Research Article

Frequency Equation of Flexural Vibrating Cantilever Beam Considering the Rotary Inertial Moment of an Attached Mass

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The major goal of this paper is to address the derivation of the frequency equation of flexural vibrating cantilever beam considering the bending moment generated by an additional mass at the free end of beam, not just the shear force. It is a transcendental equation with two unambiguous physical meaning parameters. And the influence of the two parameters on the characteristics of frequency and shape mode was made. The results show that the inertial moment of the mass has the significant effect on the natural frequency and the shape mode. And it is more reasonable using this frequency equation to analyze vibration and measure modulus.

1. Introduction

The cantilever beam is a simple structure, and it is an important simplified model for many engineering problem in the fields of mechanical engineering, civil engineering, and so forth. However, there is a vast number of papers concerned with the determination of the eigenfrequencies of the cantilever beam subject to various boundary conditions, which can be found in the classic book [1]. And more and more issues of cantilever beam with complicated boundary conditions [2] or external load conditions [3, 4] have been studied by theoretical deduction [5] or numerical method [6], or it is application to determine Young's modulus [7].

The cantilever beam model is also used in geotechnical earthquake engineering as a simplified model [8] for the ground responses due to earthquake. It is also a basic principle of measuring the soil's dynamic shear modulus which is an indispensable parameter for analyzing the earthquake response of site caused by far-field ground. The apparatus used to obtain the dynamic shear modulus is well known as resonant column apparatus. The soil column, installed in this apparatus, is driven by electromagnetic force at free end [9] producing its torsional vibration. If flexural vibration

of the soil column occurs, the dynamic Young modulus can be obtained, which is also an important parameter for dynamic analysis of site suffered by near-field ground motion. Essentially, the frequency equation of flexural vibrating cantilever beam with an additional mass is needed. However, the vibrating frequency and shape mode of soil column are effected by not only the shear force but also the moment force, generated by the motion of the additional mass attached at the free end of the soil column. So the influence of these two forces on the vibrating of soil column must be discussed. Cascante et al. [10] derived a frequency equation of this vibrational problem using Rayleigh's method assuming that the mode shape is a third-order polynomial. Laura [11] derived the frequency equation of a cantilever beam attaching an additional mass, which is considered as shear force acted on the free end of beam but did not consider the moment force generated by the mass. And recently, Gürgöze [12–14] studies the eigenfrequencies of a cantilever beam carrying a tip mass or spring-mass.

The paper tried to derive the frequency equation of the cantilever beam in order to obtain more accurate solution for the soil column and to analyze the frequency characteristics
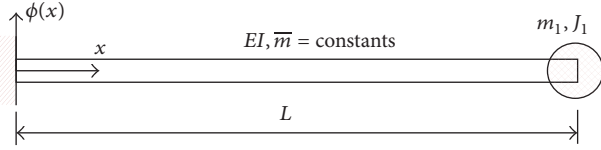
FIGURE 1: Diagram of the cantilever beam with an additional mass.

of this system effected by the bending moment generated by the additional mass.

2. Flexural Motion Equation of Cantilever Beam with an Additional Mass

The flexural cantilever beam to be considered is the straight, uniform beam with an additional rigid mass attached by fixed connection. The significant physical properties of this beam are assumed to be the flexural stiffness EI and the mass per unit length $\overline{m}$, both of which are constant along the span L. The transverse displacement response $y(x,t)$ is a function of position and time. The free vibration equation of motion for the system as shown in Figure 1 is easily formulated by directly expressing the equilibrium of all forces acting on the differential segment of beam. And it becomes [15]

$$EI\frac{\partial^4 y(x,t)}{\partial x^4} + \overline{m}\frac{\partial^2 y(x,t)}{\partial t^2} = 0. \tag{1}$$

Since EI and $\overline{m}$ are constant, one form of solution of this equation can be obtained by separation of variables using

$$y(x,t) = \phi(x)Y(t) \tag{2}$$

which indicates that the free vibration motion is of a specific shape $\phi(x)$ having a time-dependent amplitude $Y(t)$. Substituting this equation into (1) can yield two ordinary differential equations.

$$\ddot{Y}(t) + \omega^2 Y(t) = 0,$$
$$\phi''''(x) - \beta^4\phi(x) = 0 \tag{3}$$

in which $\beta^4 = \omega^2\overline{m}/EI$, and ω is the circular frequency. These equations have the solution separately as follows:

$$Y(t) = B_1\sin(\omega t) + B_2\cos(\omega t), \tag{4}$$

$$\phi(x) = A_1\cos\beta x + A_2\sin\beta x + A_3\mathrm{ch}\beta x + A_4\mathrm{sh}\beta x \tag{5}$$

in which constants B_1 and B_2 depend upon the initial displacement and velocity conditions; and real constants A_i must be evaluated so as to satisfy the known boundary conditions at the ends of the beam.

The cantilever beam considered has a fixed end, so its two known boundary conditions are

$$\phi(0) = 0,$$
$$\phi'(0) = 0. \tag{6}$$

Making use of (5) and its first partial derivative with respect to x, from (6) one obtains $A_3 = -A_1$ and $A_4 = -A_2$, So (5) becomes

$$\phi(x) = A_1\left(\cos\beta x - \mathrm{ch}\beta x\right) + A_2\left(\sin\beta x - \mathrm{sh}\beta x\right). \tag{7}$$

An additional rigid mass m_1 having a rotary mass moment of inertia J_1 is attached by fixed connection to its free end as also shown in Figure 2. These internal force components are along with the translational and rotary inertial force components $m_1\omega^2\phi(L)$ and $J_1\omega^2\phi'(L)$, respectively. So the force and moment equilibrium of the additional mass requires the boundary conditions

$$M(L) = EI\phi''(L) = -\omega^2\phi'(L)J_1,$$
$$V(L) = EI\phi'''(L) = -\omega^2\phi(L)m_1. \tag{8}$$

Making use of (5) and its first, second, and third partial derivative with respect to x and substituting them into (8) yield

$$\begin{bmatrix} \beta^2\left(\cos\beta L + \mathrm{ch}\beta L\right) + \dfrac{\omega^2 J_1}{EI}\beta\left(\sin\beta L + \mathrm{sh}\beta L\right) & \beta^2\left(\sin\beta L + \mathrm{sh}\beta L\right) - \dfrac{\omega^2 J_1}{EI}\beta\left(\cos\beta L - \mathrm{ch}\beta L\right) \\ \beta^3\left(\sin\beta L - \mathrm{sh}\beta L\right) + \dfrac{\omega^2 m_1}{EI}\left(\cos\beta L - \mathrm{ch}\beta L\right) & -\beta^3\left(\cos\beta L + \mathrm{ch}\beta L\right) + \dfrac{\omega^2 m_1}{EI}\left(\sin\beta L - \mathrm{sh}\beta L\right) \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{9}$$

For coefficients A_1 and A_2 to be nonzero, the determinant of the square matrix in this equation must equal zero, thus giving the frequency equation

$$\left[\beta^2\left(\cos\beta L + \mathrm{ch}\beta L\right) + \frac{\omega^2 J_1}{EI}\beta\left(\sin\beta L + \mathrm{sh}\beta L\right)\right]$$

$$\cdot\left[-\beta^3\left(\cos\beta L + \mathrm{ch}\beta L\right) + \frac{\omega^2 m_1}{EI}\left(\sin\beta L - \mathrm{sh}\beta L\right)\right]$$

$$-\left[\beta^2\left(\sin\beta L + \mathrm{sh}\beta L\right) - \frac{\omega^2 J_1}{EI}\beta\left(\cos\beta L - \mathrm{ch}\beta L\right)\right]$$

$$\cdot\left[\beta^3\left(\sin\beta L - \mathrm{sh}\beta L\right) + \frac{\omega^2 m_1}{EI}\left(\cos\beta L - \mathrm{ch}\beta L\right)\right]$$
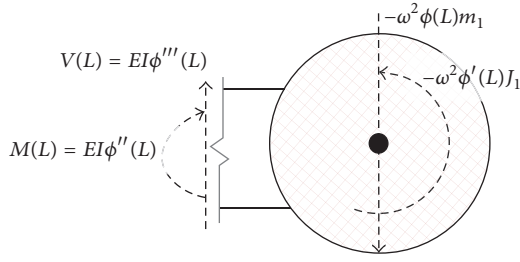
$$= 0 \tag{10}$$

FIGURE 2: Diagram of force equilibrium analysis at the free end.

which can reduce to the following form making use of $\omega^2/EI = \beta^4/\overline{m}$

$$\frac{J_1 m_1}{\overline{m}^2}\beta^4 \left(1 - \cos\beta L \mathrm{ch}\beta L\right)$$

$$+ \frac{m_1}{\overline{m}}\beta \left(\sin\beta L \mathrm{ch}\beta L - \cos\beta L \mathrm{sh}\beta L\right)$$

$$= \frac{J_1}{\overline{m}}\beta^3 \left(\sin\beta L \mathrm{ch}\beta L + \cos\beta L \mathrm{sh}\beta L\right) \tag{11}$$

$$+ \left(1 + \cos\beta L \mathrm{ch}\beta L\right).$$

Setting

$$z = \beta L = \sqrt[4]{\frac{\omega^2 \overline{m}}{EI}}L \tag{12}$$

is a power function of the frequency of the flexural vibrating cantilever beam. And setting $\alpha_J = J_1/\overline{m}L^3 = J_1/m_T L^2$ is a ratio of the rotary mass moment of inertia for the additional mass and for a rigid mass of m_T having a rotary arm length of L; and $\alpha_m = m_1/\overline{m}L = m_1/m_T$ is a ratio of the mass between the additional mass and the cantilever beam. So the frequency equation becomes

$$\alpha_J \alpha_m z^4 \left(1 - \cos z \mathrm{ch}z\right) + \alpha_m z \left(\sin z \mathrm{ch}z - \cos z \mathrm{sh}z\right)$$

$$= \alpha_J z^3 \left(\sin z \mathrm{ch}z + \cos z \mathrm{sh}z\right) + \left(1 + \cos z \mathrm{ch}z\right). \tag{13}$$

This frequency equation is a transcendental equation that contained two parameters with unambiguous physical meaning. The solution of this equation, which is the frequency of the system of the flexural vibrating cantilever beam, can be only obtained by numerical method for now.

3. Solution of the Frequency Equation

In order to solve the frequency equation, the term of the right side of (13) can be moved to the left side, and then suppose the left side equals $f(z, \alpha_J, \alpha_m)$. And $f = 0$ is equivalent to (13). Suppose the parameters α_J and α_m are constant, and then the relationship between $f(z)$ and z can be calculated and drawn as shown in Figure 3, in which the parameters $\alpha_J = 10$ and $\alpha_m = 1$. However, the value of $f(z)$ rises abruptly. So the y-axis alters to $\log f(z)$ in order to show the curve is up and down across the x-axis obviously. The intersection points of
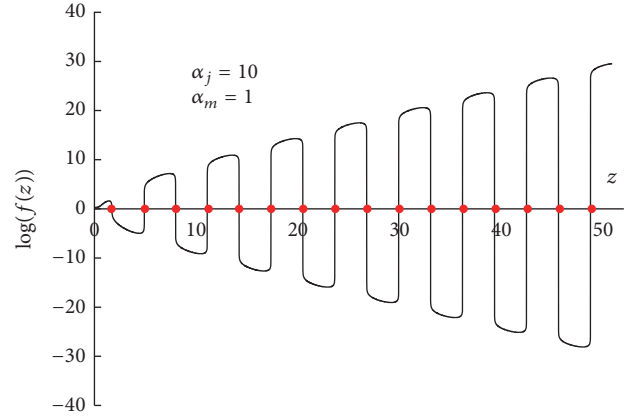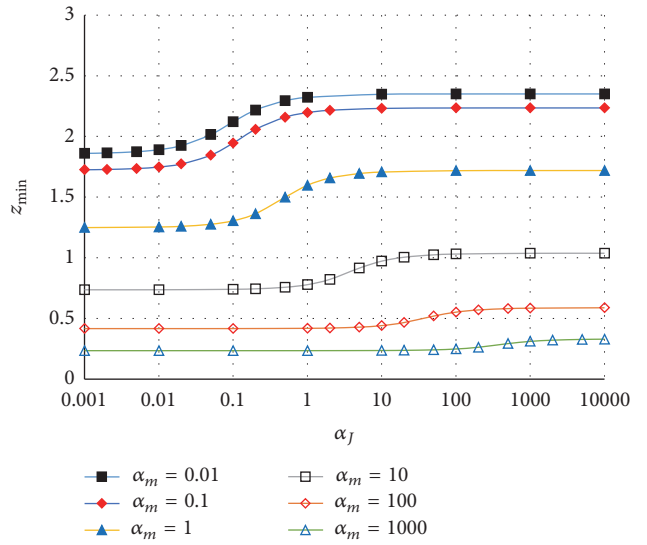

FIGURE 3: Relationship between $f(z)$ and z.

FIGURE 4: Varied z with α_J.

the curve and x-axis as marked in Figure 3 with circle points are the roots of the equation $f(z) = 0$, where its approximate roots can be obtained using numerical methods for solving system of nonlinear equations, such as method of bisection and Newton's method.

For a specific system, where the α_J and α_m are determined, the solution of the frequency equation (13) can be obtained. And the roots of the parameter z are varied with the parameters α_J and α_m. Figures 4 and 5 show their effects. With the increment of the parameter α_J, the root z is increased identically with a sigmoid function as shown in Figure 4. It means that the frequency of the system will increase if the rotary mass moment of inertia for the additional mass is increased. And the rate of the increment is larger around $\alpha_J = \alpha_m$ than others. For example, the increment of z at the range of $\alpha_J = 0.1 \sim 10$ is fastest while $\alpha_m = 1$. With the increment of the parameter α_m, the root z is decreasing, as shown in Figure 5, which means that the frequency of the system will decrease with the additional mass increasing.

TABLE 1: The first five natural frequencies of the cantilever beam considering the bending moment.

Frequency number (i)	Nondimensional natural frequency $\omega_i' = \sqrt{EI/\overline{m}L^4}z_i$									$\omega_i'*$
	$\alpha_m = 100$ $\alpha_J = 1$	$\alpha_m = 10$ $\alpha_J = 1$	$\alpha_m = 1$ $\alpha_J = 100$	$\alpha_m = 1$ $\alpha_J = 10$	$\alpha_m = 1$ $\alpha_J = 1$	$\alpha_m = 1$ $\alpha_J = 0.1$	$\alpha_m = 1$ $\alpha_J = 0.01$	$\alpha_m = 1$ $\alpha_J = 0.001$	$\alpha_m = 1$ $\alpha_J = 0$	$\alpha_m = 1$ (ref [11])
1	0.1749603	0.6054297	2.950242	2.91158	2.55044	1.70226	1.571098	1.55867	1.557298	1.557298
2	22.30366	22.47413	23.93841	23.93129	23.86031	23.17405	19.13831	16.57872	16.25009	16.25009
3	61.66046	61.83822	63.43232	63.42955	63.40190	63.1286	60.72695	53.50208	50.89584	50.89584
4	120.90684	121.0850	122.7265	122.7251	122.71062	122.5668	121.2014	113.2826	105.1983	105.1983
5	199.86943	200.0480	201.7208	201.71991	201.71107	201.6229	200.7639	194.2422	179.2320	179.2320

*Notes. The values of this column were calculated by authors using the frequency equation from [11].
And the results were carefully checked by comparing the values of z_i (using the character y_i in this reference).
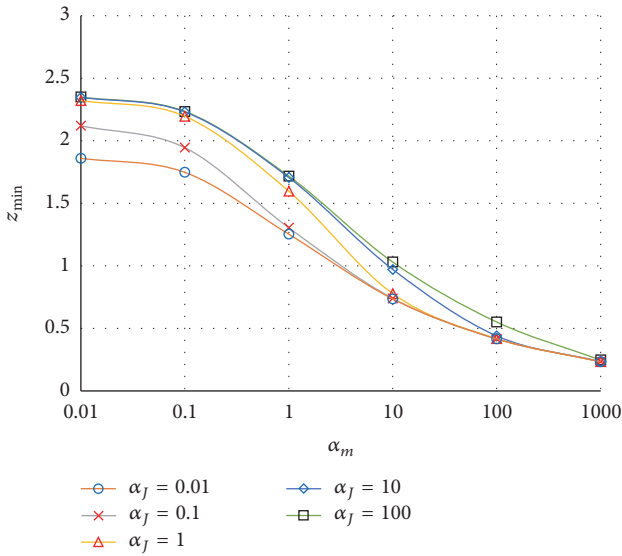

FIGURE 5: Varied z with α_m.

4. Natural Frequencies and Shape Modes Characteristics

The natural frequency ω_i of the flexural vibration of cantilever beam considering the bending moment generated by the attached mass can be calculated using the transformation of (12). Table 1 lists the results of the first five natural frequencies of this system with the varied parameters values α_j and α_m. The natural frequency increases with the parameter α_m decreasing, which means the quality of the additional mass reduction compared to that of the cantilever beam. This is consistent with the result of others [7, 16]. And the frequency also increases with the increment of the rotary mass moment of inertia for the additional mass compared to the beam, which is defined as the parameter α_j. In order to make comparisons, the results of frequencies when the parameters $\alpha_j = 0$ and $\alpha_m = 1$ are also listed, which means the free end of beam has no moment force generated by the additional mass. As expected, these natural frequency values are the same as that obtained from [11], in which only the shear force is considered generated by the attached mass. And the larger value of α_j causes the higher frequency of the cantilever beam as shown in Table 1.

However, the interval between the two frequencies for α_j equal to zero and for α_j unequal to zero is considerable, especially that between the two higher frequency numbers. So the ignored rotary mass moment of inertia may be too ideal for engineering problem and cause obviously error.

The vibration shape mode can also be obtained by substituting the ratio of A_1 and A_2 into the shape equation (7). And Figure 6(a) shows the first four mode shapes of the cantilever beam considering the moment force with the parameters $\alpha_m = 1$ and $\alpha_J = 10$. The first four mode shapes of the cantilever beam are shown without considering the moment force in [11]. It is clearly shown that the vibration shape is effected by the moment force from the additional mass significantly. However, the existence of the additional mass restrains the development of displacement at the free end of beam.

In order to analyze the influence of α_J on the mode shape, the 3rd to 5th mode shapes of considering the moment force with $\alpha_m = 1$ and $\alpha_J = 0.001$ and without considering situation are given, as shown in Figure 6(b). Even if the parameter α_J is of very small value, the 5th mode shape of these two situations has obviously divergence, though the smaller the serial number of mode shape is, the less obvious the divergence becomes.

5. Conclusion

The frequency equation of cantilever beam with an additional mass exciting flexural vibration was derived considering the rotary inertial moment of inertia of an attached mass, including the shear force. It is a transcendental equation, and it contains two parameters with unambiguous physical meaning, which can be defined as the ratio of rotary mass moment of inertia and the ratio of the mass, respectively.

These two parameters effect both the natural frequency and the shape mode of the beam. As the ratio of rotary mass moment of inertia increases, the natural frequency climbs. Even a little increment of the ratio may cause higher variance between considering and not considering the rotary mass moment of inertia, especially for the high natural frequency. And the ratio of the rotary mass moment of inertia also

Considering moment force at free end: ($\alpha_m = 1; \alpha_j = 10$)
- ■— 1st mode
- ▲— 3rd mode
- ●— 2nd mode
- ▼— 4th mode

Without considering moment force at free end
- –□– 1st mode
- –△– 3rd mode
- –○– 2nd mode
- –▽– 4th mode

(a)

Considering moment force at free end: ($\alpha_m = 1; \alpha_j = 0.001$)
- –△– 3rd mode
- –◇– 5th mode
- –▽– 4th mode

Without considering moment force at free end
- ▲— 3rd mode
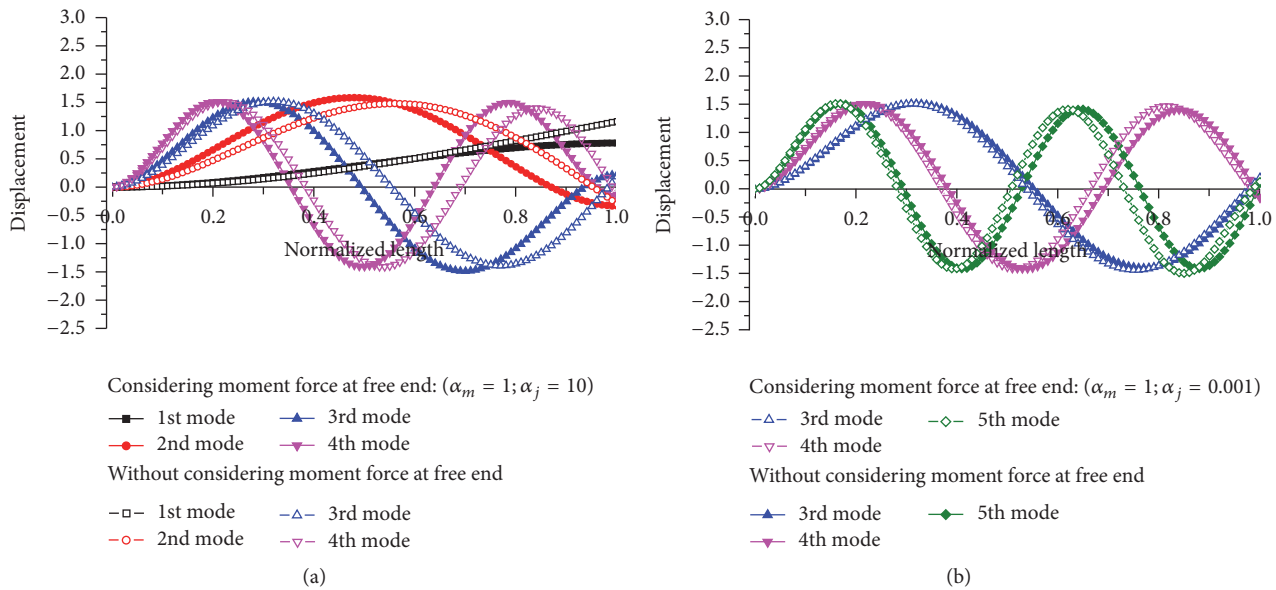- ◆— 5th mode
- ▼— 4th mode

(b)

FIGURE 6: The first five mode shapes of considering the moment force with $\alpha_m = 1$ and $\alpha_J = 10$ and without considering situation (a) and the 3rd to 5th mode shapes of considering the moment force with $\alpha_m = 1$ and $\alpha_J = 0.001$ and without considering situation (b).

effects the mode shape of this system. The higher the serial number of mode shape investigated is, the more obvious the divergence becomes.

Competing Interests

There are no competing interests regarding this paper.

Acknowledgments

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