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A GENERAL EXISTENCE PRINCIPLE FOR FIXED POINT THEOREMS IN D-METRIC SPACES

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ABSTRACT. We establish two general principles for fixed point theorems in D-metric spaces, and then show that several theorems in D-metric spaces follow as corollaries of these general principles.

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1. Introduction. The concept of a D-metric space was introduced by the first author in [1]. A nonempty set X, together with a function $D : X \times X \times X \to [0,\infty)$ is called a D-metric space, denoted by (X,D) if D satisfies

(i) $D(x,y,z) = 0$ if and only if $x = y = z$ (coincidence),

(ii) $D(x,y,z) = D(p\{x,y,z\})$, where p is a permutation of x,y,z (symmetry),

(iii) $D(x,y,z) \le D(x,y,a)+D(x,a,z)+D(a,y,z)$ for all $x,y,z,a \in X$ (tetrahedral inequality).

The nonnegative real function D is called a D-metric on X. Some specific examples of D-metrics appear in [2]. A D-metric is a continuous function on X^3 in the topology of D-metric convergence, which is Hausdorff (see [5]).

In this paper, we establish two general fixed point principles for mappings in a D-metric space, which yield several fixed point theorems as corollaries.

2. Preliminaries. Let $f : X \to X$. The orbit of f at the point $x \in X$ is the set $O(x) = \{x, fx, f^2x, \dots\}$. An orbit of x is said to be bounded if there exists a constant $K > 0$ such that $D(u,v,w) \le K$ for all $u,v,w \in O(x)$. The constant K is called a D-bound of $O(x)$. A D-metric space X is said to be f-orbitally bounded if $O(x)$ is bounded for each $x \in X$. A sequence $x_n \subset X$ is said to be D-Cauchy if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that, for all $m > n$, $p \ge n_0$, $D(x_m,x_n,x_p) < \varepsilon$. A sequence $\{x_n\} \subset X$ is said to be D-convergent to a point $x \in X$ if, for each $\varepsilon > 0$ there exists a positive integer n_0 such that, for all $m,n \ge n_0$, $D(x_m,x_n,x) < \varepsilon$. An orbit $O(x)$ is called f-orbitally complete if every D-Cauchy sequence in $O(x)$ converges to a point in X.

Lemma 2.1 [4]. *Let $\{x_n\} \subset X$ be a bounded sequence with D-bound K satisfying*

$$D(x_n,x_{x+1},x_m) \le \lambda^n K \tag{2.1}$$

for all positive integers $m > n$, and some $0 \le \lambda < 1$. Then $\{x_n\}$ is D-Cauchy.

3. Main results

Theorem 3.1. *Let (X,D) be a D-metric spaces, f a selfmap of X. Suppose that there exists an $x_0 \in X$ such that $O(x_0)$ is D-bounded and f-orbitally complete. Suppose also that f satisfies*

$$D(fx,fy,fz) \le \lambda \max\{D(x,y,z), D(x,fx,z)\} \quad \text{for } x,y,z \in \overline{O(x_0)} \tag{3.1}$$

for some $0 \le \lambda < 1$. Then f has a unique fixed point in X.

Proof. Suppose there exists an n such that $x_n = x_{n+1}$. Then f has x_n as a fixed point in X. Therefore we may assume that all of the x_n are distinct.

We wish to show that, for any positive integers m,n, $m > n$, that $D(x_{n+1},x_{n+2},x_m) \le \lambda^n K$, where K is the D-bound of $O(x_0)$. The proof is by induction. From (3.1), for any m,

$$D(x_1,x_2,x_m) \le \lambda \max\{D(x_0,x_1,x_{m-1}), D(x_0,x_1,x_{m-1})\} \le \lambda K. \tag{3.2}$$

Again using (3.1),

$$D(x_2,x_3,x_m) \le \lambda \max\{D(x_2,x_3,x_{m-1}), D(x_1,x_2,x_{m-1})\}. \tag{3.3}$$

Using (3.2),

$$D(x_2,x_3,x_m) \le \lambda \max\{D(x_2,x_3,x_{m-1}), \lambda K\}. \tag{3.4}$$

Inequality (3.4) can be regarded as a recursion formula in m. Therefore

$$D(x_2,x_3,x_m) \le \lambda \max\{\lambda \max\{D(x_2,x_3,x_{m-2}),\lambda K\},\lambda K\} \le \lambda^2 K. \tag{3.5}$$

Assume the induction hypothesis. Then, from (3.1),

$$\begin{aligned} D(x_{n+1},x_{n+2},x_m) &\le \lambda \max\{D(x_{n+1},x_{n+2},x_{m-1}), D(x_n,x_{n+1},x_{m-1})\} \\ &\le \lambda \max\{D(x_{n+1},x_{n+2},x_{m-1}),\lambda^n K\}. \end{aligned} \tag{3.6}$$

Inequality (3.6) can be regarded as a recursion formula in m. Therefore,

$$\begin{aligned} D(x_{n+1},x_{n+2},x_m) &\le \lambda \max\{\lambda \max\{D(x_{n+1},x_{n+2},x_{m-2}),\lambda^n K\},\lambda^n K\} \\ &= \max\{\lambda^2 D(x_{n+1},x_{n+2},x_{m-2}),\lambda^{n+2}K,\lambda^{n+1}K\} \\ &= \max\{\lambda^2 D(x_{n+1},x_{n+2},x_{m-2}),\lambda^{n+1}K\} \\ &\le \max\{\lambda^2 \cdot \lambda \max\{D(x_{n+1},x_{n+2},x_{m-3}),\lambda^{n+1}K\},\lambda^{n+1}K\} \\ &= \max\{\lambda^3 D(x_{n+1},x_{n+2},x_{m-3}),\lambda^{n+1}K\} \le \cdots \\ &\le \max\{\lambda^n D(x_{n+1},x_{n+2},x_{m-n}),\lambda^{n+1}K\} \\ &\le \max\{\lambda^n \cdot \lambda \max\{D(x_{n+1},x_{n+2},x_{m-n-1}),\lambda^{n+1}K\},\lambda^{n+1}K\} \\ &= \lambda^{n+1}K, \end{aligned} \tag{3.7}$$

and $\{x_n\}$ is D-Cauchy by Lemma 2.1. Since X is x_0-orbitally complete, there exists a $p \in X$ with $\lim x_n = p$.

In (3.1) set $x = x_n$, $z = p$ to obtain

$$D(x_{n+1}, x_{n+1}, fp) \le \lambda \max\{D(x_n, x_n, p), D(x_n, x_{n+1}, p)\}. \tag{3.8}$$

Taking the limit of (3.8) as $n \to \infty$ yields $D(p,p,fp) \le \lambda D(p,p,p) = 0$, and $p = fp$.
To prove uniqueness, suppose that q is also a fixed point of f. Then, from (3.1),

$$D(p,p,q) = D(fp,fp,fq) \le \lambda \max\{D(p,p,q), D(p,fp,q)\} = \lambda D(p,p,q), \tag{3.9}$$

which implies that $p = q$. □

Corollary 3.2 [2, Theorem 2.1]. *Let f be a selfmap of a complete and bounded D-metric space X satisfying*

$$D(fx, fy, fz) \le \lambda D(x,y,z) \tag{3.10}$$

for all $x,y,z \in X$, for some $0 \le \lambda < 1$. Then f has a unique fixed point p, and f is continuous at p.

Proof. In (3.10) set $y = fx$ to obtain (3.1). Then, from Theorem 3.1, f has a unique fixed point p.

To prove continuity, let $\{z_n\} \subset X$ with $\lim z_n = p$. From (3.10),

$$D(p,p,fz_n) = D(fp,fp,fz_n) \le \lambda D(p,p,z_n). \tag{3.11}$$

Taking the limit as $n \to \infty$ gives $\limsup D(p,p,fz_n) = 0$, and $\liminf D(p,p,fz_n) = 0$ which implies that $\lim fz_n = p = fp$, and f is continuous at p. □

Corollary 3.3 [2, Corollary 1.1]. *Let f be a selfmap of a complete and bounded D-metric space satisfying the condition that there exists a positive integer q such that*

$$D(f^q x, f^q y, f^q z) \le \lambda D(x,y,z) \tag{3.12}$$

for all $x,y,z \in X$, for some $0 \le \lambda < 1$. Then f has a unique fixed point p, and f is f-orbitally continuous at p.

Proof. Define $T = f^q$. Then (3.12) reduces to (3.10), and T has a unique fixed point p by Corollary 3.2; i.e., $p = Tp = f^q p$. Thus $fp = f^{q+1}p = T(fp)$, and fp is also a fixed point of T. Uniqueness implies that $fp = p$, and p is a fixed point of f. Condition (3.12) implies the uniqueness of p as a fixed point of f.

For the continuity, let $\{z_n\} \subset O(f)$, with $\lim z_n = p$. From (3.12),

$$D(f^q p, f^q p, f^q z_n) \le \lambda D(p,p,z_n). \tag{3.13}$$

Taking the limit as $n \to \infty$ shows that $\lim f^q z_n = p = f^q p$, and f^q is f-orbitally continuous at p. But, since each $z_n \in O(f)$, $\lim f^q z_n = \lim f z_{n+q-1}$, and f is f-orbitally continuous at p. □

Corollary 3.4. *Let f be a selfmap of X, X an f-orbitally bounded and complete D-metric space satisfying*

$$D(fx,fy,fz) \le \alpha \left[\frac{1 + D(x,fx,z)}{1 + D(x,y,z)}\right] D(y,fy,z) + \beta D(x,y,z) \tag{3.14}$$

for all $x,y,z \in X$, $\alpha,\beta \geq 0$, $\alpha+\beta < 1$. *Then* f *has a unique fixed point* p *and* f *is continuous at* p.

Proof. In (3.14) set $y = fx$ to obtain

$$\begin{aligned} D(fx,f^2x,fz) &\leq \alpha D(fx,f^2x,z)+\beta D(x,fx,z) \\ &\leq \lambda \max\{D(fx,f^2x,z),D(x,fx,z)\}, \end{aligned} \tag{3.15}$$

where $\lambda = \alpha+\beta < 1$, and (3.1) is satisfied. The conclusion follows from Theorem 3.1.

To prove the continuity of f at p, let $\{z_n\} \subset X$ with $\lim z_n = p$. In (3.14) set $x = z = p$, $y = z_n$, to obtain

$$\begin{aligned} D(p,fz_n,p) &= D(fp,fz_n,fp) \\ &\leq \alpha\left[\frac{1+D(p,fp,p)}{1+D(p,z_n,p)}\right]D(z_n,fz_n,p)+\beta D(p,z_n,p) \\ &\leq \alpha D(z_n,fz_n,p)+\beta D(p,z_n,p). \end{aligned} \tag{3.16}$$

Taking the lim sup of both sides of (3.16) as $n \to \infty$ yields

$$D(p,\limsup fz_n,p) \leq \alpha D(p,\limsup fz_n,p), \tag{3.17}$$

which implies that $\limsup fz_n = p$. Similarly, taking the lim inf of both sides of (3.16) as $n \to \infty$ yields

$$D(p,\liminf fz_n,p) \leq \alpha D(p,\liminf fz_n,p), \tag{3.18}$$

which implies that $\liminf fz_n = p$. Therefore $\lim fz_n = p = fp$, and f is continuous at p. □

Corollary 3.5. *Let* f *be a selfmap of an* f*-orbitally bounded and complete* D*-metric space* X, q *a fixed positive integer. Suppose that* f *satisfies*

$$D(f^qx,f^qy,f^qz) \leq \alpha\left[\frac{1+D(x,f^qx,z)}{1+D(x,y,z)}\right]D(y,f^qy,z)+\beta D(x,y,z) \tag{3.19}$$

for all $x,y,z \in X$, *where* $\alpha,\beta \geq 0$, $\alpha+\beta < 1$. *Then* f *has a unique fixed point* p *and* f *is* f*-orbitally continuous at* p.

Proof. Set $T = f^q$. Then T satisfies (3.14). Therefore T has a unique fixed point at p, and is continuous at p. A standard argument then verifies that f has p as a unique fixed point. As in the proof of Corollary 3.3, f is f-orbitally continuous at p. □

4. α-condensing maps.

For any set A in a D-metric space X, the D-diameter of $A, \delta(A)$, is defined by $\delta(A) = \sup_{x,y,z\in A} D(x,y,z)$. The measure of noncompactness of a bounded set A in a D-metric space X is a nonnegative real number $\alpha(A)$ defined by

$$\alpha(A) = \inf\{\gamma > 0 : A = \cup_{i=1}^{n} : A_i \text{ for which } \delta(A_i) \leq \gamma \text{ for } i = 1,2,\ldots,n\}. \tag{4.1}$$

Definition 4.1. A selfmap f of X is called α-condensing if, for any bounded set A in X, $f(A)$ is bounded and $\alpha(f(A)) < \alpha(A)$ if $\alpha(A) > 0$.

Some authors refer to α-condensing maps as densifying maps.

Lemma 4.2. *Let $f : X \to X$, X an f-orbitally bounded and complete D-metric space, be α-condensing. Then $\overline{O(x)}$ is compact for each $x \in X$.*

Proof. Let $x \in X$ and define $A \subset X$ by $A = \{x_n\}$, where $x_n = f^n x$. Then

$$A = \{x, fx, f^2x, \dots\} = \{x\} \cup \{fx, f^2x, \dots\} = \{x\} \cup f(A). \tag{4.2}$$

Therefore, if A is not precompact, then $\alpha(A) = \alpha(f(A)) < \alpha(A)$, a contradiction. Therefore $\bar{A} = \overline{O(x)}$ is compact, since $\bar{A}$ is a complete D-metric space. □

Define $\delta(x,y,z) = \delta(O(x) \cup O(y) O(z))$

Theorem 4.3. *Let f be a continuous compact selfmap of a bounded D-metric space X, satisfying*

$$D(f^r x, f^s y, f^t z) < \delta(x,y,z) \text{ for each } x,y,z \in X, \text{ with two of } \{x,y,z\} \text{ distinct}, \tag{4.3}$$

where r, s, and t are fixed positive integers. Then f has a unique fixed point in X.

Proof. Since f is compact, there exists a compact subset Y of X containing fX. Then $fY \subset Y$ and $A := \cap_{n=1}^{\infty} f^n Y$ is a nonempty compact f-invariant subset of X which is mapped by f onto itself. A has the same properties with respect to f^r, f^s, and f^t.

Suppose that $\delta(A) > 0$. Since A is compact there exist $x,y,z \in A$ such that $\delta(A) = D(x,y,z)$. Since $fA = A$, there exist x', y', and z' in A such that $x = f^r x'$, $y = f^s y'$, and $z = f^t z'$. Then, from (4.3),

$$\delta(A) = D(x,y,z) = D(f^r x', f^s y', f^t z') < \delta(x,y,z) = \delta(A), \tag{4.4}$$

a contradiction. Therefore A consists of a single point, which is a fixed point of f.

Suppose p and q are fixed points of f, $p \neq q$. Then, from (4.3),

$$0 < D(p,p,q) = D(f^r p, f^s p, f^t q) < D(p,p,q), \tag{4.5}$$

a contradiction. Therefore the fixed point is unique. □

Corollary 4.4 [8, Theorem 2]**.** *Let X be a compact D-metric space, f a continuous selfmap of X satisfying*

$$\begin{aligned} &D(fx,fy,fz) \\ &\quad < \max\{D(x,y,z), D(x,fx,z), D(y,fy,z), D(x,fy,z), D(y,fx,z)\} D(p,p,q) \end{aligned} \tag{4.6}$$

for all $x,y,z \in X$ with $x \neq fx$, $y \neq fy$, or $z \neq fz$. Then f has a unique fixed point p in X.

Proof. Inequality (4.6) implies that $D(fx,fy,fz) < \delta(x,y,z)$, and the existence and uniqueness of a fixed point p follows from Theorem 4.3.

For continuity, let $\{z_n\} \subset X$ with $z_n \neq p$ for each n and $\lim z_n = p$. From (4.6)

$$D(p,p,fz_n) = D(fp,fp,fz_n) < D(p,fp,z_n). \tag{4.7}$$

Taking the limit as $n \to \infty$ implies that f is continuous at p. □

Theorem 4.5. *Let f be an f-orbitally continuous α-condensing selfmap of a complete bounded D-metric space X. Let $a \in X$. If (4.3) holds on $\overline{O(a)}$, then f has a unique fixed point $p \in \overline{O(a)}$, and $\lim_n f^n x = p$ for each $x \in \overline{O(a)}$.*

Proof. From Lemma 4.2, $\overline{O(a)}$ is compact. Since f is a continuous α-condensing selfmap of $\overline{O(a)}$, f is compact. Now apply Theorem 4.3. □

Corollary 4.6. *Let f be a continuous α-condensing selfmap of a complete bounded D-metric space X satisfying (4.6) for all $x,y,z \in X$ with $x \neq fx$, $y \neq fy$, or $z \neq fz$. Then f has a unique fixed point p in X.*

As in the proof of Corollary 4.4, $D(fx,fy,fz) < \delta(x,y,z)$ and the result follows from Theorem 4.5.

Theorem 4.7. *Let f be a selfmap of a D-metric space X. Suppose that there exists a point $a \in X$ with $\overline{O(a)}$ bounded and complete. Suppose that f is continuous and α-condensing on $\overline{O(a)}$ and satisfies (4.3) for each $x,y,z \in \overline{O(a)}$ with two of $\{x,y,z\}$ distinct, and $x \neq fx$, $y \neq fy$, $z \neq fz$. Then f has a fixed point in $\overline{O(a)}$.*

Proof. By Lemma 4.2 $\overline{O(a)}$ is compact. If there exists some integer n for which $f^n a = f^{n+1} a$, then f has a fixed point in $\overline{O(a)}$. Assume that $f^n a \neq f^{n+1} a$ for each n. Note that f, restricted to $\overline{O(a)}$ is a continuous compact selfmap of $\overline{O(a)}$. Suppose that $u \neq fu$ for each cluster point u of $\overline{O(a)}$. Then f satisfies condition (4.3) for all $x,y,z \in \overline{O(a)}$, with two of $\{x,y,z\}$ distinct. Therefore, by Theorem 4.3, f, restricted to $\overline{O(a)}$, has a unique fixed point $p \in \overline{O(a)}$. This contradicts the assumption that $u \neq fu$ for each cluster point u of $\overline{O(a)}$. Therefore $u = fu$ for some cluster point $u \in \overline{O(a)}$.

The proofs of Theorems 4.3, 4.5, and 4.7 are very similar to their metric space counterparts in [6] and [7], but have been given here for completeness.

The following results are proved using the proof technique analogous to the corresponding metric space theorems. □

Theorem 4.8. *Let f be a selfmap of X, an f-orbitally bounded and complete D-metric space. Suppose that f is α-condensing, f-orbitally continuous and satisfies*

$$D(fx,fy,fz) < \alpha\left[\frac{1+D(x,fx,z)}{1+D(x,y,z)}\right]D(y,fy,z) + \beta D(x,y,z) = M(x,y,z) \tag{4.8}$$

for all $x,y,z \in X$ with $x \neq fx$, $y \neq fy$, $z \neq fz$, where $\alpha,\beta > 0$, $\alpha+\beta \leq 1$. Then f has a unique fixed point $p \in X$ and f is continuous at p.

Proof. If $\alpha+\beta < 1$, the result follows from Corollary 3.4. Therefore we assume that $\alpha+\beta = 1$. Let $x_0 \in X$ and define $x_{n+1} = fx_n$, $n \geq 0$. From Lemma 4.2 it follows that $\overline{O(x_0)}$ is compact. Obviously $f : \overline{O(x_0)} \to \overline{O(x_0)}$.

Case I. There exists some $x,y,z \in \overline{O(x_0)}$ for which $M = 0$. Then $y = fy = z = x$, and y is a fixed point of f. Inequality (4.8) implies uniqueness.

Case II. $M \neq 0$ for all $x,y,z \in \overline{O(x_0)}$. Define a function $F : (\overline{O(x_0)})^3 \to [0,\infty)$ by

$$F(x,y,z) = \frac{D(fx,fy,fz)}{M(x,y,z)}. \tag{4.9}$$

The function F is well defined on $(\overline{O(x_0)})^3$ since $M \neq 0$ on $\overline{O(x_0)}$.

Since F is continuous on $\overline{O(x_0)}$, it attains its maximum value at some point $(u,v,w) \in \overline{O(x_0)}$. We call this maximum value c. From (4.8) it follows that $0 < c < 1$. Therefore

$$\begin{aligned} D(fx,fy,fz) &\leq cM(x,y,z) \\ &\leq \alpha' \left[\frac{1+D(x,fx,z)}{1+D(x,y,z)}\right] D(y,fy,z) + \beta' D(x,y,z) \end{aligned} \tag{4.10}$$

for all $x,y,z \in \overline{O(x_0)}$, where $\alpha' = c\alpha > 0$, $\beta' = c\beta > 0$, and $\alpha' + \beta' = c(\alpha+\beta) < 1$. Since $\overline{O(x_0)}$ is compact, it is bounded and complete. The result follows from Corollary 3.4. □

Corollary 4.9. *Let f be a selfmap of a complete and f-orbitally bounded D-metric space. Suppose that f is α-condensing and f-orbitally continuous. Let q be a positive integer. Suppose that f satisfies*

$$D(f^q x, f^q y, f^q z) < \alpha \left[\frac{1+D(x,f^q x,z)}{1+D(x,y,z)}\right] D(y,f^q y,z) + \beta D(x,y,z) \tag{4.11}$$

for all $x,y,z \in X$ for which the right-hand side of (4.11) is not zero, where $\alpha,\beta > 0$, $\alpha+\beta \leq 1$. Then f has a unique fixed point p and f is f-orbitally continuous at p.

Proof. Set $T = f^q$. Then T satisfies (4.8), and the existence and uniqueness of the fixed point p, for T, follows from Theorem 4.8. It then follows that p is the unique fixed point for f. The continuity argument is the same as that used in the proof of Corollary 3.3. □

Corollary 4.10. *Let f be a continuous selfmap of a compact D-metric space satisfying (4.8). Then f has a unique fixed point p, and f is continuous at p.*

This result is an immediate consequence of Theorem 4.8.

Corollary 4.10 includes [3, Theorem 2.2] *as a special case.*

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