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COMPATIBLE MAPPINGS AND COMMON FIXED POINTS "REVISITED"

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ABSTRACT. A fixed point theorem involving a Meir-Keeler type contraction principle is refined by diminishing continuity requirements.

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1. INTRODUCTION.

In [1], the concept of compatible maps was introduced as a generalization of commuting $(fg = gf)$ maps and weakly commuting maps (see [2]). Self maps f and g of a metric space (X,d) are **compatible** iff $lim_n d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $fx_n, gx_n \to t$ for some $t \in X$. To demonstrate the utility of this concept, a Meir-Keeler type theorem of Park and Bae [3] was generalized by replacing the commutativity requirement by compatibility and extending the concept of (ϵ,δ)-f-contractions for two functions as given in [3] to four functions as follows.

DEFINITION 1.1. [1] Let A,B,S,T be self maps of a metric space (X,d). A and B are (ϵ,δ)-S,T-contractions iff $A(X) \subseteq T(X), B(X) \subseteq S(X)$, and there exists a function $\delta: (0,\infty) \to (0,\infty)$ such that $\delta(\epsilon) > \epsilon$ for all $\epsilon > 0$ and for $x,y \in X$:

(i) $\epsilon \leq d(Sx,Ty) < \delta(\epsilon)$ implies $d(Ax,By) < \epsilon$, and

(ii) $Ax = By$ whenever $Sx = Ty$.

As the preceding suggests, if $A: X \to X$, we shall use Ax to denote $A(x)$ when convenient and no confusion is likely. We also let N denote the set of natural numbers.

Of interest to us is the following result which combines the two main theorems proved in [1].

THEOREM 1.1. Let S and T be continuous self maps of a complete metric space (X,d), and let A and B be (ϵ,δ)-S,T-contractions such that the pairs A,S and B,T are compatible. Then A,B,S,T have a unique common fixed point if one of the following conditions (a) or (b) is satisfied:

(a) A and B are Continuous.

(b) δ is lower semi-continuous.

Our purpose in "revisiting" [1] is to show that the preceding theorem can be appreciably generalized by using property (ii) in Definition 1.1 more extensively. In fact, we shall show that condition (b) can be dropped and that only one of the functions $A,B,S,$ or T need be continuous. By so doing, we answer question 4.1 in [1], and highlight the role played by "compatibility" in producing common fixed points.

2. RESULTS.

We need the following from Proposition 2.2 in [1].

PROPOSITION 2.1. Let $A,B:(X,d)\to(X,d)$ be compatible. If $At=Bt$, then $ABt=BAt$. Moreover, if $Ax_n, Bx_n\to t$ for some $t\in X$ and if A is continuous, then $BAx_n\to At$.

The next result contributes to economy of effort.

PROPOSITION 2.2. Let A,B,S and T be self maps of a complete metric space (X,d) such that the pairs A,S and B,T are compatible. Suppose that for $x,y\in X$,

$$Sx\neq Ty \text{ implies } d(Ax,By)<d(Sx,Ty). \tag{2.1}$$

If $\exists\ p,u,v\in X$ such that $(*)$ $p=Au=Su=Bv=Tv$, then $p=Ap=Sp=Bp=Tp$.

PROOF. Since $p=Au=Su$ and A and S are compatible, $Sp=SAu=ASu=Ap$. But then, if $p\neq Ap$, $Tv\neq Sp$ by $(*)$, so that (2.1) implies

$$d(p,Ap)=d(Bv,Ap)<d(Tv,Sp)=d(p,Ap),$$

a contradiction. Therefore, $p=Ap=Sp$. By symmetry, $p=Bp=Tp$. □

We now state and prove our main result.

THEOREM 2.1. Let S and T be self maps of a complete metric space (X,d) and let A and B be (ϵ,δ)-S,T-contractions such that the pairs A,S and B,T are compatible. If one of A,B,S, or T is continuous, then A,B,S,T have a unique common fixed point.

PROOF. Since A and B are (ϵ,δ)-S,T-contractions, $A(X)\subseteq T(X), B(X)\subseteq S(X)$, and as a consequence of (i) and (ii) in the definition we know that

$$Sx=Ty \text{ implies } Ax=By, \text{ and } d(Ax,By)<d(Sx,Ty) \text{ if } Sx\neq Ty. \tag{2.2}$$

In particular, $d(Ax,By)\leq d(Sx,Ty)$ for $x,y\in X$.

Let $x_0\in X$. For $n\in N$, let $y_{2n-1}=Tx_{2n-1}=Ax_{2n-2}$ and $y_{2n}=Sx_{2n}=Bx_{2n-1}$. Since $A(X)\subseteq T(X)$ and $B(X)\subseteq S(X)$, the x_i can be so chosen. By Lemma 3.1 in [1], the sequence $\{y_n\}$ so defined is a Cauchy sequence and therefore converges to $z\in X$ since (X,d) is complete. But then

$$Ax_{2n},\ Sx_{2n},\ Bx_{2n-1},\ Tx_{2n-1}\to z. \tag{2.3}$$

Now suppose that A is continuous. Since A and S are compatible, we have

$$ASx_{2n},\ SAx_{2n},\ AAx_{2n}\to Az, \tag{2.4}$$

by Proposition 2.1. Since $A(X)\subseteq T(X)$, $Tu=Az$ for some $u\in X$. Then $d(SAx_{2n},Tu)\to 0$ and thus $d(AAx_{2n},Bu)\to d(Az,Bu)=0$, by (2.2) and (2.4). Consequently,

$$Bu=Tu=Az. \tag{2.5}$$

Moreover, since $B(X)\subseteq S(X)$, there exists $v\in X$ such that $\underline{Sv}=Bu=\underline{Tu}$, so that $Av=Bu$ by (2.2). From the preceding we infer, $Az=Bu=Tu=Sv=Av$, and we conclude by Proposition (2.2) that $p=Az$ is the desired common fixed point of A,B,S, and T. Of course, a common fixed point is assured by symmetry if B is continuous.

Now suppose that one of S or T, say S, is continuous. As above, (2.3) and Proposition 2.1 imply that

$$ASx_{2n},\ SAx_{2n},\ SSx_{2n}\to Sz. \tag{2.6}$$

In this instance, we use the fact that $A(X)\subseteq T(X)$ to produce $v_n\in X$ for each $n\in N$ such that

$Tv_n = ASx_{2n}$. Then (2.6) implies $d(SSx_{2n}, Tv_n) \to d(Sz, Sz) = 0$, so that (2.2) implies that $d(Sz, Bv_n) \leq d(Sz, ASx_{2n}) + d(ASx_{2n}, Bv_n) \to 0$; i.e., $Bv_n, Tv_n \to Sz$. Consequently, by (2.2) we can write $d(Bv_n, Az) \leq d(Tv_n, Sz) \to 0$, so that $Bv_n \to Az$ and $Bv_n \to Sz$; we conclude that $Az = Sz$ by "uniqueness of limits".

Again, since $A(X) \subseteq T(X)$, there exists $u \in X$ such that $Tu = Az = Sz$. Thus, $Bu = Az$ by (2.2). We have: $Az = Sz = Bu = Tu$, so that Proposition 2.2 implies that A, B, S, T have a common fixed point. By symmetry, the conclusion also holds if T is continuous.

We conclude by noting that the uniqueness of the common fixed point p follows easily from (2.2). □

COROLLARY 2.1 Let A, B, S, T be self maps of a complete metric space (X, d) such that the pairs A, S and B, T are compatible, and $A(X) \subseteq T(X), B(X) \subseteq S(X)$. If $\exists\ r \in (0,1)$ such that $d(Ax, By) \leq r\ d(Sx, Ty)$ for $x, y \in X$, then A, B, S, and T have a unique common fixed point provided one of these four functions is continuous.

PROOF. Let $\delta(\epsilon) = \epsilon / r$ for $\epsilon > 0$. Then $\delta: (0, \infty) \to (0, \infty)$ and $\delta(\epsilon) > \epsilon$ since $r < 1$. Also, $d(Sx, Ty) < \delta(\epsilon)$ implies $d(Ax, By) < r(\epsilon / r) = \epsilon$. □

Corollary 2.1 improves Corollary 3.2 in [1], which required that S and T be continuous. The following interesting corollary is Corollary 2.6 in [3] without the requirement that f be continuous.

COROLLARY 2.2. Let f be a bijective self map of a complete metric space (X, d). Suppose that for any $\epsilon > 0$, $\exists\ \delta > 0$ such that for all $x, y \in X$

$$\epsilon \leq d(fx, fy) < \epsilon + \delta \text{ implies } d(x, y) < \epsilon,$$

then f has a unique fixed point.

PROOF. The conclusion follows from Theorem 2.1 with $\delta(\epsilon) = \epsilon + \delta$, $f = S = T$, and $A = B = I$, the identity map, which is continuous and commutes with, and is therefore compatible with, any self map of X. □

3. AN EXAMPLE AND CONCLUSION.

It is natural to ask if we could drop all continuity requirements in Theorem 2.1 and still obtain the conclusion. The following example shows that this would be impossible.

EXAMPLE 3.1. Let $X = [0,1]$ and let d be the absolute value metric. Let

$Ax = Bx = (x/2$ for $x \in (0,1]$ and $1/2$ if $x = 0)$

$Sx = Tx = (x$ for $x \in (0,1]$ and 1 if $x = 0)$.

Then $A(X) = B(X) = (0, 1/2] \subseteq S(X) = T(X) = (0,1]$. A and S are compatible, since A and S commute. Moreover, $|Ax - By| = \frac{1}{2}|Sx - Ty|$ for all $x, y \in X$. Consequently, Corollary 2.1, and hence, Theorem 2.1, is false without continuity requirement on at least one function.

The literature abounds with attempts to generalize theorems which use inequalities of the form (i) in Definition 1.1 by substituting more elaborate expressions $M(x,y)$ for $d(Sx, Ty)$. In the instance in which only one function is continuous, as in Theorem 2.1, care should be exercised. For example, if for $d(Sx, Ty)$ in (i) we substitute $M(x,y) = max\ \{d(Sx, Ty),\ d(Sx, Ax),\ d(By, Ty)\}$, Theorem 2.1 is false. To see this, modify the above example by letting $T(0) = S(0) = 0$ and $A(0) = B(0) = 1$. (See also the paper [4] by Rao). In fact, this modified example is a counterexample to the main theorem, Theorem 1, in [5]. It is shown in a paper [6], which is yet to appear, that if the contractive definition in [5] is modified by introducing the function δ of our Definition 1.1 and requiring that δ be lower-semicontinuous, then Theorem 1 of [5] is valid. All of which suggests that the hypothesis of our Theorem 2.1 is quite tight.

We conclude by noting that Theorem 2.1 is true if we replace the continuity hypothesis with the requirement that one of $S(X)$ or $T(X)$ is complete in X. We refer you to Theorem 2.1 in [7]. In this context also the above example has a message, since $S(X) = T(X) = (0,1]$, which is not complete in $[0,1] = X$.

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