}[g(u)-\rho Tu]\rangle \\
&\quad + \rho\varphi(J_{\varphi(g(u))}[g(u)-\mu Tu], J_{\varphi(g(u))}[g(u)-\rho Tu]) \\
&\quad - \rho\varphi(J_{\varphi(g(u))}[g(u)-\rho Tu], J_{\varphi(g(u))}[g(u)-\rho Tu]) \geq 0,
\end{aligned} \tag{3.29}$$

which implies that

$$\begin{aligned}
&\langle Tu, R_\rho(u) - R_\mu(u)\rangle + \varphi(J_{\varphi(g(u))}[g(u)-\mu Tu], J_{\varphi(g(u))}[g(u)-\rho Tu]) \\
&\qquad - \varphi(J_{\varphi(g(u))}[g(u)-\rho Tu], J_{\varphi(g(u))}[g(u)-\rho Tu]) \\
&\quad \geq \frac{1}{2\rho}\langle R_\rho(u), R_\rho(u) - R_\mu(u)\rangle.
\end{aligned} \tag{3.30}$$

From (3.27) and (3.30), we have

$$\begin{aligned}
D_{\rho,\mu}(u) &\geq \frac{1}{2\rho}\langle R_\rho(u), R_\rho(u) - R_\mu(u)\rangle + \frac{1}{2\mu}||R_\mu(u)||^2 - \frac{1}{2\rho}||R_\rho(u)||^2 \\
&= \frac{1}{2}\left(\frac{1}{\mu} - \frac{1}{\rho}\right)||R_\mu(u)||^2 + \frac{1}{2\rho}\langle R_\rho(u), R_\rho(u) - R_\mu(u)\rangle \\
&\quad - \frac{1}{2\rho}||R_\rho(u) - R_\mu(u)||^2 - \frac{1}{2\rho}\langle R_\mu(u), R_\rho(u) - R_\mu(u)\rangle \\
&= \frac{1}{2}\left(\frac{1}{\mu} - \frac{1}{\rho}\right)||R_\mu(u)||^2 + \frac{1}{2\rho}||R_\rho(u) - R_\mu(u)||^2 \\
&\geq \frac{1}{2}\left(\frac{1}{\mu} - \frac{1}{\rho}\right)||R_\mu(u)||^2,
\end{aligned} \tag{3.31}$$

which implies the rightmost inequality in (3.28).

In a similar way, by taking $g(u) = J_{\varphi(g(u))}[g(u)-\mu Tu]$, $g(z) = g(u) - \mu Tu$, and $g(v) = J_{\varphi(g(u))}[g(u)-\rho Tu]$ in (2.18), we have

$$\begin{aligned}
&\langle J_{\varphi(g(u))}[g(u)-\mu Tu] - g(u) + \mu Tu, J_{\varphi(g(u))}[g(u)-\rho Tu] - J_{\varphi(g(u))}[g(u)-\mu Tu]\rangle \\
&\quad + \rho\varphi(J_{\varphi(g(u))}[g(u)-\rho Tu], J_{\varphi(g(u))}[g(u)-\mu Tu]) \\
&\quad - \rho\varphi(J_{\varphi(g(u))}[g(u)-\mu Tu], J_{\varphi(g(u))}[g(u)-\mu Tu]) \geq 0,
\end{aligned} \tag{3.32}$$

which implies that

$$\langle Tu, R_\rho(u) - R_\mu(u)\rangle + \varphi(J_{\varphi(g(u))}[g(u) - \mu Tu], J_{\varphi(g(u))}[g(u) - \rho Tu]) \\ - \varphi(J_{\varphi(g(u))}[g(u) - \rho Tu], J_{\varphi(g(u))}[g(u) - \rho Tu]) \geq \frac{1}{2\rho}\langle R_\rho(u), R_\rho(u) - R_\mu(u)\rangle. \tag{3.33}$$

Consequently, from (3.27) and (3.33), we obtain

$$\begin{aligned} D_{\rho,\mu}(u) &\leq \frac{1}{\mu}\langle R_\mu(u), R_\rho(u) - R_\mu(u)\rangle + \frac{1}{2\mu}||R_\mu(u)||^2 - \frac{1}{2\rho}||R_\rho(u)||^2 \\ &= \frac{1}{2}\left(\frac{1}{\mu} - \frac{1}{\rho}\right)||R_\mu(u)||^2 + \frac{1}{2\rho}\langle R_\rho(u), R_\rho(u) - R_\mu(u)\rangle \\ &\quad - \frac{1}{2\rho}||R_\rho(u) - R_\mu(u)||^2 - \frac{1}{2\rho}\langle R_\mu(u), R_\rho(u) - R_\mu(u)\rangle \\ &= \frac{1}{2}\left(\frac{1}{\mu} - \frac{1}{\rho}\right)||R_\rho(u)||^2 - \frac{1}{2}\mu||R_\rho(u) - R_\mu(u)||^2 \\ &\leq \frac{1}{2}\left(\frac{1}{\mu} - \frac{1}{\rho}\right)||R_\rho(u)||^2, \end{aligned} \tag{3.34}$$

which implies the leftmost inequality in (3.28).

Combining (3.31) and (3.34), we obtain (3.28), the required result. □

Using essentially the technique of Theorem 3.5, we can obtain the following result.

Theorem 3.7. *Let $\overline{u} \in H$ be a solution of (2.1). If the operator T is strongly monotone with constant $\alpha > 0$ and g is strongly nonexpanding with constant $\sigma > 0$, then*

$$||u - \overline{u}||^2 \leq \frac{(2\rho\mu)}{(\rho(2\mu\alpha + 1) - \mu)\sigma D_{\rho,\mu}}, \quad \forall u \in H. \tag{3.35}$$

Proof. Let $\overline{u} \in H$ be a solution of (2.1). Then, taking $v = u$ in (3.5), we have

$$\langle T\overline{u}, g(u) - g(\overline{u})\rangle + \varphi(g(u), g(\overline{u})) - \varphi(g(\overline{u}), g(\overline{u})) \geq 0. \tag{3.36}$$

Also from (3.26), (3.36), and strong monotonicity of T, we have

$$\begin{aligned} D_{\rho,\mu}(u) &\geq \langle Tu, g(u) - g(\overline{u})\rangle - \varphi(g(\overline{u}), g(\overline{u})) + \varphi(g(u), g(\overline{u})) \\ &\quad + \frac{1}{2\mu}||g(u) - g(\overline{u})||^2 - \frac{1}{2\rho}||g(u) - g(\overline{u})||^2 \\ &\geq \langle T\overline{u}, g(u) - g(\overline{u})\rangle - \varphi(g(\overline{u}), g(\overline{u})) + \varphi(g(u), g(\overline{u})) \\ &\quad + \alpha||g(u) - g(\overline{u})||^2 + \frac{1}{2\mu}||g(u) - g(\overline{u})||^2 - \frac{1}{2\rho}||g(u) - g(\overline{u})||^2 \\ &\geq \sigma\left(\alpha + \frac{1}{2\mu} - \frac{1}{2\rho}\right)||u - \overline{u}||^2, \end{aligned} \tag{3.37}$$

from which the required result (3.35) follows, where $\sigma > 0$ is a strongly expansivoty constant of g. □

Acknowledgement

The author expresses his sincere appreciation to the referee for valuable comments and constructive suggestions.

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