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RADIUS PROBLEMS FOR A SUBCLASS OF CLOSE-TO-CONVEX UNIVALENT FUNCTIONS

KHALIDA INAYAT NOOR

Mathematics Department
College of Science
King Saud University, Riyadh 11451
Saudi Arabia

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ABSTRACT. Let $P[A,B]$, $-1 \leq B < A \leq 1$, be the class of functions p such that $p(z)$ is subordinate to $\frac{1+Az}{1+Bz}$. A function f, analytic in the unit disk E is said to belong to the class $K^*_\beta[A,B]$ if, and only if, there exists a function g with $\frac{zg'(z)}{g(z)} \in P[A,B]$ such that $\mathrm{Re}\frac{(zf'(z))'}{g'(z)} > \beta$, $0 \leq \beta < 1$ and $z \in E$. The functions in this class are close-to-convex and hence univalent. We study its relationship with some of the other subclasses of univalent functions. Some radius problems are also solved.

KEY WORDS AND PHRASES. Close-to-convex, starlike univalent, convex, radius of convexity.
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1. INTRODUCTION.

Let f be analytic in $E = \{z: |z| < 1\}$ and be given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

A function g, analytic in E, is called subordinate to a function G if there exists a Schwarz function $w(z)$, analytic in E with $w(0) = 0$ and $|w(z)| < 1$ in E, such that $g(z) = G(w(z))$.

In [1], Janowski introduced the class $P[A,B]$. For A and B, $-1 \leq B < A \leq 1$, a function p, analytic in E with $p(0) = 1$ belongs to the class $P[A,B]$ if $p(z)$ is subordinate to $\frac{1+Az}{1+Bz}$. When $A = 1$, $B = -1$, we obtain the class P of functions with positive real part in E. Also for $A = 1-2\beta$, $B = -1$, $0 \leq \beta < 1$, we have the class $P(\beta)$. A function $h \in P(\beta)$, $0 \leq \beta < 1$ if and only if $\mathrm{Re}\, h(z) > \beta$, $z \in E$.

Let $S^*[A,B]$ and $C[A,B]$ denote the classes of functions, analytic in E, and given by (1.1) such that $\frac{zf'(z)}{f(z)} \in P[A,B]$ and $\frac{(zf'(z))'}{f'(z)} \in P[A,B]$ respectively. Also, for $B = -1$ and $A = 1-2\gamma$, $0 \leq \gamma < 1$, we have $S^*(\gamma)$ and $C(\gamma)$ the classes of starlike and convex functions of order γ, see [2].

Now we have the following:

DEFINITION 1.1: Let f be analytic in E and be given by (1.1). Then f is said to be in the class $K_\beta[A,B]$, $-1 \leq B < A \leq 1$ if and only if there exists a $g \in S^*[A,B]$ such that, for $z \in E$. $\frac{zf'(z)}{g(z)} \in P(\beta)$.

This class has been defined and studied by Silvia [3] in a more general way. When $B = -1$, $A = 1$ and $\beta = 0$, we have the class K of close-to-convex univalent functions.

DEFINITION 1.2.: Let f be analytic in E and be given by (1.1). Then $f \epsilon K^*_\beta[A,B]$ if and only there exists a $g \in S^*[A,B]$ such that $\frac{(zf'(z))'}{g'(z)} \in P(\beta)$ for $z \in E$.

For $\beta = 0$, $A = 1$ and $B = -1$, we obtain the class K^* discussed in [4].

If we take $g \in C[A,B]$ in Definition 1.2, we obtain the class $C^*_\beta[A,B]$. The special cases of this class have been investigated in [5, 6, 7].

We shall focus on the class $K^*_\beta[A,B]$ and establish the relationship of this class with some other subclasses of close-to-convex functions. It is clear that

$$C[A,B] \subset S^*[A,B] \subset K_\beta[A,B] \subset K$$

and

$$C[A,B] \subset C^*_\beta[A,B] \subset K^*_\beta[A,B] \subset K_\beta[A,B] \subset K$$

We shall also solve some radius problems for the functions in $K^*_\beta[A,B]$.

2. PRELIMINARY RESULTS.

We shall need the following:

LEMMA 2.1 [8]: If $f \in C(\gamma)$, then $f(z)$ is analytic, univalent and starlike of order $\lambda(\gamma)$ where, for $0 \leq \gamma < 1$,

$$\lambda(\gamma) = \begin{cases} \dfrac{4^\gamma(1-2\gamma)}{4-2^{2\gamma+1}}, & \gamma \neq \frac{1}{2} \\ (\log 4)^{-1}, & \gamma = \frac{1}{2} \end{cases}$$

This result is sharp.

LEMMA 2.2. Let $p \in P(\beta)$, $0 \leq \beta < 1$. Then

i) $p(z) = (1-\beta)h(z) + \beta$, $h \in P$ (see [2]).

ii) $|p'(z)| \leq \dfrac{2[\operatorname{Re} p(z) - \beta]}{1-r^2}$

iii) $\left|\dfrac{p'(z)}{p(z)}\right| \leq \dfrac{2(1-\beta)}{(1-r)((1-2\beta)r+1)}$

For (ii) and (iii), we refer to [9].

LEMMA 2.3. The radius of convexity of $S^*[A,B]$ is given by the smallest root r_o in $(0,1)$ of

i) $A^2r^2 - (3A-B)r + 1 = 0$ if $R_1 \leq R_2$

ii) $[(A-B) + 4A(1-A)]r^4 + 2[(A-B) + 2(1-A)^2]r^2 + (A-B)r - 4(1-A) = 0,$ if $R_2 \leq R_1$, where

$$R_1 = \left(\frac{L}{K}\right)^{1/2}, \quad R_2 = \frac{1-Ar}{1-Br}, \quad L = (1-A)(1+Ar^2),$$

and

$$K = (A-B)(1-r^2) + (1-B)(1+Br^2).$$

LEMMA 2.4. Let $p \in P[A,B]$. Then

$$\frac{1-Ar}{1-Br} \leq \operatorname{Re} p(z) \leq |p(z)| \leq \frac{1+Ar}{1+Br}$$

LEMMA 2.5. Let N and D be analytic in E, D maps onto a many-sheeted starlike region. $N(0) = 0 = D(0)$ and $\frac{N'(z)}{D'(z)} \in P[A,B]$. Then $\frac{N(z)}{D(z)} \in [A,B]$.

For the above two lemmas we refer to [11].

3. MAIN RESULTS.

From Definition 1.2 and Lemma 2.5, we clearly see that the function f belonging to $K^*_\beta[A,B]$ is close-to-convex and hence univalent. In fact, we can prove the following:

THEOREM 3.1. Let $f \in K^*_\beta[A,B]$, $0 \leq \beta < 1$. Then $f \in K_\sigma[A,B]$, where $\sigma(\beta)$ is given as

$$\sigma(\beta) = \begin{cases} \dfrac{4^\beta(1-2\beta)}{4-2^{2\beta+1}}, & \beta \neq \frac{1}{2} \\ (\log 4)^{-1}, & \beta = \frac{1}{2} \end{cases} \tag{3.1}$$

This result is sharp for $A = 1 - b\beta, \beta = 1$.

PROOF. Since $f \in K^*_\beta[A,B]$, there exists a $g \in S^*[A,B]$ such that, for $z \in E$,

$$\begin{aligned} \frac{(zf'(z))'}{g'(z)} &= (1-\beta)h(z) + \beta, \qquad h \in P \\ &= (1-\beta)\frac{z\phi'(z)}{\phi(z)} + \beta, \qquad \text{for some } \phi \in S^* \\ &= \frac{N'(z)}{D'(z)} \end{aligned} \tag{3.2}$$

So

$$\begin{aligned} \frac{N(z)}{D(z)} = \frac{zf'(z)}{g(z)} &= \frac{z\left(\frac{\phi(z)}{z}\right)^{1-\beta}}{\int_0^z \left(\frac{\phi(t)}{t}\right)^{1-\beta} dt} \\ &= \frac{1}{\int_0^z \left(\frac{z}{t}\right)^{1-\beta}\left[\frac{\phi(t)}{\phi(z)}\right]^{1-\beta} \frac{dt}{z}}, \end{aligned} \tag{3.3}$$

where we integrate along the straight line segment [0,2], $z \in E$. Using Lemma 2.5 for $B = -1$ and

$A = 1-2\beta$, we conclude that $\text{Re}\frac{N(z)}{D(z)} = \text{Re}\frac{zf'(z)}{g(z)} > \beta \geq 0$, and since $\frac{zf'(z)}{g(z)} = 1$ at $z = 0$, we have

$$\left|\frac{zf'(z)}{g(z)} - \frac{1+r^2}{1-r^2}\right| \leq \frac{2r}{1-r^2}, \tag{3.4}$$

$|z| = r, \quad z \in E$; see [12].

From (3.4) it is clear that

$$\underset{f \in K^*_\beta[A,B]}{\text{Min}} \ \underset{|z|=r}{\text{Min}} \ \text{Re}\frac{zf'(z)}{g(z)}$$

$$= \underset{f \in K^*_\beta[A,B]}{\text{Min}} \ \underset{|z|=r}{\text{Min}} \left|\frac{zf'(z)}{g(z)}\right|,$$

and hence it is sufficient to find the minimum of the right hand side of (3.3). Then from [8], we have

$$\sigma(\beta) = \min\left[\left|\int_o^z \left(\frac{z}{t}\right)^{1-\beta}\left(\frac{\phi(t)}{\phi(z)}\right)^{1-\beta}\frac{dt}{z}\right|\right]^{-1},$$

for $\phi \in S^*$, $z \in E$ and $\sigma(\beta)$ is as given in (3.1). This proves our result.

Sharpness for $A = 1-2\beta$, $B = 1$ follows by taking

$$f_\beta(z) = g_\beta(z) = \begin{cases} \dfrac{1-(1-z)^{2\beta-1}}{2\beta-1}, & \beta \neq \frac{1}{2} \\ \log(1-z)^{-1}, & \beta = \frac{1}{2} \end{cases}$$

Using Definition 1.2 and Lemma 2.1, we immediately have the following:

THEOREM 3.2. Let $f \in C^*_\beta[1-2\gamma, -1]$. Then $f \in K^*_\beta[1-2\lambda, 1]$, where $\lambda(\gamma)$ is as given in Lemma 2.1.

THEOREM 3.3. Let $f \in K^*_\beta[A,B]$. Then there exists a $g \in C[A,B]$ such that h defined by

$$h'(z) = \frac{(zf'(z))'}{1+\frac{zg''(z)}{g'(z)}}$$

belongs to $K_\beta[A,B]$, for $z \in E$.

PROOF. Since $f \in K^*_\beta[A,B]$, we have $\frac{(zf'(z))'}{G'(z)} \in P(\beta)$, $G \in S^*[A,B]$. Let $G(z) = zg'(z)$, so $g \in C[A,B]$. Now

$$G'(z) = (zg'(z))' = g'(z)\left[1+\frac{zg''(z)}{g'(z)}\right]$$

Thus

$$\frac{(zf'(z))'}{G'(z)} = \frac{(zf'(z))'}{g'(z)\left[1+\frac{zg''(z)}{g'(z)}\right]} = \frac{h'(z)}{g'(z)}$$

and this implies $h \in K_\beta[A,B]$.

We now deal with the radius problems.

THEOREM 3.4. Let $f \in K_\beta[A,B]$, $z \in E$. Then $f \in K^*_\beta[A,B]$ for $|z| < r_1$, where r_1 is the least positive root in $(0,1)$ of the equation

$$1-(A+2)r+(2B-1)r^2+Ar^3=0$$

PROOF. For $z \in E$, we can write

$$zf'(z)=g(z)h(z), \qquad h \in P(\beta) \text{ and } g \in S^*[A,B]\,.$$

Then

$$\frac{(zf'(z))'}{g'(z)}=h(z)+\frac{g(z)}{g'(z)}h'(z)\,,$$

from which it follows that

$$\mathrm{Re}\left[\frac{(zf'(z))'}{g'(z)}-\beta\right] \geq \mathrm{Re}\,h(z)-\beta-\left|\frac{g(z)}{g'(z)}h'(z)\right|.$$

Now, since $g \in S^*[A,B]$, it follows from Lemma 2.4 that

$$\left|\frac{g(z)}{g'(z)}\right| \leq \frac{r(1-Br)}{1-Ar}\,. \tag{3.5}$$

Using (3.5) and Lemma 2.2(ii) we have

$$\mathrm{Re}\left[\frac{(zf'(z))'}{g'(z)}-\beta\right] \geq \left[\mathrm{Re}\,h(z)-\beta\right]\left\{1-\frac{2r}{1-r^2}\frac{1-Br}{1-Ar}\right\}$$

$$=\left[\mathrm{Re}\,h(z)-\beta\right]\left[\frac{1-(A+2)r+(2B-1)r^2+Ar^3}{(1-r^2)\,(1-Ar)}\right]$$

and this gives us the required result.

THEOREM 3.5. Let $f \in K^*_\beta[A,B]$. Then $f \in C^*_\beta[1,-1]$ for $|z| < r_o$, where r_o is as given in Lemma 2.3.

PROOF. Since $f \in K^*_\beta[A,B]$ implies that $\frac{(zf'(z))'}{g'(z)} \in P(\beta)$, $g \in S^*[A,B]$, $z \in E$. To show that $f \in C^*_\beta[1,-1]$ for $|z| < r_o$, it is sufficient to prove that $g \in C[1,-1] \equiv C$ for $|z| < r_o$ and this follows immediately from Lemma 2.3. Hence the theorem.

THEOREM 3.6. Let $F = zf'$ and let $f \in K^*_\beta[A,B]$. Then F maps $|z| < r_2$ onto a convex domain, where r_2 is the least positive root in $(0,1)$ of the equation

$$(1-2\beta)r^3+(r_o+2)\,(2\beta-1)r^2-(2r_o+1)r+r_o=0\,,$$

and r_o as given in Lemma 2.3.

PROOF. $zF'(z) = z(zf'(z))' = zg'(z)h(z), \qquad h \in P(\beta),\ g \in S^*[A,B]$

Thus

$$\frac{(zF'(z))'}{F'(z)} = \frac{(zg'(z))'}{g'(z)} + \frac{zh'(z)}{h(z)},$$

and

$$\mathrm{Re}\frac{(zF'(z))'}{F'(z)} \geq \mathrm{Re}\frac{(zg'(z))'}{g'(z)} - \left|\frac{zh'(z)}{h(z)}\right|$$

Since $g \in S^*[A,B]$, it follows from Lemma 2.3 that $g \in C[1,-1] \equiv C$ for $|z| < r_o$. So we have, see [12],

$$\mathrm{Re}\frac{(zg'(z))'}{g'(z)} \geq \frac{r_o - r}{r_o + r} \tag{3.6}$$

Using (3.6) and Lemma 2.2(iii), we have

$$\mathrm{Re}\frac{(zF'(z))'}{F'(z)} \geq \frac{r_o - r}{r_o + r} - \frac{2r(1-\beta)}{(1-r)((1-2\beta)r+1)}$$

$$= \frac{(r_o - r)(1-r)((1-2\beta)r+1) - 2r(1-\beta)(r_o+r)}{(r_o+r)(1-r)((1-2\beta)r+1)}$$

After simplification we obtain the required result.

THEOREM 3.7. Let $F \in K^*_\beta[A,B]$ with respect to $G \in S^*[A,B]$, $0 \leq \beta < 1$. Let, for $0 < \alpha \leq \frac{1}{2}$,

$$f(z) = (1-\alpha)F(z) + \alpha zF'(z), \tag{3.7}$$

and

$$g(z) = (1-\alpha)G(z) + \alpha zG'(z). \tag{3.8}$$

Then $f \in K^*_\beta[A,B]$ with respect to g for $|z| < r$, where $r = \min(r_4, r_3)$ with $r_4 = \dfrac{1}{2\alpha + \sqrt{4\alpha^2 - 2\alpha + 1}}$ and r_3 the least positive root in $(0,1)$ of the equation

$$r_o + [1 - 2\alpha(1+r_o)]r - (r_o + 2\alpha)r^2 - (1-2\alpha)r^3 = 0 \tag{3.9}$$

The number $r_o \in (0,1)$ is given in Lemma 2.3.

PROOF. We can write (3.7) as

$$F(z) = \frac{1}{\alpha} z^{1-\frac{1}{\alpha}} \int_0^z z^{\frac{1}{\alpha}-2} f(z)\,dz.$$

So

$$zF'(z) = \frac{1}{\alpha} z^{1-\frac{1}{\alpha}}\left[(1-\tfrac{1}{\alpha})\int_0^z z^{\frac{1}{\alpha}-2} f(z)dz + z^{\frac{1}{\alpha}-1} f(z)\right]$$

$$= \frac{1}{\alpha} z^{1-\frac{1}{\alpha}}\left[\int_0^z z^{\frac{1}{\alpha}-1} f'(z)dz\right].$$

Thus

$$\frac{(aF'(z))'}{G'(z)} = \frac{z^{\frac{1}{\alpha}} f(z) - (\frac{1}{\alpha}-1)\int_0^z z^{\frac{1}{\alpha}-1} f'(z)dz}{(\frac{1}{\alpha}-1)\int_0^z z^{\frac{1}{\alpha}-1} g'(z)dz}$$

$$= (1-\beta)h(z) + \beta, \quad h \in P.$$

Differentiating both sides and simplifying, we obtain

$$\operatorname{Re}\left[\frac{(zf'(z))'}{g'(z)} - \beta\right] \geq (1-\beta)\operatorname{Re}h(z)\left[1 - \frac{2}{1-r^2}\left|\frac{\int_0^z z^{\frac{1}{\alpha}-1} g'(z)dz}{z^{\frac{1}{\alpha}-1} g'(z)}\right|\right] \tag{3.10}$$

Now

$$\frac{z^{\frac{1}{\alpha}-1} g'(z)}{\int_0^z z^{(\frac{1}{\alpha}-1)} g'(z)dz} = (\frac{1}{\alpha}-1) + \frac{(zG'(z))'}{G'(z)} \tag{3.11}$$

Using (3.6) and (3.11), the relation (3.10) yields

$$\operatorname{Re}\left[\frac{(zf'(z))'}{g'(z)} - \beta\right] \geq (1-\beta)\operatorname{Re}h(z)\left[1 - \frac{2}{1-r^2}\,\frac{\alpha r(r_o+r)}{r_o+(1-2\alpha)r}\right]$$

$$= (1-\beta)\operatorname{Re}h(z)\left[\frac{r_o(1-2\alpha-2\alpha\, r_o)r - (r_o+2\alpha)r^2 - (1-2\alpha)r^3}{(1-r^2)\left[r_o+(1-2\alpha)r\right]}\right] \tag{3.12}$$

Since it is known [13] that $g \in S^*[A,B]$ for $|z| < r_4 = \dfrac{1}{2\alpha + \sqrt{4\alpha^2 - 2\alpha + 1}}$, we obtain from (3.12) that $f \in K^*_\beta[A,B]$ for $|z| < r = \min(r_4, r_3)$, where r_3 is the least positive root of (3.9).

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