Research Article

Entropy Measures for Interval-Valued Intuitionistic Fuzzy Sets and Their Application in Group Decision-Making

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Entropy measure is an important topic in the fuzzy set theory and has been investigated by many researchers from different points of view. In this paper, two new entropy measures based on the cosine function are proposed for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets. According to the features of the cosine function, the general forms of these two kinds of entropy measures are presented. Compared with the existing ones, the proposed entropy measures can overcome some shortcomings and be used to measure both fuzziness and intuitionism of these two fuzzy sets; as a result, the uncertain information of which can be described more sufficiently. These entropy measures have been applied to assess the experts' weights and to solve multicriteria fuzzy group decision-making problems.

1. Introduction

Since Zadeh introduced fuzzy set theory [1], some generalized forms have been proposed and studied to treat imprecision and uncertainty [2–10]. Atanassov proposed the notions of intuitionistic fuzzy sets (IFSs) [2] and interval-valued intuitionistic fuzzy sets (IVIFSs) [3]; Gau and Buehrer [4] introduced the notion of vague sets. And some authors [5–7] pointed out that the IFS theory and the vague set theory are equivalent to the interval-value fuzzy set (IVFS) theory proposed by Zadeh [11].

As two important topics in the FS theory, entropy measures and similarity measures of fuzzy sets have been investigated widely by many researchers from different points of view. The entropy of a fuzzy set describes the fuzziness degree of the fuzzy set. de Luca and Termini [12] introduced some axioms which captured people's intuitive comprehension to describe the fuzziness degree of a fuzzy set. Kaufmann [13] proposed a method for measuring the fuzziness degree of a fuzzy set by a metric distance between its membership function and the membership function of its nearest crisp set. Yager [14] suggested the entropy measure being expressed by the distance between a fuzzy set and its complement. Chiu and Wang [15] gave simple calculation for entropies of fuzzy numbers in addition and extension principle. They [16] also investigated the entropy relationship between the original fuzzy set and the image fuzzy set. Hong and Kim [17] introduced a simple method for calculating the entropy of the image fuzzy set without calculating its membership function. Zeng and Li [18] showed that similarity measures and entropies of fuzzy sets can be transformed to each other based on their axiomatic definitions.

Aimed at these important numerical indexes in the fuzzy set theory, some researchers extended these concepts to the IVFS theory and IFS theory and investigated their related topics from different points of view [19–24]. We review some generalization study on entropy measure. Burillo and Bustince [25] introduced the notions of entropy on IVFSs and IFSs to measure the degree of intuitionism of an IVFS or an IFS. Szmidt and Kacprzyk [26] proposed a nonprobabilistic-type entropy measure with a geometric interpretation of IFSs. Hung and Yang [27] gave their axiomatic definitions of entropies of IFSs and IVFSs by exploiting the concept of probability. Farhadinia [20] generalized some results on

the entropy of IVFSs based on the intuitionistic distance and its relationship with similarity measure. After that, many authors also proposed different entropy formulas for IFSs [27–31], IVFSs [32, 33], and vague sets [34]. For IVIFSs, Liu et al. in [35] proposed a set of axiomatic requirements for entropy measures, which extended Szmidt and Kacprzyk's axioms formulated for entropy of IFSs [26]. Wei et al. in [36] extended the entropy measure of IFSs proposed in [26] to IVIFSs and gave an approach to construct similarity measures by entropy measures of IVIFSs. By this approach, the proposed entropy measure can yield a similarity measure of IVIFSs, which has been applied in the context of pattern recognition, multiple-criteria fuzzy decision-making, and medical diagnosis. For entropy measures of IVIFSs, we refer to [37–39].

In [31], Vlachos and Sergiadis revealed an intuitive and mathematical connection between the notions of entropy for FSs and IFSs in terms of fuzziness and intuitionism. They pointed out that entropy for FSs is indeed a measure of fuzziness, while for IFSs, entropy can measure both fuzziness and intuitionism. Recall that the fuzziness is dominated by the difference between membership degree and nonmembership degree, and the intuitionism is dominated by the hesitation degree. Hence it is very interesting to construct entropy formulas measuring both fuzziness and intuitionism. We propose an entropy measure, as well as its general form, for IFSs and then generalize it to IVIFSs. Our entropy measures are compared with some existing ones in [29–31, 37]. As an application in multicriteria fuzzy group decision-making, we propose a method to assess the experts' weights by the proposed entropy measures.

The rest of the paper is organized as follows. Section 2 reviews some necessary concepts of IFSs and IVIFSs. In Section 3, we propose a new entropy measure and its general form for IFSs. Then we compare the proposed entropy measure with some existing ones and give some conditions under which these existing entropy measures may not work as desired, while the proposed entropy measure can do well. In Section 4, we extend the entropy measures defined in Section 3 to IVIFSs and propose entropy measures for IVIFSs. These entropy measures are compared with the ones defined by Ye in [37]. Section 5 gives the application of the proposed entropy measures in assessing the weights of experts. Concluding remarks are drawn in Section 6.

2. Preliminaries

Some basic concepts of intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets are reviewed from Atanassov [2], Atanassov and Gargov [3], and Xu [40].

Definition 1 (see [2]). Let X be a universe of discourse. An intuitionistic fuzzy set in X is an object having the form

$$A = \{\langle x, u_A(x), v_A(x) \rangle \mid x \in X\}, \tag{1}$$

where

$$u_A : X \longrightarrow [0, 1], \qquad v_A : X \longrightarrow [0, 1] \tag{2}$$

with the condition

$$0 \leq u_A(x) + v_A(x) \leq 1, \quad \forall x \in X. \tag{3}$$

The numbers $u_A(x)$ and $v_A(x)$ denote the degree of membership and nonmembership of x to A, respectively.

For convenience of notations, we abbreviate "intuitionistic fuzzy set" to IFS and denote by IFS(X) the set of all IFSs in X.

For each IFS A in X, we call $\pi_A(x) = 1 - u_A(x) - v_A(x)$ the intuitionistic index of x in A, which denotes the hesitancy degree of x to A. The complementary set of A is defined as

$$A^C = \{\langle x, v_A(x), u_A(x) \rangle \mid x \in X\}. \tag{4}$$

Definition 2 (see [2]). For two IFSs $A = \{\langle x, u_A(x), v_A(x) \rangle \mid x \in X\}$ and $B = \{\langle x, u_B(x), v_B(x) \rangle \mid x \in X\}$, their relations are defined as follows:

(1) $A \subseteq B$ if and only if $u_A(x) \leq u_B(x)$, $v_A(x) \geq v_B(x)$, for each $x \in X$;

(2) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Consider that, sometimes, it is not approximate to assume that the membership degrees for certain elements of an IFS are exactly defined, but a value range can be given. In such cases, Atanassov and Gargov [3] introduced the following notion of interval-valued intuitionistic fuzzy sets.

Definition 3 (see [3]). Let X be a universe of discourse and int$(0, 1)$ denote all closed subintervals of the interval $[0, 1]$. An interval-valued intuitionistic fuzzy set A in X is an object having the form

$$A = \{\langle x, u_A(x), v_A(x) \rangle \mid x \in X\}, \tag{5}$$

where

$$u_A : X \longrightarrow \text{int}(0, 1), \qquad v_A : X \longrightarrow \text{int}(0, 1) \tag{6}$$

with the condition

$$0 \leq \sup(u_A(x)) + \sup(v_A(x)) \leq 1. \tag{7}$$

The intervals $u_A(x)$ and $v_A(x)$ denote the degree of membership and nonmembership of x to A, respectively.

For convenience, let $u_A(x) = [u_A^-(x), u_A^+(x)]$, $v_A(x) = [v_A^-(x), v_A^+(x)]$, so

$$A = \{\langle x, [u_A^-(x), u_A^+(x)], [v_A^-(x), v_A^+(x)] \rangle \mid x \in X\} \tag{8}$$

with the condition $0 \leq u_A^+(x) + v_A^+(x) \leq 1$.

We abbreviate "interval-valued intuitionistic fuzzy set" to IVIFS and denote by IVIFS(X) the set of all IVIFSs in X.

We call the interval

$$[1 - u_A^+(x) - v_A^+(x), 1 - u_A^-(x) - v_A^-(x)], \tag{9}$$

abbreviated by $[\pi_A^-(x), \pi_A^+(x)]$ or $\pi_A(x)$, the interval-valued intuitionistic index of x in A, which is a hesitancy degree of x to A.

Clearly, if $u_A(x) = u_A^-(x) = u_A^+(x)$ and $v_A(x) = v_A^-(x) = v_A^+(x)$, then the given IVIFS A is reduced to an ordinary IFS.

Definition 4 (see [25]). Let int(0, 1) denote all closed subintervals of the interval [0, 1]. For $[a_1, b_1], [a_2, b_2] \in$ int(0, 1), we define

$$[a_1, b_1] \leq [a_2, b_2] \text{ if and only if } a_1 \leq a_2, b_1 \leq b_2;$$

$$[a_1, b_1] \preceq [a_2, b_2] \text{ if and only if } a_1 \leq a_2, b_1 \geq b_2;$$

$$[a_1, b_1] = [a_2, b_2] \text{ if and only if } a_1 = a_2, b_1 = b_2.$$

Definition 5 (see [3]). For two IVIFSs $A = \{\langle x, [u_A^-(x), u_A^+(x)], [v_A^-(x), v_A^+(x)] \rangle \mid x \in X\}$ and $B = \{\langle x, [u_B^-(x), u_B^+(x)], [v_B^-(x), v_B^+(x)] \rangle \mid x \in X\}$, their relations and operations are defined as follows:

(1) $A \subseteq B$ if and only if $[u_A^-(x), u_A^+(x)] \leq [u_B^-(x), u_B^+(x)]$, $[v_A^-(x), v_A^+(x)] \geq [v_B^-(x), v_B^+(x)]$, for each $x \in X$;

(2) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$;

(3) $A^C = \{\langle x, [v_A^-(x), v_A^+(x)], [u_A^-(x), u_A^+(x)] \rangle \mid x \in X\}$.

In [40], $\alpha = ([a, b], [c, d])$ is called an interval-valued intuitionistic fuzzy value (IVIFV), where $[a, b] \in$ int(0, 1), $[c, d] \in$ int(0, 1), and $b + d \leq 1$. Let Θ be the universal set of IVIFVs.

Based on two functions, Xu [40] provided a method to compare two IVIFVs.

Definition 6 (see [40]). Let $\alpha = ([a_1, b_1], [c_1, d_1])$ and $\beta = ([a_2, b_2], [c_2, d_2])$ be two IVIFVs. Let $s(\alpha) = (1/2)(a_1 - c_1 + b_1 - d_1)$ and $s(\beta) = (1/2)(a_2 - c_2 + b_2 - d_2)$ be the score degrees of α and β, respectively; let $h(\alpha) = (1/2)(a_1 + b_1 + c_1 + d_1)$ and $h(\beta) = (1/2)(a_2 + b_2 + c_2 + d_2)$ be the accuracy degrees of α and β, respectively. Then we have the following.

(1) If $s(\alpha) < s(\beta)$, then α is smaller than β, denoted by $\alpha < \beta$.

(2) If $s(\alpha) = s(\beta)$, then the following hold:

   (a) if $h(\alpha) = h(\beta)$, then α is indifferent to β, denoted by $\alpha \sim \beta$;

   (b) if $h(\alpha) < h(\beta)$, then α is smaller than β, denoted by $\alpha < \beta$;

   (c) if $h(\alpha) > h(\beta)$, then α is bigger than β, denoted by $\alpha > \beta$.

Definition 7 (see [40]). Let $\alpha = ([a_1, b_1], [c_1, d_1])$ and $\beta = ([a_2, b_2], [c_2, d_2])$ be two IVIFVs. Then three operational laws of IVIFVs are given as follows:

(1) $\alpha \oplus \beta = ([a_1 + a_2 - a_1 a_2, b_1 + b_2 - b_1 b_2], [c_1 c_2, d_1 d_2])$;

(2) $\lambda \alpha = ([1 - (1 - a_1)^\lambda, 1 - (1 - b_1)^\lambda], [c_1^\lambda, d_1^\lambda]), \lambda \geq 0$;

(3) $\alpha^C = ([c_1, d_1], [a_1, b_1])$.

With the thorough research of IVIFS theory and the continuous expansion of its application scope, it is more and more important to aggregate intuitionistic fuzzy information effectively. Xu [40] proposed interval-valued intuitionistic fuzzy weighted averaging operator to aggregate the interval-valued intuitionistic fuzzy information.

Definition 8 (see [40]). Let $\alpha_i = ([a_i, b_i], [c_i, d_i])$ $(i = 1, 2, \ldots, n)$ be a collection of IVIFVs. An interval-valued intuitionistic fuzzy weighted averaging (IVIFWA) operator is a mapping $\Theta^n \to \Theta$, such that

$$\begin{aligned}
&\text{IVIFWA}\,(\alpha_1, \alpha_2, \ldots, \alpha_n) \\
&= w_1 \alpha_1 \oplus w_2 \alpha_2 \oplus \cdots \oplus w_n \alpha_n \\
&= \left(\left[1 - \prod_{j=1}^n (1 - a_j)^{w_j}, 1 - \prod_{j=1}^n (1 - b_j)^{w_j} \right], \right. \\
&\qquad \left. \left[\prod_{j=1}^n c_j^{w_j}, \prod_{j=1}^n d_j^{w_j} \right] \right),
\end{aligned} \tag{10}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the weighting vector of α_i $(i = 1, 2, \ldots, n)$ with $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$.

In Definition 8, if $a_i = b_i$ and $c_i = d_i$ $(i = 1, 2, \ldots, n)$, then IVIFVs α_i reduce to IFVs and the IVIFWA operator reduces the IFWA operator.

In many practical problems, such as, multicriteria decision-making, the study objects are finite. So in the rest of the paper, we assume that the universe X is a finite set, listed by $\{x_1, x_2, \ldots, x_n\}$.

3. Entropy Measures for IFSs

In this section we will propose a concrete entropy measure for IFSs and demonstrate its efficiency through comparisons with some existing entropy measures in [28–31].

3.1. A New Entropy Measure for IFSs. Szmidt and Kacprzyk [26] extended the axioms of de Luca and Termini [12] to propose the following definition of an entropy measure for IFSs.

Definition 9 (see [26]). A real-valued function E: IFS(X) $\to$ [0, 1] is called an entropy measure for IFSs if it satisfies the following axiomatic requirements:

(E1) $E(A) = 0$ if and only if A is a crisp set; that is, $u_A(x_i) = 1$ or $u_A(x_i) = 0$ for any $x_i \in X$;

(E2) $E(A) = 1$ if and only if $u_A(x_i) = v_A(x_i)$ for all $x_i \in X$;

(E3) $E(A) = E(A^C)$;

(E4) $E(A) \leq E(B)$ if $u_A(x_i) \geq u_B(x_i)$ and $v_B(x_i) \geq v_A(x_i)$ for $u_B(x_i) \geq v_B(x_i)$ or $u_A(x_i) \leq u_B(x_i)$ and $v_B(x_i) \leq v_A(x_i)$ for $u_B(x_i) \leq v_B(x_i)$ for any $x_i \in X$.

In this subsection, we introduce a new entropy measure for IFSs. For each $A \in$ IFS(X), define $E(A)$ by

$$\frac{1}{n} \sum_{i=1}^n \cos \frac{u_A(x_i) - v_A(x_i)}{2(1 + \pi_A(x_i))} \pi. \tag{11}$$

Then we have the following theorem.

Theorem 10. *The mapping E, defined by (11), is an entropy measure for IFSs.*

Proof. It is sufficient to show that the mapping $E(\cdot)$, defined by (11), satisfies the conditions (E1)–(E4) in Definition 9.

Let $E_i(A) = \cos((u_A(x_i) - v_A(x_i))/2(1 + \pi_A(x_i)))\pi$ for $i = 1, 2, \ldots, n$. From $0 \le u_A(x_i) \le 1$, $0 \le v_A(x_i) \le 1$, and $0 \le \pi_A(x_i) \le 1$, we have $-\pi/2 \le ((u_A(x_i) - v_A(x_i))/2(1 + \pi_A(x_i)))\pi \le \pi/2$. Thus $0 \le E_i(A) \le 1$.

(E1) Let A be a crisp set; that is, $u_A(x_i) = 0$, $v_A(x_i) = 1$, or $u_A(x_i) = 1$, $v_A(x_i) = 0$ for any $x_i \in X$. No matter in which case, we have $E_i(A) = 0$. Hence $E(A) = 0$.

Conversely, suppose now that $E(A) = 0$. Since $0 \le E_i(A) \le 1$, it follows that $E_i(A) = 0$. Also since $-\pi/2 \le ((u_A(x_i) - v_A(x_i))/2(1 + \pi_A(x_i)))\pi \le \pi/2$, we have $((u_A(x_i) - v_A(x_i))/2(1 + \pi_A(x_i)))\pi = \pi/2$ or $((u_A(x_i) - v_A(x_i))/2(1 + \pi_A(x_i)))\pi = -\pi/2$. Thus we can obtain $u_A(x_i) = 1$, $v_A(x_i) = 0$ or $u_A(x_i) = 0$, $v_A(x_i) = 1$. So A is a crisp set.

(E2) Let $u_A(x_i) = v_A(x_i)$ for each $x_i \in X$. Applying this condition to (11), we easily obtain $E(A) = 1$.

Conversely, we suppose that $E(A) = 1$. From (11) and $0 \le E_i(A) \le 1$, we obtain that $E_i(A) = 1$ for each $x_i \in X$. Also from $-\pi/2 \le ((u_A(x_i) - v_A(x_i))/2(1 + \pi_A(x_i)))\pi \le \pi/2$, we have $u_A(x_i) = v_A(x_i)$ for each $x_i \in X$.

(E3) For $A^C = \{\langle x_i, v_A(x_i), u_A(x_i)\rangle \mid x_i \in X\}$, we can easily get that $E(A^C) = E(A)$.

(E4) Suppose that $u_A(x_i) \ge u_B(x_i)$ and $v_B(x_i) \ge v_A(x_i)$ for $u_B(x_i) \ge v_B(x_i)$. Since $1 - u_A(x_i) \ge 0$ and $v_A(x_i) - 1 \le 0$, we have $v_B(x_i)(1 - u_A(x_i)) \ge v_A(x_i)(1 - u_A(x_i))$ and $u_B(x_i)(v_A(x_i) - 1) \ge u_A(x_i)(v_A(x_i) - 1)$. Thus

$$v_B(x_i)(1 - u_A(x_i)) + u_B(x_i)(v_A(x_i) - 1)$$
$$\ge v_A(x_i)(1 - u_A(x_i)) + u_A(x_i)(v_A(x_i) - 1). \tag{12}$$

It follows that

$$v_B(x_i)(1 - u_A(x_i)) + u_B(x_i)(v_A(x_i) - 1)$$
$$+ u_A(x_i) - v_A(x_i) \ge 0. \tag{13}$$

Thus

$$(u_A(x_i) - v_A(x_i))(2 - u_B(x_i) - v_B(x_i))$$
$$\ge (u_B(x_i) - v_B(x_i))(2 - u_A(x_i) - v_A(x_i)), \tag{14}$$

which implies that $(u_A(x_i) - v_A(x_i))/(1 + \pi_A(x_i)) \ge (u_B(x_i) - v_B(x_i))/(1 + \pi_B(x_i))$. Thus $E_i(A) \le E_i(B)$.

Similarly, when $u_A(x_i) \le u_B(x_i)$ and $v_B(x_i) \le v_A(x_i)$ for $u_B(x_i) \le v_B(x_i)$, we can also prove that $E_i(A) \le E_i(B)$. Hence we have $E(A) \le E(B)$. □

Analyzing the features of the cosine function, we give the following general form of the entropy measure E defined in (11), which is suggested by the referee.

Theorem 11. *Let $f : [-1, 1] \rightarrow [0, 1]$ be an even function such that f is strictly monotone increasing on $[0, 1]$, $f(-1) = f(1) = 0$, and $f(0) = 1$. For an IFS $A = \{\langle x_i, u_A(x_i), v_A(x_i)\rangle \mid x_i \in X\}$, let*

$$E_f(A) = \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{u_A(x_i) - v_A(x_i)}{1 + \pi_A(x_i)}\right). \tag{15}$$

Then E_f is an entropy measure for IFSs.

Proof. The process of the proof is similar to that for Theorem 10. We omit it. □

There are many functions, for example, $f(x) = 1 - |x|$, $f(x) = 1 - x^2$, or $f(x) = \cos(x/2)$, satisfying the requirements in Theorem 11. Clearly, different functions give rise to different entropy measures for IFSs.

3.2. Comparison with Existing Entropy Measures. For an IFS A in X, Ye [29] proposed two entropy measures $J_1(A)$ and $J_2(A)$:

$$J_1(A) = \frac{1}{n}\sum_{i=1}^{n}\left\{\left[\sin\frac{1 + u_A(x_i) - v_A(x_i)}{4}\pi\right.\right.$$
$$+ \sin\frac{1 - u_A(x_i) + v_A(x_i)}{4}\pi - 1\right]$$
$$\left.\times\frac{1}{\sqrt{2} - 1}\right\},$$

$$J_2(A) = \frac{1}{n}\sum_{i=1}^{n}\left\{\left[\cos\frac{1 + u_A(x_i) - v_A(x_i)}{4}\pi\right.\right.$$
$$+ \cos\frac{1 - u_A(x_i) + v_A(x_i)}{4}\pi - 1\right]$$
$$\left.\times\frac{1}{\sqrt{2} - 1}\right\}. \tag{16}$$

The following proposition shows that these two formulas are the same.

Proposition 12. *For each $A = \{\langle x_i, u_A(x_i), v_A(x_i)\rangle \mid x_i \in X\}$ in IFS(X), let*

$$J(A) = \frac{1}{n}\sum_{i=1}^{n}\left\{\left[\sqrt{2}\cos\frac{u_A(x_i) - v_A(x_i)}{4}\pi - 1\right]\times\frac{1}{\sqrt{2} - 1}\right\}. \tag{17}$$

Then $J_1(A) = J_2(A) = J(A)$.

Proof. By the properties of trigonometric functions, we have, for each $A = \{\langle x_i, u_A(x_i), v_A(x_i)\rangle \mid x_i \in X\}$,

$$J_1(A) = \frac{1}{n}\sum_{i=1}^{n}\left\{\left[\sin\frac{1+u_A(x_i)-v_A(x_i)}{4}\pi\right.\right.$$

$$+\cos\left(\frac{\pi}{2}-\frac{1-u_A(x_i)+v_A(x_i)}{4}\pi\right)-1\right]$$

$$\left.\times\frac{1}{\sqrt{2}-1}\right\}$$

$$=\frac{1}{n}\sum_{i=1}^{n}\left\{\left[\sin\frac{1+u_A(x_i)-v_A(x_i)}{4}\pi\right.\right.$$

$$+\cos\frac{1+u_A(x_i)-v_A(x_i)}{4}\pi-1\right]$$

$$\left.\times\frac{1}{\sqrt{2}-1}\right\},$$

$$J_2(A) = \frac{1}{n}\sum_{i=1}^{n}\left\{\left[\cos\frac{1+u_A(x_i)-v_A(x_i)}{4}\pi\right.\right.$$

$$+\sin\left(\frac{\pi}{2}-\frac{1-u_A(x_i)+v_A(x_i)}{4}\pi\right)-1\right]$$

$$\left.\times\frac{1}{\sqrt{2}-1}\right\}$$

$$=\frac{1}{n}\sum_{i=1}^{n}\left\{\left[\sin\frac{1+u_A(x_i)-v_A(x_i)}{4}\pi\right.\right.$$

$$+\cos\frac{1+u_A(x_i)-v_A(x_i)}{4}\pi-1\right]$$

$$\left.\times\frac{1}{\sqrt{2}-1}\right\}.$$

$$(18)$$

It follows that $J_1(A) = J_2(A)$. Next, we can simplify it as follows:

$$J_1(A) = J_2(A)$$

$$=\frac{1}{n}\sum_{i=1}^{n}\left\{\left[\sin\frac{1+u_A(x_i)-v_A(x_i)}{4}\pi\right.\right.$$

$$+\cos\frac{1+u_A(x_i)-v_A(x_i)}{4}\pi-1\right]$$

$$\left.\times\frac{1}{\sqrt{2}-1}\right\}$$

$$=\frac{1}{n}\sum_{i=1}^{n}\left\{\sqrt{2}\left[\frac{\sqrt{2}}{2}\sin\frac{1+u_A(x_i)-v_A(x_i)}{4}\pi\right.\right.$$

$$+\frac{\sqrt{2}}{2}\cos\frac{1+u_A(x_i)-v_A(x_i)}{4}\pi-\frac{\sqrt{2}}{2}\right]$$

$$\left.\times\frac{1}{\sqrt{2}-1}\right\}$$

$$=\frac{1}{n}\sum_{i=1}^{n}\left\{\sqrt{2}\left[\sin\frac{2+u_A(x_i)-v_A(x_i)}{4}\pi-\frac{\sqrt{2}}{2}\right]\right.$$

$$\left.\times\frac{1}{\sqrt{2}-1}\right\}$$

$$=\frac{1}{n}\sum_{i=1}^{n}\left\{\left[\sqrt{2}\cos\frac{u_A(x_i)-v_A(x_i)}{4}\pi-1\right]\times\frac{1}{\sqrt{2}-1}\right\}$$

$$= J(A).$$

$$(19)$$

$\square$

The following example shows that the entropy measure J can produce some counterintuitive cases.

Example 13. Let $A_1 = \{\langle x_i, 0.1, 0.3\rangle \mid x_i \in X\}$, $A_2 = \{\langle x_i, 0.3, 0.5\rangle \mid x_i \in X\}$, and $A_3 = \{\langle x_i, 0.4, 0.6\rangle \mid x_i \in X\}$ be three IFSs in X. Now we calculate their entropies.

The absolute differences between the membership degrees and the nonmembership degrees of each x_i to A_1, A_2, and A_3 are the same; thus, by formula (17), we can obtain that $J(A_1) = J(A_2) = J(A_3) = 0.9580$. But we can see that the hesitancy degrees of the element x_i to A_1, A_2, and A_3 are different. Intuitively, the uncertain information of A_1 is more than that of A_2, and the uncertain information of A_3 is the least. Obviously, the results obtained by using Ye's formula are not in accordance with our intuition.

Now, let us calculate the entropies of A_1, A_2, and A_3 by formula (11). We have $E(A_1) = 0.9808$, $E(A_2) = 0.9659$, and $E(A_3) = 0.9511$, so that $E(A_1) > E(A_2) > E(A_3)$. This is consistent with our intuition according to the above analysis. The following theorem is a straightforward exercise.

Theorem 14. *Let $X = \{x\}$. For a constant a in $(0, 1]$, let $\mathscr{F}_a$ be the set of all IFSs $\{\langle x, u_A(x), v_A(x)\rangle\}$ in X with $|u_A(x) - v_A(x)| = a$. Then E is strictly monotone increasing with respect to $\pi_A(x)$ on $\mathscr{F}_a$.*

Comparing the entropy measures E and J, we find that E could measure not only the degree of fuzziness, but also the degree of intuitionism of IFSs, which overcomes the shortcoming that J could only measure the degree of fuzziness of IFSs. The entropy measures in [28–31] could also measure both fuzziness and intuitionism, but in some cases,

some of them may not work well as desired. Next we compare the entropy measure E with them.

We first recall these entropy measures. Xia and Xu [28] derived a cross-entropy measure E_{XM}^q:

$$
\begin{aligned}
E_{XM}^q(A) = -\frac{1}{nT}\sum_{i=1}^{n} & \big\{ \left[1 + q\left(1 - u_A(x_i)\right)\right] \\
& \cdot \ln\left[1 + q\left(1 - u_A(x_i)\right)\right] \\
& + \left[1 + q\left(1 - v_A(x_i)\right)\right] \\
& \cdot \ln\left[1 + q\left(1 - v_A(x_i)\right)\right] \\
& - \left[1 + q\left(1 - u_A(x_i)\right) + 1 \right. \\
& \left. + q\left(1 - v_A(x_i)\right)\right] \\
& \cdot \left[\ln\left(1 + q\left(1 - u_A(x_i)\right) + 1 \right.\right. \\
& \left.\left. + q\left(1 - v_A(x_i)\right)\right) - \ln 2\right] \big\} + 1,
\end{aligned}
$$
(20)

where $T = (1 + q)\ln(1 + q) - (2 + q)(\ln(2 + q) - \ln 2)$, $q > 0$.

In [30, 31], Vlachos and Sergiadis proposed an entropy measure E_{VS1} according to a cross-entropy measure and an entropy measure E_{VS2} based on the product of two vectors:

$$
\begin{aligned}
E_{VS1}(A) = -\frac{1}{n\ln 2}\sum_{i=1}^{n} & \big[u_A(x_i)\ln u_A(x_i) + v_A(x_i)\ln v_A(x_i) \\
& - \left(1 - \pi_A(x_i)\right)\ln\left(1 - \pi_A(x_i)\right) \\
& - \pi_A(x_i)\ln 2 \big],
\end{aligned}
$$

$$
E_{VS2}(A) = \frac{1}{n}\sum_{i=1}^{n}\frac{2u_A(x_i)v_A(x_i) + \pi_A^2(x_i)}{u_A^2(x_i) + v_A^2(x_i) + \pi_A^2(x_i)}.
$$
(21)

These three entropy measures satisfy the set of requirements in Definition 9. The following examples show that the entropy measures E_{VS1} and E_{VS2} may give inconsistent information in some cases.

Example 15. Let

$$
\begin{aligned}
A_1 &= \{\langle x, 0.1, 0.9\rangle \mid x \in X\}, \\
A_2 &= \{\langle x, 0.2, 0.7\rangle \mid x \in X\}, \\
A_3 &= \{\langle x, 0.2, 0.5\rangle \mid x \in X\}, \\
A_4 &= \{\langle x, 0.2, 0.4\rangle \mid x \in X\}, \\
A_5 &= \{\langle x, 0.4, 0.5\rangle \mid x \in X\}, \\
A_6 &= \{\langle x, 0.3, 0.4\rangle \mid x \in X\}, \\
A_7 &= \{\langle x, 0.1, 0.2\rangle \mid x \in X\}, \\
A_8 &= \{\langle x, 0.5, 0.5\rangle \mid x \in X\}.
\end{aligned}
$$
(22)

Table 1 gives us the entropies of A_i by E, E_{XM}^1, E_{VS1}, and E_{VS2}.

TABLE 1: Comparison with existing entropies.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
E	0.3090	0.7557	0.9350	0.9749	0.9898	0.9927	0.9957	1
E_{XM}^1	0.3645	0.7616	0.9196	0.9654	0.9905	0.9911	0.9920	1
E_{VS1}	0.4690	0.7878	0.9042	0.9510	**0.9920**	**0.9897**	**0.9755**	1
E_{VS2}	0.2195	0.5370	0.7632	0.8889	**0.9762**	**0.9706**	0.9815	1

From Table 1, we can see that, for E and E_{XM}^1, the closer the membership degree and the nonmembership degree, or the bigger the hesitation degree, the greater its entropy. Particularly, when the membership degree is equal to the nonmembership degree, the entropy reaches the maximum value 1. The results obtained by using the entropy measures E and E_{XM}^1 are in accordance with our intuition.

For A_5, A_6, and A_7, we can see that the absolute differences between the membership degrees and the nonmembership degrees are the same, and hesitancy degrees of the element x to A_5, A_6, and A_7 are increasing. Intuitively, the uncertain degree of the three IFSs should be increasing. In fact, from Table 1, we have $E(A_5) < E(A_6) < E(A_7)$ and $E_{XM}^1(A_5) < E_{XM}^1(A_6) < E_{XM}^1(A_7)$, which are consistent with our intuition. But, by entropy measures E_{VS1} and E_{VS2}, we have $E_{VS1}(A_5) > E_{VS1}(A_6) > E_{VS1}(A_7)$, $E_{VS2}(A_5) > E_{VS2}(A_6)$, and $E_{VS2}(A_6) < E_{VS2}(A_7)$. Obviously, the results obtained by using the two entropy measures are not in accordance with our intuition. Furthermore, for the entropy measures E_{VS1} and E_{VS2}, we have the following conclusions.

Theorem 16. *Let $X = \{x\}$. For a constant a in $(0, 1)$, let $\mathcal{F}_a$ be the set of all IFSs $\{\langle x, u_A(x), v_A(x)\rangle\}$ in X with $|u_A(x) - v_A(x)| = a$. Then*

(1) *$E_{VS1}(A)$ is strictly monotone decreasing with respect to $\pi_A(x)$ on $\mathcal{F}_a$;*

(2) *when $0 \leq \pi_A(x) < 1/3$, $E_{VS2}(A)$ is strictly monotone decreasing with respect to $\pi_A(x)$ on $\mathcal{F}_a$; when $1/3 < \pi_A(x) \leq 1$, $E_{VS2}(A)$ is strictly monotone increasing with respect to $\pi_A(x)$ on $\mathcal{F}_a$.*

Proof. Since $\pi_A(x) = 1 - u_A(x) - v_A(x)$ and $|u_A(x) - v_A(x)| = a$, we have $u_A(x) = (1 - \pi_A(x) + a)/2$ and $v_A(x) = (1 - \pi_A(x) - a)/2$ or $v_A(x) = (1 - \pi_A(x) + a)/2$ and $u_A(x) = (1 - \pi_A(x) - a)/2$.

(1) Applying the above conditions, we have

$$
\begin{aligned}
E_{VS1}(A) &= -\frac{1}{\ln 2}\left[\frac{1 - \pi_A(x) + a}{2}\ln\frac{1 - \pi_A(x) + a}{2}\right. \\
& \quad + \frac{1 - \pi_A(x) - a}{2}\ln\frac{1 - \pi_A(x) - a}{2} \\
& \quad \left. - \left(1 - \pi_A(x)\right)\ln\left(1 - \pi_A(x)\right) - \pi_A(x)\ln 2\right],
\end{aligned}
$$
(23)

where we let $0\ln 0 = 0$ if necessary.

Let $f(t) = ((1 - t + a)/2) \ln((1 - t + a)/2) + ((1 - t - a)/2) \ln((1 - t - a)/2) - (1 - t) \ln(1 - t) - t \ln 2$ for $t \in (0, 1 - a)$. Then $f'(t) = -(1/2) \ln((1 - t + a)/2) - (1/2) \ln((1 - t - a)/2) + \ln 1 + t - \ln 2 = -(1/2) \ln(((1 - t)^2 - a^2)/(1 - t)^2) > 0$, so that $f(t)$ is strictly monotone increasing with respect to t. Thus $E_{\text{VS1}}(A)$ is strictly monotone decreasing with respect to $\pi_A(x)$ on $\mathscr{F}_a$.

(2) Since $u_A(x) = (1 - \pi_A(x) + a)/2$ and $v_A(x) = (1 - \pi_A(x) - a)/2$ or $v_A(x) = (1 - \pi_A(x) + a)/2$ and $u_A(x) = (1 - \pi_A(x) - a)/2$, we have

$$E_{\text{VS2}}(A) = \left(2 \frac{1 - \pi_A(x) - a}{2} \frac{1 - \pi_A(x) + a}{2} + \pi_A^2(x) \right)$$
$$\cdot \left(\left(\frac{1 - \pi_A(x) - a}{2} \right)^2 + \left(\frac{1 - \pi_A(x) + a}{2} \right)^2 \right.$$
$$\left. + \pi_A^2(x) \right)^{-1}$$
$$= \frac{3\pi_A^2(x) - 2\pi_A(x) + 1 - a^2}{3\pi_A^2(x) - 2\pi_A(x) + 1 + a^2}. \tag{24}$$

Let

$$f(\pi_A(x)) = \frac{3\pi_A^2(x) - 2\pi_A(x) + 1 - a^2}{3\pi_A^2(x) - 2\pi_A(x) + 1 + a^2}. \tag{25}$$

Then $f'(\pi_A(x)) = 2a^2(6\pi_A(x) - 2)/(3\pi_A^2(x) - 2\pi_A(x) + 1 + a^2)^2$. Clearly, when $0 \leq \pi_A(x) < 1/3$, we have $f'(\pi_A(x)) < 0$, so that $E_{\text{VS3}}(A)$ is strictly monotone decreasing with respect to $\pi_A(x)$ on $\mathscr{F}_a$; when $1/3 < \pi_A(x) \leq 1$, $f'(\pi_A(x)) > 0$, which implies that $E_{\text{VS3}}(A)$ is strictly monotone increasing with respect to $\pi_A(x)$ on $\mathscr{F}_a$. $\square$

Hence, compared with the above entropy measures, the entropy measure E defined by formula (11) is more effective and reasonable to measure the uncertain information of IFSs.

4. An Entropy Measure for Interval-Valued Intuitionistic Fuzzy Sets

In this section we will extend the entropy measure E to IVIFSs and define a new entropy measure for IVIFSs which is compared to the entropy measures defined in [37].

Liu et al. [35] and Zhang et al. [38] proposed a set of axiomatic requirements for an entropy measure of IVIFSs, which extends Szmidt and Kacprzyk's axioms formulated for entropy of IFSs [26].

Definition 17 (see [35]). A real-valued function E: IVIFS$(X) \to [0, 1]$ is called an entropy measure for an IVIFS A, if it satisfies the following axiomatic requirements:

(E1) $E(A) = 0$ if and only if A is a crisp set;

(E2) $E(A) = 1$ if and only if $[u_A^-(x_i), u_A^+(x_i)] = [v_A^-(x_i), v_A^+(x_i)]$ for each $x_i \in X$;

(E3) $E(A) = E(A^C)$;

(E4) $E(A) \leq E(B)$ if A is less fuzzy than B, which is defined as

$$[u_A^-(x_i), u_A^+(x_i)] \leq [u_B^-(x_i), u_B^+(x_i)],$$
$$[v_A^-(x_i), v_A^+(x_i)] \geq [v_B^-(x_i), v_B^+(x_i)]$$
$$\text{for } [u_B^-(x_i), u_B^+(x_i)] \leq [v_B^-(x_i), v_B^+(x_i)];$$
$$[u_A^-(x_i), u_A^+(x_i)] \geq [u_B^-(x_i), u_B^+(x_i)],$$
$$[v_A^-(x_i), v_A^+(x_i)] \leq [v_B^-(x_i), v_B^+(x_i)]$$
$$\text{for } [u_B^-(x_i), u_B^+(x_i)] \geq [v_B^-(x_i), v_B^+(x_i)];$$
$$[u_A^-(x_i), u_A^+(x_i)] \preceq [u_B^-(x_i), u_B^+(x_i)],$$
$$[v_A^-(x_i), v_A^+(x_i)] \succeq [v_B^-(x_i), v_B^+(x_i)]$$
$$\text{for } [u_B^-(x_i), u_B^+(x_i)] \preceq [v_B^-(x_i), v_B^+(x_i)];$$
$$[u_A^-(x_i), u_A^+(x_i)] \succeq [u_B^-(x_i), u_B^+(x_i)],$$
$$[v_A^-(x_i), v_A^+(x_i)] \preceq [v_B^-(x_i), v_B^+(x_i)]$$
$$\text{for } [u_B^-(x_i), u_B^+(x_i)] \succeq [v_B^-(x_i), v_B^+(x_i)]. \tag{26}$$

In this section we will introduce a formula to calculate the entropy of an IVIFS based on the entropy measure for IFSs defined by formula (11). For any $A \in \text{IVIFS}(X)$ we define

$$\overline{E}(A) = \frac{1}{n} \sum_{i=1}^{n} \cos \frac{|u_A^-(x_i) - v_A^-(x_i)| + |u_A^+(x_i) - v_A^+(x_i)|}{2(2 + \pi_A^+(x_i) + \pi_A^-(x_i))} \pi. \tag{27}$$

Then we have the following theorem.

Theorem 18. *The mapping $\overline{E}$, defined by (27), is an entropy measure for IVIFSs.*

Proof. In order to prove that the mapping $\overline{E}$ is an entropy measure, it is sufficient to show that $\overline{E}$ satisfies the conditions (E1)–(E4) in Definition 17. Suppose that

$$\overline{E}_i(A) = \cos \frac{|u_A^-(x_i) - v_A^-(x_i)| + |u_A^+(x_i) - v_A^+(x_i)|}{2(2 + \pi_A^+(x_i) + \pi_A^-(x_i))} \pi. \tag{28}$$

Since $u_A^-(x_i), u_A^+(x_i), v_A^-(x_i), v_A^+(x_i), \pi_A^-(x_i)$, and $\pi_A^+(x_i)$ are all in the interval $[0, 1]$, we have $0 \leq ((|u_A^-(x_i) - v_A^-(x_i)| + |u_A^+(x_i) - v_A^+(x_i)|)/2(2 + \pi_A^+(x_i) + \pi_A^-(x_i)))\pi \leq \pi/2$. Thus $0 \leq \overline{E}_i(A) \leq 1$.

(E1) Let A be a crisp set; that is, $[u_A^-(x_i), u_A^+(x_i)] = [0, 0]$, $[v_A^-(x_i), v_A^+(x_i)] = [1, 1]$, or $[u_A^-(x_i), u_A^+(x_i)] = [1, 1]$, $[v_A^-(x_i), v_A^+(x_i)] = [0, 0]$ for any $x_i \in X$. No matter in which case, we have $\overline{E}_i(A) = 0$. Hence $\overline{E}(A) = 0$.

On the other hand, suppose that $\overline{E}(A) = 0$. Since $0 \leq \overline{E}_i(A) \leq 1$, we have $\overline{E}_i(A) = 0$. Also since $0 \leq ((|u_A^-(x_i) - v_A^-(x_i)| + |u_A^+(x_i) - v_A^+(x_i)|)/2(2 + \pi_A^+(x_i) + \pi_A^-(x_i)))\pi \leq \pi/2$, we have $((|u_A^-(x_i) - v_A^-(x_i)| + |u_A^+(x_i) - v_A^+(x_i)|)/2(2 + \pi_A^+(x_i) + \pi_A^-(x_i)))\pi = \pi/2$. From the following four possible relations of $[u_A^-(x_i), u_A^+(x_i)]$ and $[v_A^-(x_i), v_A^+(x_i)]$:

$$
\begin{aligned}
[u_A^-(x_i), u_A^+(x_i)] &\leq [v_A^-(x_i), v_A^+(x_i)], \\
[u_A^-(x_i), u_A^+(x_i)] &\geq [v_A^-(x_i), v_A^+(x_i)], \\
[u_A^-(x_i), u_A^+(x_i)] &\preceq [v_A^-(x_i), v_A^+(x_i)], \\
[u_A^-(x_i), u_A^+(x_i)] &\succeq [v_A^-(x_i), v_A^+(x_i)],
\end{aligned}
\tag{29}
$$

we can easily obtain $u_A^-(x_i) = u_A^+(x_i) = 1$, $v_A^-(x_i) = v_A^+(x_i) = 0$, or $v_A^-(x_i) = v_A^+(x_i) = 1$, $u_A^-(x_i) = u_A^+(x_i) = 0$, so A is a crisp set.

(E2) Let $[u_A^-(x_i), u_A^+(x_i)] = [v_A^-(x_i), v_A^+(x_i)]$. Applying this condition to (27) yields $\overline{E}(A) = 1$.

Now suppose that $\overline{E}(A) = 1$. It follows that $\overline{E}_i(A) = 1$ for each $x_i \in X$. Since $0 \leq ((|u_A^-(x_i) - v_A^-(x_i)| + |u_A^+(x_i) - v_A^+(x_i)|)/2(2 + \pi_A^+(x_i) + \pi_A^-(x_i)))\pi \leq \pi/2$, we have $|u_A^-(x_i) - v_A^-(x_i)| + |u_A^+(x_i) - v_A^+(x_i)| = 0$. Thus $u_A^-(x_i) = v_A^-(x_i)$, $u_A^+(x_i) = v_A^+(x_i)$ for each $x_i \in X$.

(E3) For $A^C = \{\langle x_i, [v_A^-(x_i), v_A^+(x_i)], [u_A^-(x_i), u_A^+(x_i)]\rangle \mid x_i \in X\}$, we can easily get that $\overline{E}(A^C) = \overline{E}(A)$.

(E4) Suppose that $[u_A^-(x_i), u_A^+(x_i)] \leq [u_B^-(x_i), u_B^+(x_i)]$, $[v_A^-(x_i), v_A^+(x_i)] \geq [v_B^-(x_i), v_B^+(x_i)]$ for $[u_B^-(x_i), u_B^+(x_i)] \leq [v_B^-(x_i), v_B^+(x_i)]$. Then we have the following relations:

$$
\begin{aligned}
u_A^-(x_i) &\leq u_B^-(x_i) \leq v_B^-(x_i) \leq v_A^-(x_i), \\
u_A^+(x_i) &\leq u_B^+(x_i) \leq v_B^+(x_i) \leq v_A^+(x_i).
\end{aligned}
\tag{30}
$$

In order to prove $\overline{E}_i(A) \leq \overline{E}_i(B)$, we need to prove that

$$
\frac{|u_A^-(x_i) - v_A^-(x_i)| + |u_A^+(x_i) - v_A^+(x_i)|}{2 + \pi_A^+(x_i) + \pi_A^-(x_i)}
\tag{31}
$$

$$
\geq \frac{|u_B^-(x_i) - v_B^-(x_i)| + |u_B^+(x_i) - v_B^+(x_i)|}{2 + \pi_B^+(x_i) + \pi_B^-(x_i)}.
$$

By the assumption, it is equivalent to prove that

$$
\frac{u_A^-(x_i) - v_A^-(x_i) + u_A^+(x_i) - v_A^+(x_i)}{2 + \pi_A^+(x_i) + \pi_A^-(x_i)}
\tag{32}
$$

$$
\leq \frac{u_B^-(x_i) - v_B^-(x_i) + u_B^+(x_i) - v_B^+(x_i)}{2 + \pi_B^+(x_i) + \pi_B^-(x_i)},
$$

which can be simplified to prove that

$$
\begin{aligned}
&\left(u_A^-(x_i) + u_A^+(x_i)\right)\left(2 - v_B^-(x_i) - v_B^+(x_i)\right) \\
&\quad + 2\left(v_B^-(x_i) + v_B^+(x_i)\right) \\
&\quad + \left(v_A^-(x_i) + v_A^+(x_i)\right)\left(u_B^-(x_i) + u_B^+(x_i) - 2\right) \\
&\quad - 2\left(u_B^-(x_i) + u_B^+(x_i)\right) \leq 0.
\end{aligned}
\tag{33}
$$

Indeed, since $2 - v_B^-(x_i) - v_B^+(x_i) \geq 0$ and $u_B^-(x_i) + u_B^+(x_i) - 2 \leq 0$, we have

$$
\begin{aligned}
&\left(u_A^-(x_i) + u_A^+(x_i)\right)\left(2 - v_B^-(x_i) - v_B^+(x_i)\right) \\
&\leq \left(u_B^-(x_i) + u_B^+(x_i)\right)\left(2 - v_B^-(x_i) - v_B^+(x_i)\right), \\
&\left(v_A^-(x_i) + v_A^+(x_i)\right)\left(u_B^-(x_i) + u_B^+(x_i) - 2\right) \\
&\leq \left(v_B^-(x_i) + v_B^+(x_i)\right)\left(u_B^-(x_i) + u_B^+(x_i) - 2\right).
\end{aligned}
\tag{34}
$$

Then, by computing the sum of the left (resp., right) terms of the above two inequalities, we have (33). Hence $\overline{E}_i(A) \leq \overline{E}_i(B)$ holds.

By a similar way, we can also prove that $\overline{E}_i(A) \leq \overline{E}_i(B)$ for the other three cases. Since $\overline{E}_i(A) \leq \overline{E}_i(B)$ for each i, we have $\overline{E}(A) \leq \overline{E}(B)$.   □

Similar to Theorem 11, we give the following general form of the entropy measure $\overline{E}$ defined in (27).

Theorem 19. *Let $f : [-1, 1] \to [0, 1]$ be an even function such that f is strictly monotone increasing on $[0, 1]$, $f(-1) = f(1) = 0$, and $f(0) = 1$. For any $A \in \text{IVIFS}(X)$, let*

$$
\overline{E}_f(A) = \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{|u_A^-(x_i) - v_A^-(x_i)| + |u_A^+(x_i) - v_A^+(x_i)|}{(2 + \pi_A^+(x_i) + \pi_A^-(x_i))}\right).
\tag{35}
$$

Then $\overline{E}_f$ is an entropy measure for IVIFSs.

Proof. The process of the proof is similar to that for Theorem 18. We omit it.   □

In the following, we will compare the entropy measure $\overline{E}$ defined by (27) with the entropy measures defined in [37, 41]:

$$
\begin{aligned}
E(A) = \frac{1}{n}\sum_{i=1}^{n} \Big(&\left(2 - |u_A^-(x_i) - v_A^-(x_i)| - |u_A^+(x_i) - v_A^+(x_i)| \right. \\
&\left. + \pi_A^-(x_i) + \pi_A^+(x_i)\right) \\
&\cdot \left(2 + |u_A^-(x_i) - v_A^-(x_i)| + |u_A^+(x_i) - v_A^+(x_i)| \right. \\
&\left. + \pi_A^-(x_i) + \pi_A^+(x_i)\right)^{-1} \Big).
\end{aligned}
\tag{36}
$$

Let $A = \{\langle x, [u_A^-(x), u_A^+(x)], [v_A^-(x), v_A^+(x)]\rangle \mid x \in X\}$ be an IVIFS in X. Chen et al. [41] proposed a concrete entropy measure:

$$
\begin{aligned}
E_{CX}(A) = -\frac{1}{n \ln 4} \sum_{i=1}^{n} & \left[u_A^-(x_i) \ln u_A^-(x_i) + u_A^+(x_i) \ln u_A^+(x_i) \right. \\
& + v_A^-(x_i) \ln v_A^-(x_i) + v_A^+(x_i) \ln v_A^+(x_i) \\
& - \left(1 - \pi_A^-(x_i)\right) \ln \left(1 - \pi_A^-(x_i)\right) \\
& - \pi_A^-(x_i) \ln 2 \\
& - \left(1 - \pi_A^+(x_i)\right) \ln \left(1 - \pi_A^+(x_i)\right) \\
& \left. - \pi_A^+(x_i) \ln 2 \right].
\end{aligned}
\tag{37}
$$

Ye [37] introduced two entropy measures L_1 and L_2 as follows:

$$
\begin{aligned}
L_1(A) \\
= \frac{1}{n} \sum_{i=1}^{n} & \left\{ \left[\sin \frac{1 + u_A^-(x_i) + pW_u(x_i) - v_A^-(x_i) - qW_v(x_i)}{4} \pi \right. \right. \\
& + \sin \frac{1 - u_A^-(x_i) - pW_u(x_i) + v_A^-(x_i) + qW_v(x_i)}{4} \\
& \left. \left. \cdot \pi - 1 \right] \times \frac{1}{\sqrt{2}-1} \right\},
\end{aligned}
$$

$$
\begin{aligned}
L_2(A) \\
= \frac{1}{n} \sum_{i=1}^{n} & \left\{ \left[\cos \frac{1 + u_A^-(x_i) + pW_u(x_i) - v_A^-(x_i) - qW_v(x_i)}{4} \pi \right. \right. \\
& + \cos \frac{1 - u_A^-(x_i) - pW_u(x_i) + v_A^-(x_i) + qW_v(x_i)}{4} \\
& \left. \left. \cdot \pi - 1 \right] \times \frac{1}{\sqrt{2}-1} \right\},
\end{aligned}
\tag{38}
$$

where $W_u(x_i) = u_A^+(x_i) - u_A^-(x_i)$, $W_v(x_i) = v_A^+(x_i) - v_A^-(x_i)$, and $p, q \in [0, 1]$ are two fixed numbers.

Theorem 20. *For each A in IVIFS(X), let*

$$
\begin{aligned}
L(A) \\
= \frac{1}{n} \sum_{i=1}^{n} & \left\{ \left[\sqrt{2} \cos \frac{u_A^-(x_i) + pW_u(x_i) - v_A^-(x_i) - qW_v(x_i)}{4} \right. \right. \\
& \left. \left. \cdot \pi - 1 \right] \times \frac{1}{\sqrt{2}-1} \right\},
\end{aligned}
\tag{39}
$$

where $W_u(x_i) = u_A^+(x_i) - u_A^-(x_i)$, $W_v(x_i) = v_A^+(x_i) - v_A^-(x_i)$, and $p, q \in [0, 1]$.
Then $L_1(A) = L_2(A) = L(A)$.

Proof. The process of the proof is similar to that for Proposition 12. We omit it. $\qquad\square$

Example 21. Let us calculate entropies for the following IVIFSs:

$$
\begin{aligned}
A_1 &= \{\langle x_i, [0.1, 0.2], [0.1, 0.2]\rangle \mid x_i \in X\}, \\
A_2 &= \{\langle x_i, [0.3, 0.3], [0.1, 0.5]\rangle \mid x_i \in X\}, \\
A_3 &= \{\langle x_i, [0.1, 0.2], [0.3, 0.4]\rangle \mid x_i \in X\}, \\
A_4 &= \{\langle x_i, [0.2, 0.3], [0.4, 0.5]\rangle \mid x_i \in X\}, \\
A_5 &= \{\langle x_i, [0.3, 0.4], [0.5, 0.6]\rangle \mid x_i \in X\}, \\
A_6 &= \{\langle x_i, [0.2, 0.3], [0.5, 0.6]\rangle \mid x_i \in X\}, \\
A_7 &= \{\langle x_i, [0.1, 0.2], [0.5, 0.6]\rangle \mid x_i \in X\}, \\
A_8 &= \{\langle x_i, [0.1, 0.2], [0.7, 0.8]\rangle \mid x_i \in X\}.
\end{aligned}
\tag{40}
$$

By the entropy measure E_{CX}, we get

$$
\begin{aligned}
E_{CX}(A_1) &= 1, & E_{CX}(A_2) &= 0.944, \\
E_{CX}(A_3) &= 0.9377, & E_{CX}(A_4) &= 0.9573, \\
E_{CX}(A_5) &= 0.9627, & E_{CX}(A_6) &= 0.9153, \\
E_{CX}(A_7) &= 0.8195, & E_{CX}(A_8) &= 0.6784.
\end{aligned}
\tag{41}
$$

The difference between the membership degrees and nonmembership degrees of A_4 and A_5 is the same, and the hesitant degree of A_4 is bigger than that of A_5, so the entropy of A_4 should be bigger than that of A_5. However, by the entropy measure E_{CX}, we have $E_{CX}(A_4) < E_{CX}(A_5)$.

Using (43), we have

$$
\begin{aligned}
L(A_1) &= L(A_2) = 1, \\
L(A_3) &= L(A_4) = L(A_5) = 0.9580, \\
L(A_6) &= 0.9057, \qquad L(A_7) = 0.8329, \\
L(A_8) &= 0.6279.
\end{aligned}
\tag{42}
$$

We suppose that $p = q = 0.5$ in formula (39); then formula (39) reduces to

$$
\begin{aligned}
L(A) = \frac{1}{n} \sum_{i=1}^{n} & \left\{ \left[\sqrt{2} \cos \frac{u_A^-(x_i) + u_A^+(x_i) - v_A^-(x_i) - v_A^+(x_i)}{8} \right. \right. \\
& \left. \left. \cdot \pi - 1 \right] \times \frac{1}{\sqrt{2}-1} \right\}.
\end{aligned}
\tag{43}
$$

From these results, we can see that the entropy formula defined by (43) has the following two drawbacks.

(1) It does not satisfy the necessary condition of (E2) in Definition 17. In fact, for any IVIFS A satisfying $u_A^-(x_i) + u_A^+(x_i) = v_A^-(x_i) + v_A^+(x_i)$ for each x_i, we can obtain $J(A) = 1$.

(2) It only reflects the difference between the membership degree and nonmembership degree. Thus, for any two IVIFS A and B satisfying $|u_A^-(x_i) + u_A^+(x_i) - v_A^-(x_i) - v_A^+(x_i)| = |u_B^-(x_i) + u_B^+(x_i) - v_B^-(x_i) - v_B^+(x_i)|$, for all $x_i \in X$, we have $J(A) = J(B)$.

Now by the entropy formula defined by (27), we can obtain

$$\overline{E}(A_1) = 1, \qquad \overline{E}(A_2) = 0.9749,$$

$$\overline{E}(A_3) = 0.9781, \qquad \overline{E}(A_4) = 0.9709,$$

$$\overline{E}(A_5) = 0.9595, \qquad \overline{E}(A_6) = 0.9239, \qquad (44)$$

$$\overline{E}(A_7) = 0.8855, \qquad \overline{E}(A_8) = 0.6547.$$

The results show that $\overline{E}(A_1) \neq \overline{E}(A_2)$ and $\overline{E}(A_4) > \overline{E}(A_5)$. The proposed entropy measure $\overline{E}$ can overcome the above shortcomings of the entropy measures E_{CX} and L.

5. The Application of Entropy Measures in Multicriteria Decision-Making

Entropy measures have been applied in many problems such as optimizing the distinguishability of input space partitioning [42] and assessing the weights of experts or criteria in intuitionistic fuzzy decision-making [43–45]. In this section we propose a method to determinate experts' weights in multicriteria group decision-making with interval-valued intuitionistic fuzzy information by using the proposed entropy measures.

The group decision-making problem which is considered in this paper can be represented as follows. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of evaluation alternatives, $D = \{d_1, d_2, \ldots, d_s\}$ a set of experts, $U = \{C_1, C_2, \ldots, C_m\}$ a criterion set, and $w = (w_1, w_2, \ldots, w_m)^T$ the weighting vector of criteria such that $w_j \in [0, 1]$ and $\sum_{j=1}^m w_j = 1$. Let $A_k = (s_{ij}^{(k)})_{n \times m}$ $(k = 1, 2, \ldots, s)$ be interval-valued intuitionistic fuzzy decision matrices, where $s_{ij}^{(k)} = ([a_{ij}^k, b_{ij}^k], [c_{ij}^k, d_{ij}^k])$ is an IVIFV, provided by the decision maker $d_k \in D$ for the alternative $x_i \in X$ with respect to the criterion $C_j \in U$. Decision maker's goal is to obtain the ranking order of the alternatives.

According to [46], if criteria include cost criteria and benefit criteria in multicriteria decision-making process, we should transform the criterion values of cost type into those of benefit type. Hence decision-making matrices $A_k = (s_{ij}^{(k)})_{n \times m}$ $(k = 1, 2, \ldots, s)$ are transformed into normalized decision-making matrices $R_k = (r_{ij}^{(k)})_{n \times m}$ $(k = 1, 2, \ldots, s)$, where

$$r_{ij}^{(k)} = \begin{cases} s_{ij}^{(k)}, & \text{for benefit criterion } C_j, \\ s_{ij}^{(k)C}, & \text{for cost criterion } C_j, \end{cases} \qquad (45)$$

and $s_{ij}^{(k)C} = ([c_{ij}^k, d_{ij}^k], [a_{ij}^k, b_{ij}^k])$ $(i = 1, 2, \ldots, n, j = 1, 2, \ldots, m)$.

For a given weighting vector of criteria, we can use the IVIFWA operator to derive the individual overall evaluation values $z_i^{(k)} = ([a_i^{(k)}, b_i^{(k)}], [c_i^{(k)}, d_i^{(k)}])$ of alternatives x_i $(i = 1, 2, \ldots, n)$ by experts d_k $(k = 1, 2, \ldots, s)$:

$$
\begin{aligned}
z_i^{(k)} &= \text{IVIFWA}_w \left(r_{i1}^{(k)}, r_{i2}^{(k)}, \ldots, r_{im}^{(k)} \right) \\
&= w_1 r_{i1}^{(k)} \oplus w_2 r_{i2}^{(k)} \oplus \cdots \oplus w_m r_{im}^{(k)},
\end{aligned}
\qquad (46)
$$

where $w = (w_1, w_2, \ldots, w_m)^T$ is the weighting vector of criteria C_j $(j = 1, 2, \ldots, m)$, with $w_j \in [0, 1]$ and $\sum_{j=1}^m w_j = 1$.

5.1. Determining the Weights of Experts. In many practical group decision-making problems, it is an important research topic to determine the weights of experts according to experts' evaluation information. In this subsection, we propose a method to derive the weights of experts based on the proposed entropy measures.

It is known that entropies can measure the uncertainty degrees of IVIFSs. $Z_k = \{z_i^{(k)} = ([a_i^{(k)}, b_i^{(k)}], [c_i^{(k)}, d_i^{(k)}]) \mid i = 1, 2, \ldots, n\}$ is actually an IVIFS in alternative sets X, which includes the overall assessment values for all alternatives x_i $(i = 1, 2, \ldots, n)$ by experts $d_k \in D$. By formula (27), the entropy of Z_k can be calculated, which is denoted by $\overline{E}_k$. $\overline{E}_k$ indicates the uncertainty degree of assessment information provided by expert d_k. During the practical group decision-making process, we usually expect that the uncertainty degree of the assessment information is as small as possible. Thus, the bigger $\overline{E}_k$ is, the smaller the weight should be given to d_k. Conversely, the smaller $\overline{E}_k$ is, the bigger the weight should be given to d_k. Therefore, the weights of experts are defined as follows:

$$\lambda_k = \frac{1 - e_k}{\sum_{i=1}^s (1 - e_k)}, \quad \text{where } e_k = \frac{\overline{E}_k}{\sum_{i=1}^s \overline{E}_k}, \; k = 1, 2, \ldots, s. \qquad (47)$$

In the next subsection, we will aggregate the overall assessment information of individual experts to get the evaluations of the group for alternatives.

5.2. An Approach to Solve Interval-Valued Intuitionistic Fuzzy Group Decision-Making Problems. According to the analysis in Section 5.1, we develop the following steps to get the ranking of alternatives.

Step 1. Use formula (45) to transform decision matrices $A_k = (s_{ij}^{(k)})_{n \times m}$ into normalized decision matrices $R_k = (r_{ij}^{(k)})_{n \times m}$.

Step 2. Use formula (46) to derive the individual overall assessment values $z_i^{(k)}$ of alternatives x_i by experts d_k $(i = 1, 2, \ldots, n, \; k = 1, 2, \ldots, s)$.

Step 3. Use formulas (27) and (47) to derive experts' weighting vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)^T$.

Step 4. Use the IVIFWA operator to derive the overall assessment values z_i $(i = 1, 2, \ldots, n)$ of the alternatives x_i $(i = 1, 2, \ldots, n)$:

$$
\begin{aligned}
z_i &= \text{IVIFWA}_\lambda \left(z_i^{(1)}, z_i^{(2)}, \ldots, z_i^{(s)} \right) \\
&= \lambda_1 z_i^{(1)} \oplus \lambda_2 z_i^{(2)} \oplus \cdots \oplus \lambda_s z_i^{(s)},
\end{aligned}
\tag{48}
$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)^T$ is the weighting vector of experts with $\lambda_k \in [0, 1]$ and $\sum_{k=1}^s \lambda_k = 1$.

Step 5. Use Definition 6 to compare the overall assessment values z_i $(i = 1, 2, \ldots, n)$ and rank the alternatives x_i $(i = 1, 2, \ldots, n)$.

For interval-valued intuitionistic fuzzy decision matrices $A_k = (s_{ij}^{(k)})_{n \times m}$ $(k = 1, 2, \ldots, s)$ with $s_{ij}^{(k)} = ([a_{ij}^k, b_{ij}^k], [c_{ij}^k, d_{ij}^k])$, if $a_{ij}^k = b_{ij}^k$ and $c_{ij}^k = d_{ij}^k$, then these interval-valued intuitionistic fuzzy decision matrices reduce to intuitionistic fuzzy decision matrices. So the above method is also suitable to solve intuitionistic fuzzy group decision-making problems with unknown experts' weights.

In the following, we give an example which was adapted from Xu and Cai [47] to illustrate the above approach.

Example 22 (see [47]). Consider an air-condition system selection problem. Suppose that there exist three air-condition systems x_i $(i = 1, 2, 3)$ to be selected, and the following is the list of five criteria C_j $(j = 1, 2, 3, 4, 5)$: good quality (C_1), easiness to operate (C_2), being economical (C_3), good service after selling (C_4), and cost (C_5). Among these criteria, C_j $(j = 1, 2, 3, 4)$ are of benefit type; C_5 is of cost type. $w = (0.200, 0.299, 0.106, 0.156, 0.239)^T$ is the weighting vector of criteria. The evaluations of experts d_k $(k = 1, 2, 3)$ for the air-condition systems x_i $(i = 1, 2, 3)$ under criteria C_j $(j = 1, 2, 3, 4, 5)$ are represented by IFVs $s_{ij}^{(k)}$, which construct the decision matrices $A_k = (s_{ij}^{(k)})_{3 \times 5}$ $(k = 1, 2, 3)$:

$$
A_1 = \begin{pmatrix} (0.8, 0.1) & (0.7, 0.1) & (0.7, 0.2) & (0.9, 0.0) & (0.4, 0.5) \\ (0.7, 0.1) & (0.8, 0.2) & (0.6, 0.4) & (0.7, 0.1) & (0.6, 0.4) \\ (0.8, 0.2) & (0.9, 0.1) & (0.7, 0.0) & (0.7, 0.2) & (0.5, 0.5) \end{pmatrix},
$$

$$
A_2 = \begin{pmatrix} (0.9, 0.1) & (0.8, 0.1) & (0.7, 0.0) & (0.9, 0.1) & (0.3, 0.7) \\ (0.7, 0.2) & (0.8, 0.1) & (0.9, 0.1) & (0.7, 0.3) & (0.7, 0.2) \\ (0.7, 0.1) & (0.9, 0.0) & (0.8, 0.0) & (0.8, 0.2) & (0.6, 0.3) \end{pmatrix},
$$

$$
A_3 = \begin{pmatrix} (0.8, 0.0) & (0.7, 0.1) & (0.9, 0.0) & (0.8, 0.1) & (0.4, 0.6) \\ (0.8, 0.2) & (0.7, 0.3) & (0.8, 0.1) & (0.9, 0.1) & (0.6, 0.3) \\ (0.9, 0.1) & (0.8, 0.0) & (0.8, 0.1) & (0.9, 0.0) & (0.5, 0.4) \end{pmatrix}.
\tag{49}
$$

We now use the above steps to rank the three air-condition systems x_i $(i = 1, 2, 3)$.

Step 1. Using (45), we transform decision matrices $A_k = (s_{ij}^{(k)})_{3 \times 5}$ $(k = 1, 2, 3)$ into normalized decision matrices $R_k = (r_{ij}^{(k)})_{3 \times 5}$ $(k = 1, 2, 3)$:

$$
R_1 = \begin{pmatrix} (0.8, 0.1) & (0.7, 0.1) & (0.7, 0.2) & (0.9, 0.0) & (0.5, 0.4) \\ (0.7, 0.1) & (0.8, 0.2) & (0.6, 0.4) & (0.7, 0.1) & (0.4, 0.6) \\ (0.8, 0.2) & (0.9, 0.1) & (0.7, 0.0) & (0.7, 0.2) & (0.5, 0.5) \end{pmatrix},
$$

$$
R_2 = \begin{pmatrix} (0.9, 0.1) & (0.8, 0.1) & (0.7, 0.0) & (0.9, 0.1) & (0.7, 0.3) \\ (0.7, 0.2) & (0.8, 0.1) & (0.9, 0.1) & (0.7, 0.3) & (0.2, 0.7) \\ (0.7, 0.1) & (0.9, 0.0) & (0.8, 0.0) & (0.8, 0.2) & (0.3, 0.6) \end{pmatrix},
$$

$$
R_3 = \begin{pmatrix} (0.8, 0.0) & (0.7, 0.1) & (0.9, 0.0) & (0.8, 0.1) & (0.6, 0.4) \\ (0.8, 0.2) & (0.7, 0.3) & (0.8, 0.1) & (0.9, 0.1) & (0.3, 0.6) \\ (0.9, 0.1) & (0.8, 0.0) & (0.8, 0.1) & (0.9, 0.0) & (0.4, 0.5) \end{pmatrix}.
\tag{50}
$$

Step 2. By formula (46), we derive the individual overall assessment values $z_i^{(k)}$ of alternatives x_i by experts d_k $(i = 1, 2, 3, \ k = 1, 2, 3)$:

$$
z_1^{(1)} = (0.7367, 0.0000), \qquad z_2^{(1)} = (0.6767, 0.2187),
$$

$$
z_3^{(1)} = (0.7750, 0.0000),
$$

$$
z_1^{(2)} = (0.8203, 0.0000), \qquad z_2^{(2)} = (0.7010, 0.2171),
$$

$$
z_3^{(2)} = (0.7622, 0.0000),
\tag{51}
$$

$$
z_1^{(3)} = (0.7524, 0.0000), \qquad z_2^{(3)} = (0.7266, 0.2448),
$$

$$
z_3^{(3)} = (0.7968, 0.0000).
$$

Step 3. By formulas (27) (or (11)) and (47), we obtain $\overline{E}_i$ $(i = 1, 2, 3)$:

$$
\overline{E}_1 = 0.6499, \qquad \overline{E}_2 = 0.5971, \qquad \overline{E}_3 = 0.6104.
\tag{52}
$$

So the experts' weighting vector λ is equal to $(0.3250, 0.3393, 0.3357)^T$.

Step 4. By formula (48), we get the overall assessment values z_i $(i = 1, 2, 3)$ of the alternatives x_i $(i = 1, 2, 3)$:

$$
z_1 = \text{IFWA}_\lambda \left(z_1^{(1)}, z_1^{(2)}, z_1^{(3)} \right) = (0.7734, 0),
$$

$$
z_2 = \text{IFWA}_\lambda \left(z_2^{(1)}, z_2^{(2)}, z_2^{(3)} \right) = (0.7024, 0.2266),
\tag{53}
$$

$$
z_3 = \text{IFWA}_\lambda \left(z_3^{(1)}, z_3^{(2)}, z_3^{(3)} \right) = (0.7748, 0).
$$

Step 5. From Definition 6, we get

$$
s(z_1) = 0.7734, \qquad s(z_2) = 0.4758, \qquad s(z_3) = 0.7784.
\tag{54}
$$

So the ranking is $x_3 \succ x_1 \succ x_2$.

In fact, Xu's method [47] considered the consensus of experts' opinions. In this paper, we take into account

the uncertain degree of individual expert's evaluation information to assess the weights of experts. For a practical decision-making problem, we can combine the two points of view to derive the relative importance weights of experts according to the requirement of decision maker.

6. Conclusions

The entropy measures for IFSs and IVIFSs have been applied in many fields such as decision-making, pattern recognition, and medical diagnosis [31, 48]. In this work, we first propose a new entropy measure for IFSs. By analyzing the features of the entropy measure, a family of entropy measures is obtained for IFSs. We then extend these entropy measures to IVIFSs. The proposed entropy formulas can measure both fuzziness and intuitionism of IFSs and IVIFSs. Numerical examples are given to show that the proposed entropy formulas are a reasonable and effective supplement of entropy measures for IFSs and IVIFSs. As an application, we use the proposed entropy measures to assess the weights of experts and solve (interval-valued) intuitionistic fuzzy group decision-making problems with unknown experts' weights.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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