

ON AN INVERSE TO THE HÖLDER INEQUALITY

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ABSTRACT. An extension is given for the inverse to Hölder's inequality obtained recently by Zhuang.

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Recently Zhuang [1] proved the following inverse of the arithmetico-geometric inequality.

THEOREM A. Let $0 < a \leq x \leq A$, $0 < b \leq y \leq B$, $1/p + 1/q = 1$, $p > 1$; then

$$\frac{x}{p} + \frac{y}{q} \leq \max \left\{ \frac{A/p + b/q}{A^{1/p}b^{1/q}}, \frac{a/p + B/q}{a^{1/p}B^{1/q}} \right\} x^{1/p}y^{1/q}, \quad (1)$$

or

$$x + y \leq \max \left\{ \frac{A + b}{A^{1/p}b^{1/q}}, \frac{a + B}{a^{1/p}B^{1/q}} \right\} x^{1/p}y^{1/q}, \quad (2)$$

the sign of equality in (1) and (2) holds if and only if either $(x, y) = (a, B)$ or $(x, y) = (A, b)$.

Moreover, if $a \geq B$, then

$$\frac{a/p + B/q}{a^{1/p}B^{1/q}} x^{1/p}y^{1/q} \leq \frac{x}{p} + \frac{y}{q} \leq \frac{A/p + b/q}{A^{1/p}b^{1/q}} x^{1/p}y^{1/q}, \quad (3)$$

the sign of equality on the right-hand side of (3) holds if and only if $(x, y) = (A, b)$, and the sign of equality on the left-hand side of (3) holds if and only if $(x, y) = (a, B)$. The sign of inequality in (3) is reversed if $b \geq A$.

This enables us to formulate the following theorem.

THEOREM 1. Suppose x, y, a, b, A, B, p, q are as in Theorem A and $\alpha, \beta > 0$. Then

$$\alpha x^p + \beta y^q \leq \max(C, D)xy, \quad (4)$$

where

$$C = (\alpha A^p + \beta b^q)/(Ab), \quad D = (\alpha a^p + \beta B^q)/(aB). \quad (5)$$

Equality occurs if and only if either $(x, y) = (a, B)$ or $(x, y) = (A, b)$. Moreover, if $\alpha p A^p \geq \beta q B^q$, then

$$Cxy \leq \alpha x^p + \beta y^q \leq Dxy, \quad (6)$$

with equality on the right-hand side if and only if $(x, y) = (A, b)$ and on the left if and only if $(x, y) = (a, B)$. The inequalities in (6) are reversed if $\alpha p A^p \leq \beta q B^q$.

PROOF. Inequalities (4) and (6) follow from (1) and (3) under the substitutions

$$x \rightarrow \alpha p x^p, \quad y \rightarrow \beta q y^q, \quad a \rightarrow \alpha p a^b, \quad b \rightarrow \beta q b^q, \quad A \rightarrow \alpha p A^p, \quad B \rightarrow \beta q B^q.$$

REMARK. Theorem 1 gives (1) and (2) together, (1) resulting from the substitutions $\alpha = 1/p$, $x \rightarrow x^{1/p}$, $A = A^{1/p}$, $a \rightarrow a^{1/p}$ and corresponding relations for β, y etc. with q in place of p , while (2) results from similar substitutions with $\alpha = 1 = \beta$.

The following result now gives an extension of the inverse to Hölder's inequality obtained in [1]. We suppose that all the integrals involved exist.

THEOREM 2. Let the functions f, g satisfy $0 < a \leq f(x) \leq A$, $0 < b \leq g(x) \leq B$ for almost all $x \in X$ with respect to a measure μ . Suppose $\alpha, \beta, p, q, C, D$ are as in Theorem 1. Then

$$\left(\int_X f^p d\mu \right)^{1/p} \left(\int_X g^q d\mu \right)^{1/q} \leq (\alpha\beta)^{-1/p} (\beta q)^{-1/q} \max(C, D) \int_X fg d\mu \tag{7}$$

and equality holds if and only if

$$\mu(E_1 \cup F_1) = \mu(X)$$

and

$$\mu(E_1) = \frac{(\alpha p A^p - \beta q b^q) \mu(X)}{\alpha p (A^p - a^p) + \beta q (B^q - b^q)},$$

where

$$E_1 = \{x \in X : f(x) = a, g(x) = B\},$$

$$F_1 = \{x \in X : f(x) = A, g(x) = b\}.$$

Moreover, if $\alpha p a^p \geq \beta q B^q$, then

$$\left(\int_X f^p d\mu \right)^{1/p} \left(\int_X g^q d\mu \right)^{1/q} \leq (\alpha p)^{-1/p} (\beta q)^{-1/q} D \int_X fg d\mu, \tag{8}$$

with equality only if $(f, g) = (a, B)$ a.e. on X and $\alpha p a^p = \beta q B^q$, and if $\alpha p A^p \leq \beta q b^q$, then

$$\left(\int_X f^p d\mu \right)^{1/p} \left(\int_X g^q d\mu \right)^{1/q} \leq (\alpha p)^{-1/p} (\beta q)^{-1/q} C \int_X fg d\mu, \tag{9}$$

with equality only if $(f, g) = (A, b)$ a.e. on X and $\alpha p A^p = \beta q b^q$.

PROOF. The first statement was proved in [1]. A simple proof of the remainder of the theorem was given for the case $\alpha = 1/p$, $\beta = 1/q$ in [2]. We give a similar simple proof for the general case.

$$\begin{aligned} \max(C, D) \int_X fg d\mu &= \int_X \max(C, D) fg d\mu \\ &\geq \int_X (\alpha f^p + \beta g^q) d\mu \\ &= \frac{1}{p} (\alpha p) \int_X f^p d\mu + \frac{1}{q} (\beta q) \int_X g^q d\mu \\ &\geq (\alpha p)^{1/p} (p q)^{1/q} \left(\int_X f^p d\mu \right)^{1/p} \left(\int_X g^q d\mu \right)^{1/q}, \end{aligned}$$

by the arithmetico-geometric inequality.

The equality conditions result from those in Theorem 1 and the arithmetico-geometric inequality.

Similarly we can prove (8). Using the second inequality in (6) we have

$$\begin{aligned} D \int_X fg \, d\mu &= \int_X Dfg \, d\mu \\ &\geq \int_X (\alpha f^p + \beta g^q) d\mu \\ &= \frac{1}{p} (\alpha p) \int_X f^p d\mu + \frac{1}{q} (\beta q) \int_X g^q d\mu \\ &\geq (\alpha p)^{1/p} (\beta q)^{1/q} \left(\int_X f^p d\mu \right)^{1/p} \left(\int_X g^q d\mu \right)^{1/q}. \end{aligned}$$

Relation (9) follows similarly.

REMARK. The simplest cases of (8) and (9) occur for $\alpha = 1/p, \beta = 1/q$. Then we have that if $a^p \geq B^q$, then

$$\left(\int_X f^p d\mu \right)^{1/p} \left(\int_X g^q d\mu \right)^{1/q} \leq D_1 \int_X fg \, d\mu$$

and if $A^p \leq b^q$, then

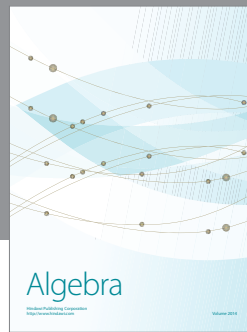
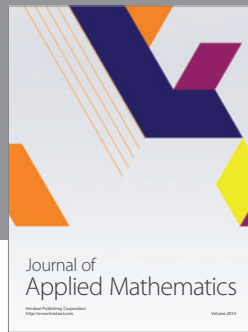
$$\left(\int_X f^p d\mu \right)^{1/p} \left(\int_X g^q d\mu \right)^{1/q} \leq C_1 \int_X fg \, d\mu,$$

where

$$\begin{aligned} D_1 &= \left(\frac{1}{p} a^p + \frac{1}{q} B^q \right) / (aB), \\ C_1 &= \left(\frac{1}{p} A^p + \frac{1}{q} b^q \right) / (Ab). \end{aligned}$$

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