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Research Article

The $(G'/G, 1/G)$-Expansion Method and Its Applications for Solving Two Higher Order Nonlinear Evolution Equations

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The two variable $(G'/G, 1/G)$-expansion method is employed to construct exact traveling wave solutions with parameters of two higher order nonlinear evolution equations, namely, the nonlinear Klein-Gordon equations and the nonlinear Pochhammer-Chree equations. When the parameters are replaced by special values, the well-known solitary wave solutions of these equations are rediscovered from the traveling waves. This method can be thought of as the generalization of well-known original (G'/G)-expansion method proposed by Wang et al. It is shown that the two variable $(G'/G, 1/G)$-expansion method provides a more powerful mathematical tool for solving many other nonlinear PDEs in mathematical physics.

1. Introduction

In the recent years, investigations of exact solutions to nonlinear PDEs play an important role in the study of nonlinear physical phenomena. Many powerful methods have been presented, such as the inverse scattering method [1], the Hirota bilinear transform method [2], the truncated Painleve expansion method [3–6], the Backlund transform method [7, 8], the exp-function method [9–13], the tanh-function method [14–17], the Jacobi elliptic function expansion method [18–20], the (G'/G)-expansion method [21–30], the modified (G'/G)-expansion method [31], and the $(G'/G, 1/G)$-expansion method [32–34]. The key idea of the one variable (G'/G)-expansion method is that the exact solutions of nonlinear PDEs can be expressed by a polynomial in one variable (G'/G) in which $G = G(\xi)$ satisfies the second order linear ODE $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, where λ and μ are constants and $' = d/d\xi$. The key idea of the two variable $(G'/G, 1/G)$-expansion method is that the exact traveling wave solutions of nonlinear PDEs can be expressed by a polynomial in two variables (G'/G) and $(1/G)$ in which $G = G(\xi)$ satisfies the second order linear ODE $G''(\xi) + \lambda G(\xi) = \mu$, where λ and μ are constants. The degree of this polynomial can be determined by considering the homogeneous balance between the highest-order derivatives and the nonlinear terms appearing in the given nonlinear PDEs. The coefficients of this polynomial can be obtained by solving a set of algebraic equations resulted from the process of using this method. Recently, Li et al. [32] have applied the $(G'/G, 1/G)$-expansion method and determined the exact solutions of Zakharov equations, while Zayed et al. [33, 34] have used this method to find the exact solutions of the combined KdV-mKdV equation and the Kadomtsev-Petviashvili equation.

The objective of this paper is to apply the two variable $(G'/G, 1/G)$-expansion method to find the exact traveling wave solutions of the higher order nonlinear Klein-Gordon equations [35]

$$u_{tt} - k^2 u_{xx} + \alpha u - \beta u^n + \gamma u^{2n-1} = 0, \quad n > 2, \tag{1}$$

and the higher order nonlinear Pochhammer-Chree equations [36]

$$u_{tt} - u_{xxtt} - \left(\alpha u + \beta u^{n+1} + \gamma u^{2n+1}\right)_{xx} = 0, \quad n \geq 1, \tag{2}$$

where α, β, γ, and k are constants.

Equation (1) plays an important role in many scientific applications, such as the solid state physics, the nonlinear optics, and the quantum field theory (see [37–39]).

Wazwaz [40, 41] investigated the nonlinear Klein-Gordon equations and found many types of exact traveling wave solutions including compact solutions, soliton solution, solitary patterns solutions, and periodic solutions using the tanh-function method. Zayed and Gepreel [35] have found the exact solutions of (1) using the (G'/G)-expansion method. Equation (2) represents nonlinear models of longitudinal wave propagation of elastic rods; see [42–47]. Zuo [36] has discussed (2) using the extended (G'/G)-expansion method and found the traveling wave solutions of these equations. The rest of this paper is organized as follows. In Section 2, we give the description of the two variable $(G'/G, 1/G)$-expansion method. In Section 3, we apply this method to solve (1) and (2). In Section 4, some conclusions are given.

2. Description of the Two Variable $(G'/G, 1/G)$-Expansion Method

Before we describe the main steps of this method, we need the following remarks (see [32–34]):

Remark 1. If we consider the second order linear ODE

$$G''(\xi) + \lambda G(\xi) = \mu, \tag{3}$$

and set $\phi = G'/G$, $\psi = 1/G$, then we get

$$\phi' = -\phi^2 + \mu\psi - \lambda, \qquad \psi' = -\phi\psi, \tag{4}$$

where λ and μ are constants.

Remark 2. If $\lambda < 0$, then the general solution of (3) has the following form:

$$G(\xi) = A_1 \sinh\left(\xi\sqrt{-\lambda}\right) + A_2 \cosh\left(\xi\sqrt{-\lambda}\right) + \frac{\mu}{\lambda}, \tag{5}$$

where A_1 and A_2 are arbitrary constants. Consequently, we have

$$\psi^2 = \frac{-\lambda}{\lambda^2\sigma_1 + \mu^2}\left(\phi^2 - 2\mu\psi + \lambda\right), \tag{6}$$

where $\sigma_1 = A_1^2 - A_2^2$.

Remark 3. If $\lambda > 0$, then the general solution of (3) has the following form:

$$G(\xi) = A_1 \sin\left(\xi\sqrt{\lambda}\right) + A_2 \cos\left(\xi\sqrt{\lambda}\right) + \frac{\mu}{\lambda}, \tag{7}$$

and hence

$$\psi^2 = \frac{\lambda}{\lambda^2\sigma_2 - \mu^2}\left(\phi^2 - 2\mu\psi + \lambda\right), \tag{8}$$

where $\sigma_2 = A_1^2 + A_2^2$.

Remark 4. If $\lambda = 0$, then the general solution of (3) has the following form:

$$G(\xi) = \frac{\mu}{2}\xi^2 + A_1\xi + A_2, \tag{9}$$

and hence

$$\psi^2 = \frac{1}{A_1^2 - 2\mu A_2}\left(\phi^2 - 2\mu\psi\right). \tag{10}$$

Suppose we have the following nonlinear evolution equation

$$F(u, u_t, u_x, u_{xx}, \ldots) = 0, \tag{11}$$

where F is a polynomial in $u(x,t)$ and its partial derivatives. In the following, we give the main steps of the $(G'/G, 1/G)$-expansion method [32–34].

Step 1. The traveling wave transformation

$$u(x,t) = u(\xi), \qquad \xi = x - Ct, \tag{12}$$

where C is a constant and reduces (11) to an ODE in the following form:

$$P\left(u, u', u'', \ldots\right) = 0, \tag{13}$$

where P is a polynomial of $u(\xi)$ and its total derivatives with respect to ξ.

Step 2. Assuming that the solution of (13) can be expressed by a polynomial in the two variables ϕ and ψ as follows:

$$u(\xi) = \sum_{i=0}^{N} a_i\phi^i + \sum_{i=1}^{N} b_i\phi^{i-1}\psi, \tag{14}$$

where a_i $(i = 0, 1, 2, \ldots, N)$ and b_i $(i = 1, 2, \ldots, N)$ are constants to be determined later.

Step 3. Determine the positive integer N in (14) by using the homogeneous balance between the highest-order derivatives and the nonlinear terms in (13). In some nonlinear equations the balance number N is not a positive integer. In this case, we make the following transformations [48]:

(a) when $N = q/p$, where q/p is a fraction in the lowest terms, we let

$$u(\xi) = v^{q/p}(\xi), \tag{15}$$

then substitute (15) into (13) to get a new equation in the new function $v(\xi)$ with a positive integer balance number;

(b) when N is a negative number, we let

$$u(\xi) = v^{N}(\xi), \tag{16}$$

and substitute (16) into (13) to get a new equation in the new function $v(\xi)$ with a positive integer balance number.

Step 4. Substituting (14) into (13) along with (4) and (6), the left-hand side of (13) can be converted into a polynomial in ϕ and ψ, in which the degree of ψ is no longer than 1. Equating each coefficients of this polynomial to zero yields a system of algebraic equations which can be solved by using the Maple or Mathematica to get the values of a_i, b_i, C, μ, A_1, A_2, and λ, where $\lambda < 0$.

Step 5. Similar to Step 4, substituting (14) into (13) along with (4) and (8) for $\lambda > 0$ (or (4) and (10) for $\lambda = 0$), we obtain the exact solutions of (13) expressed by trigonometric functions (or by rational functions), respectively.

3. Applications

In this section, we will apply the method described in Section 2 to find the exact traveling wave solutions of (1) and (2) which are very important in the mathematical physics and have been paid attention to by many researchers.

Example 5. In this example, we start with the higher order nonlinear Klein-Gordon equation (1). To this end, we see that the traveling wave variable (12) permits us to convert (1) into the following ODE:

$$\left(C^2 - k^2\right) u'' + \alpha u - \beta u^n + \gamma u^{2n-1} = 0, \quad n > 2. \tag{17}$$

Let us discuss the following two possibilities:

(I) If $\gamma \neq 0$.

By balancing between u'' and u^{2n-1} in (17) we get $N = 1/(n-1)$. According to Step 3, we use the transformation

$$u(\xi) = v^{1/(n-1)}(\xi), \tag{18}$$

where $v(\xi)$ is a new function of ξ. Substituting (18) into (17), we get the new ODE

$$\left(C^2 - k^2\right)\left[\frac{2-n}{(n-1)^2}\left(v'\right)^2 + \frac{1}{(n-1)} v v''\right] + \alpha v^2 - \beta v^3 + \gamma v^4 = 0. \tag{19}$$

Determining the balance number N of the new (19), we get $N = 1$. Consequently, we get

$$v(\xi) = a_0 + a_1 \phi(\xi) + b_1 \psi(\xi), \tag{20}$$

where a_0, a_1, and b_1 are constants to be determined later. There are three cases to be discussed as follows.

Case 1 (hyperbolic function solutions ($\lambda < 0$)). If $\lambda < 0$, substituting (20) into (19) and using (4) and (6), the left-hand side of (19) becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be zero yields a system of algebraic equations in a_0, a_1, b_1, μ, λ, and C as follows:

$$\begin{aligned}
\phi^4:\ & \gamma a_1^4 + \frac{2\left(C^2-k^2\right)a_1^2}{(n-1)} + \frac{2\left(C^2-k^2\right)a_1^2}{(n-1)^2} + \frac{\lambda^2\gamma b_1^4}{\left(\lambda^2\sigma_1+\mu^2\right)^2} \\
& - \frac{\left(C^2-k^2\right)n a_1^2}{(n-1)^2} - \frac{6\lambda\gamma a_1^2 b_1^2}{\left(\lambda^2\sigma_1+\mu^2\right)} - \frac{2\left(C^2-k^2\right)\lambda b_1^2}{(n-1)\left(\lambda^2\sigma_1+\mu^2\right)} \\
& - \frac{2\left(C^2-k^2\right)\lambda b_1^2}{(n-1)^2\left(\lambda^2\sigma_1+\mu^2\right)} + \frac{\left(C^2-k^2\right)n\lambda b_1^2}{(n-1)^2\left(\lambda^2\sigma_1+\mu^2\right)} = 0, \\
\phi^3:\ & -\beta a_1^3 + 4\gamma a_0 a_1^3 + \frac{2\left(C^2-k^2\right)a_0 a_1}{(n-1)} + \frac{3\beta\lambda a_1 b_1^2}{\left(\lambda^2\sigma_1+\mu^2\right)} \\
& - \frac{12\lambda\gamma a_0 a_1 b_1^2}{\left(\lambda^2\sigma_1+\mu^2\right)} - \frac{8\lambda^2\gamma\mu a_1 b_1^3}{\left(\lambda^2\sigma_1+\mu^2\right)^2} + \frac{4\left(C^2-k^2\right)\lambda\mu a_1 b_1}{(n-1)\left(\lambda^2\sigma_1+\mu^2\right)} \\
& + \frac{4\left(C^2-k^2\right)\lambda\mu a_1 b_1}{(n-1)^2\left(\lambda^2\sigma_1+\mu^2\right)} - \frac{2\left(C^2-k^2\right)n\lambda\mu a_1 b_1}{(n-1)^2\left(\lambda^2\sigma_1+\mu^2\right)} = 0, \\
\phi^3\psi:\ & 4\gamma a_1^3 b_1 + \frac{4\left(C^2-k^2\right)a_1 b_1}{(n-1)} + \frac{4\left(C^2-k^2\right)a_1 b_1}{(n-1)^2} \\
& - \frac{2\left(C^2-k^2\right)n a_1 b_1}{(n-1)^2} - \frac{4\lambda\gamma a_1 b_1^3}{\left(\lambda^2\sigma_1+\mu^2\right)} = 0, \\
\phi^2:\ & \alpha a_1^2 - 3\beta a_0 a_1^2 + 6\gamma a_0^2 a_1^2 + \frac{2\lambda^3\gamma b_1^4}{\left(\lambda^2\sigma_1+\mu^2\right)^2} \\
& + \frac{2\left(C^2-k^2\right)\lambda a_1^2}{(n-1)} + \frac{4\left(C^2-k^2\right)\lambda a_1^2}{(n-1)^2} - \frac{\alpha\lambda b_1^2}{\left(\lambda^2\sigma_1+\mu^2\right)} \\
& + \frac{2\beta\lambda^2\mu b_1^3}{\left(\lambda^2\sigma_1+\mu^2\right)^2} - \frac{3\left(C^2-k^2\right)\lambda^2 b_1^2}{(n-1)\left(\lambda^2\sigma_1+\mu^2\right)} \\
& - \frac{2\left(C^2-k^2\right)\lambda^2 b_1^2}{(n-1)^2\left(\lambda^2\sigma_1+\mu^2\right)} - \frac{6\lambda\gamma a_0^2 b_1^2}{\left(\lambda^2\sigma_1+\mu^2\right)} \\
& - \frac{2\left(C^2-k^2\right)n\lambda a_1^2}{(n-1)^2} - \frac{4\lambda^3\gamma\mu^2 b_1^4}{\left(\lambda^2\sigma_1+\mu^2\right)^3} - \frac{6\lambda^2\gamma a_1^2 b_1^2}{\left(\lambda^2\sigma_1+\mu^2\right)} \\
& + \frac{3\beta\lambda a_0 b_1^2}{\left(\lambda^2\sigma_1+\mu^2\right)} + \frac{\left(C^2-k^2\right)n\lambda^2 b_1^2}{(n-1)^2\left(\lambda^2\sigma_1+\mu^2\right)} \\
& - \frac{2\left(C^2-k^2\right)\lambda\mu^2 a_1^2}{(n-1)^2\left(\lambda^2\sigma_1+\mu^2\right)} - \frac{8\lambda^2\gamma\mu a_0 b_1^3}{\left(\lambda^2\sigma_1+\mu^2\right)^2} \\
& + \frac{2\left(C^2-k^2\right)\lambda^2\mu^2 b_1^2}{(n-1)\left(\lambda^2\sigma_1+\mu^2\right)^2} + \frac{\left(C^2-k^2\right)n\lambda\mu^2 a_1^2}{(n-1)^2\left(\lambda^2\sigma_1+\mu^2\right)} \\
& + \frac{\left(C^2-k^2\right)\lambda\mu a_0 b_1}{(n-1)\left(\lambda^2\sigma_1+\mu^2\right)} = 0,
\end{aligned}$$

$$\phi^2\psi:\ \frac{2\left(C^2-k^2\right)a_0b_1}{(n-1)} - 3\beta a_1^2 b_1 - \frac{3\left(C^2-k^2\right)\mu a_1^2}{(n-1)} - \frac{4\left(C^2-k^2\right)\mu a_1^2}{(n-1)^2} + 12\gamma a_0 a_1^2 b_1 + \frac{\beta\lambda b_1^3}{(\lambda^2\sigma_1+\mu^2)} - \frac{4\lambda^2\gamma\mu b_1^4}{(\lambda^2\sigma_1+\mu^2)^2} + \frac{2\left(C^2-k^2\right)n\mu a_1^2}{(n-1)^2} - \frac{4\lambda\gamma a_0 b_1^3}{(\lambda^2\sigma_1+\mu^2)} + \frac{5\left(C^2-k^2\right)\lambda\mu b_1^2}{(n-1)(\lambda^2\sigma_1+\mu^2)} + \frac{4\left(C^2-k^2\right)\lambda\mu b_1^2}{(n-1)^2(\lambda^2\sigma_1+\mu^2)} + \frac{12\lambda\gamma\mu a_1^2 b_1^2}{(\lambda^2\sigma_1+\mu^2)} - \frac{2\left(C^2-k^2\right)n\lambda\mu b_1^2}{(n-1)^2(\lambda^2\sigma_1+\mu^2)} = 0,$$

$$\phi:\ -3\beta a_0^2 a_1 + 4\gamma a_0^3 a_1 + 2\alpha a_0 a_1 + \frac{2\left(C^2-k^2\right)\lambda a_0 a_1}{(n-1)} + \frac{3\beta\lambda^2 a_1 b_1^2}{(\lambda^2\sigma_1+\mu^2)} - \frac{8\lambda^3\gamma\mu a_1 b_1^3}{(\lambda^2\sigma_1+\mu^2)^2} - \frac{12\lambda^2\gamma a_0 a_1 b_1^2}{(\lambda^2\sigma_1+\mu^2)} + \frac{4\left(C^2-k^2\right)\lambda^2\mu a_1 b_1}{(n-1)(\lambda^2\sigma_1+\mu^2)} + \frac{4\left(C^2-k^2\right)\lambda^2\mu a_1 b_1}{(n-1)^2(\lambda^2\sigma_1+\mu^2)} - \frac{2\left(C^2-k^2\right)n\lambda^2\mu a_1 b_1}{(n-1)^2(\lambda^2\sigma_1+\mu^2)} = 0,$$

$$\phi\psi:\ 2\alpha a_1 b_1 - 6\beta a_0 a_1 b_1 + 12\gamma a_0^2 a_1 b_1 - \frac{3\left(C^2-k^2\right)\mu a_0 a_1}{(n-1)} + \frac{3\left(C^2-k^2\right)\lambda a_1 b_1}{(n-1)} + \frac{4\left(C^2-k^2\right)\lambda a_1 b_1}{(n-1)^2} - \frac{4\lambda^2\gamma a_1 b_1^3}{(\lambda^2\sigma_1+\mu^2)} + \frac{16\lambda^2\gamma\mu^2 a_1 b_1^3}{(\lambda^2\sigma_1+\mu^2)^2} - \frac{6\beta\lambda\mu a_1 b_1^2}{(\lambda^2\sigma_1+\mu^2)} - \frac{2\left(C^2-k^2\right)n\lambda a_1 b_1}{(n-1)^2} - \frac{8\left(C^2-k^2\right)\lambda\mu^2 a_1 b_1}{(n-1)(\lambda^2\sigma_1+\mu^2)} - \frac{8\left(C^2-k^2\right)\lambda\mu^2 a_1 b_1}{(n-1)^2(\lambda^2\sigma_1+\mu^2)} + \frac{24\lambda\gamma\mu a_0 a_1 b_1^2}{(\lambda^2\sigma_1+\mu^2)} + \frac{4\left(C^2-k^2\right)n\lambda\mu^2 a_1 b_1}{(n-1)^2(\lambda^2\sigma_1+\mu^2)} = 0,$$

$$\psi:\ 4\gamma a_0^3 b_1 - 3\beta a_0^2 b_1 + 2\alpha a_0 b_1 + \frac{\beta\lambda^2 b_1^3}{(\lambda^2\sigma_1+\mu^2)} + \frac{\left(C^2-k^2\right)\lambda a_0 b_1}{(n-1)} - \frac{4\lambda^3\gamma\mu b_1^4}{(\lambda^2\sigma_1+\mu^2)^2} - \frac{4\lambda^2\gamma a_0 b_1^3}{(\lambda^2\sigma_1+\mu^2)} - \frac{4\left(C^2-k^2\right)\lambda\mu a_1^2}{(n-1)^2} - \frac{4\beta\lambda^2\mu^2 b_1^3}{(\lambda^2\sigma_1+\mu^2)^2} + \frac{8\lambda^3\gamma\mu^3 b_1^4}{(\lambda^2\sigma_1+\mu^2)^3} + \frac{2\alpha\lambda\mu b_1^2}{(\lambda^2\sigma_1+\mu^2)} + \frac{16\lambda^2\gamma\mu^2 a_0 b_1^3}{(\lambda^2\sigma_1+\mu^2)^2} - \frac{6\beta\lambda\mu a_0 b_1^2}{(\lambda^2\sigma_1+\mu^2)} + \frac{3\left(C^2-k^2\right)\lambda^2\mu b_1^2}{(n-1)(\lambda^2\sigma_1+\mu^2)} + \frac{4\left(C^2-k^2\right)\lambda\mu^3 a_1^2}{(n-1)^2(\lambda^2\sigma_1+\mu^2)} + \frac{12\lambda\gamma\mu a_0^2 b_1^2}{(\lambda^2\sigma_1+\mu^2)} + \frac{2\left(C^2-k^2\right)n\lambda\mu a_1^2}{(n-1)^2} - \frac{4\left(C^2-k^2\right)\lambda^2\mu^3 b_1^2}{(n-1)(\lambda^2\sigma_1+\mu^2)^2} - \frac{2\left(C^2-k^2\right)\lambda\mu^2 a_0 b_1}{(n-1)(\lambda^2\sigma_1+\mu^2)} - \frac{2\left(C^2-k^2\right)n\lambda\mu^3 a_1^2}{(n-1)^2(\lambda^2\sigma_1+\mu^2)} = 0,$$

$$\phi^0:\ \alpha a_0^2 - \beta a_0^3 + \gamma a_0^4 - \frac{\alpha\lambda^2 b_1^2}{(\lambda^2\sigma_1+\mu^2)} + \frac{\lambda^4\gamma b_1^4}{(\lambda^2\sigma_1+\mu^2)^2} + \frac{2\left(C^2-k^2\right)\lambda^2 a_1^2}{(n-1)^2} + \frac{2\beta\lambda^3\mu b_1^3}{(\lambda^2\sigma_1+\mu^2)^2} - \frac{\left(C^2-k^2\right)\lambda^3 b_1^2}{(n-1)(\lambda^2\sigma_1+\mu^2)} + \frac{3\beta\lambda^2 a_0 b_1^2}{(\lambda^2\sigma_1+\mu^2)} - \frac{4\lambda^4\gamma\mu^2 b_1^4}{(\lambda^2\sigma_1+\mu^2)^3} - \frac{6\lambda^2\gamma a_0^2 b_1^2}{(\lambda^2\sigma_1+\mu^2)} - \frac{\left(C^2-k^2\right)n\lambda^2 a_1^2}{(n-1)^2} - \frac{8\lambda^3\gamma\mu a_0 b_1^3}{(\lambda^2\sigma_1+\mu^2)^2} - \frac{2\left(C^2-k^2\right)\lambda^2\mu^2 a_1^2}{(n-1)^2(\lambda^2\sigma_1+\mu^2)} + \frac{2\left(C^2-k^2\right)\lambda^3\mu^2 b_1^2}{(n-1)(\lambda^2\sigma_1+\mu^2)^2} + \frac{\left(C^2-k^2\right)\lambda^2\mu a_0 b_1}{(n-1)(\lambda^2\sigma_1+\mu^2)} + \frac{\left(C^2-k^2\right)n\lambda^2\mu^2 a_1^2}{(n-1)^2(\lambda^2\sigma_1+\mu^2)} = 0. \tag{21}$$

On solving the above algebraic equations using the Maple or Mathematica, we get the following results.

Result 1. Consider the following:

$$\begin{aligned} a_0 &= 0, \\ a_1 &= 0, \\ b_1 &= \frac{n\beta\left(\lambda^2\sigma_1+\mu^2\right)}{\mu\lambda\gamma(n+1)}, \\ C &= \pm\sqrt{\frac{(n-1)^2\alpha}{\lambda}+k^2}, \\ \alpha &= \frac{n\beta^2\left(\lambda^2\sigma_1+\mu^2\right)}{\mu^2\gamma(n+1)^2}, \end{aligned} \tag{22}$$

where $\mu \neq 0$, $\gamma \neq 0$, and $\beta \neq 0$.

From (5), (20), and (22), we deduce the traveling wave solution of (17) as follows:

$$u(\xi) = \left[\frac{n\beta(\lambda^2\sigma_1+\mu^2)}{\mu\lambda\gamma(n+1)} \times \left(\frac{1}{A_1\sinh(\xi\sqrt{-\lambda})+A_2\cosh(\xi\sqrt{-\lambda})+\mu/\lambda}\right)\right]^{1/(n-1)}, \tag{23}$$

where $\xi = x \pm t\sqrt{(n-1)^2\alpha/\lambda + k^2}$.

Result 2 (Figure 1). Consider the following:

$$\begin{aligned} a_0 &= \frac{n\beta}{2\gamma(n+1)}, \\ a_1 &= \pm\frac{n\beta\sqrt{-1/\lambda}}{2\gamma(n+1)}, \\ b_1 &= \pm\frac{n\beta\sqrt{\lambda^2\sigma_1+\mu^2}}{2\gamma\lambda(n+1)}, \\ \alpha &= \frac{n\beta^2}{\gamma(n+1)^2}, \\ C &= \pm\sqrt{\frac{(n-1)^2\alpha}{\lambda}+k^2}, \end{aligned} \tag{24}$$

where $\gamma \neq 0$ and $\beta \neq 0$.

In this result, we deduce the traveling wave solution of (17) as follows:

$$u(\xi) = \left[\frac{n\beta}{2\gamma(n+1)} \pm \frac{n\beta}{2\gamma(n+1)} \times \left(\frac{A_1\cosh(\xi\sqrt{-\lambda})+A_2\sinh(\xi\sqrt{-\lambda})}{A_1\sinh(\xi\sqrt{-\lambda})+A_2\cosh(\xi\sqrt{-\lambda})+\mu/\lambda}\right) \pm \frac{n\beta}{2\lambda\gamma(n+1)} \times \left(\frac{\sqrt{\lambda^2\sigma_1+\mu^2}}{A_1\sinh(\xi\sqrt{-\lambda})+A_2\cosh(\xi\sqrt{-\lambda})+\mu/\lambda}\right)\right]^{1/(n-1)}, \tag{25}$$

where $\xi = x \pm t\sqrt{(n-1)^2\alpha/\lambda + k^2}$.

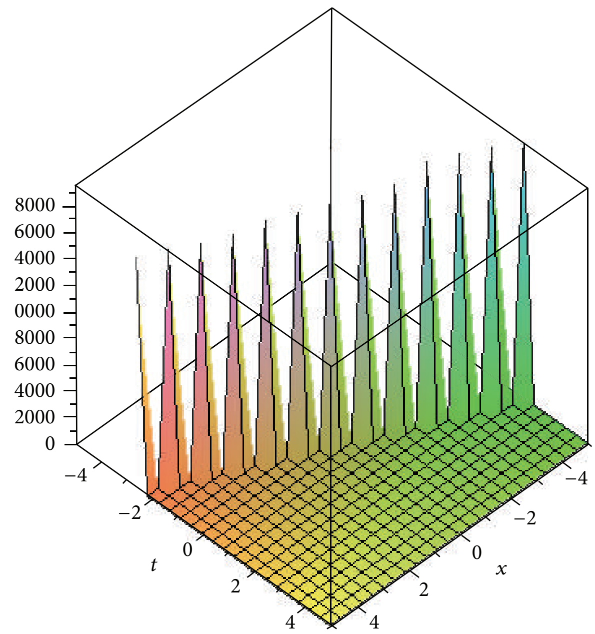

Figure 1: The plot of solution (27) when $\lambda = -1$, $\gamma = 1$, $\beta = 4$, $n = 3$, and $k = 4$.

In particular, by setting $A_1 = 0$, $A_2 \neq 0$, and $\mu = 0$ in (25), we have the solitary solution

$$u(\xi) = \left[\frac{n\beta}{2\gamma(n+1)} \pm \frac{n\beta}{2\gamma(n+1)} \times \left(\tanh(\xi\sqrt{-\lambda}) + i\,\mathrm{sech}(\xi\sqrt{-\lambda})\right)\right]^{1/(n-1)}, \quad i = \sqrt{-1}, \tag{26}$$

while if $A_1 \neq 0$, $A_2 = 0$, and $\mu = 0$, then we have the solitary solution

$$u(\xi) = \left[\frac{n\beta}{2\gamma(n+1)} \pm \frac{n\beta}{2\gamma(n+1)} \times \left(\coth(\xi\sqrt{-\lambda}) + \mathrm{csch}(\xi\sqrt{-\lambda})\right)\right]^{1/(n-1)}. \tag{27}$$

Case 2 (trigonometric function solution ($\lambda > 0$)). If $\lambda > 0$, substituting (20) into (19) and using (4) and (8), the left-hand side of (19) becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be zero yields a system of algebraic equations in $a_0, a_1, b_1, \mu, \lambda$, and C which are omitted here for simplicity. On using the Maple or Mathematica we have found the following results:

$$\begin{aligned} a_0 &= 0, \\ a_1 &= 0, \\ b_1 &= \frac{n\beta\left(\mu^2 - \lambda^2\sigma_2\right)}{\mu\lambda\gamma(n+1)}, \\ C &= \pm\sqrt{\frac{(n-1)^2\alpha}{\lambda} + k^2}, \\ \alpha &= \frac{n\beta^2\left(\mu^2 - \lambda^2\sigma_2\right)}{\mu^2\gamma(n+1)^2}, \end{aligned} \quad (28)$$

where $\mu \neq 0$, $\gamma \neq 0$, and $\beta \neq 0$.

From (7), (20), and (28), we deduce the traveling wave solution of (17) as follows:

$$u(\xi) = \left[\frac{n\beta\left(\mu^2 - \lambda^2\sigma_2\right)}{\mu\lambda\gamma(n+1)} \times \left(\frac{1}{A_1\sin(\xi\sqrt{\lambda}) + A_2\cos(\xi\sqrt{\lambda}) + \mu/\lambda}\right)\right]^{1/(n-1)}, \quad (29)$$

where $\xi = x \pm t\sqrt{(n-1)^2\alpha/\lambda + k^2}$.

Case 3 (rational function solutions ($\lambda = 0$)). If $\lambda = 0$, substituting (20) into (19) and using (4) and (10), the left-hand side of (19) becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be zero yields a system of algebraic equations in a_0, a_1, b_1, μ, and C which are omitted here for simplicity. On using the Maple or Mathematica we have found the following results:

$$\begin{aligned} a_0 &= 0, \\ a_1 &= 0, \\ b_1 &= \frac{n\beta\left(2\mu A_2 - A_1^2\right)}{\mu\gamma(n+1)}, \\ C &= \pm\sqrt{\frac{\beta^2 n(n-1)^2\left(2\mu A_2 - A_1^2\right)}{\mu^2\gamma(n+1)^2} + k^2}, \\ \alpha &= 0. \end{aligned} \quad (30)$$

From (9), (20), and (30), we deduce the traveling wave solution of (17) as follows:

$$u(\xi) = \left[\frac{n\beta(2\mu A_2 - A_1^2)}{\mu\gamma(n+1)}\left(\frac{1}{(\mu/2)\xi^2 + A_1\xi + A_2}\right)\right]^{1/(n-1)}, \quad (31)$$

where $\xi = x \pm t\sqrt{\beta^2 n(n-1)^2(2\mu A_2 - A_1^2)/\mu^2\gamma(n+1)^2 + k^2}$.

(II) If $\gamma = 0$.

In this case, (17) converts to

$$\left(C^2 - k^2\right)u'' + \alpha u - \beta u^n = 0, \quad n > 2. \quad (32)$$

By balancing between u'' and u^n in (32) we get $N = 2/(n-1)$. According to Step 3, we use the transformation

$$u(\xi) = v^{2/(n-1)}(\xi), \quad (33)$$

where $v(\xi)$ is a new function of ξ. Substituting (33) into (32), we get the new ODE

$$\left(C^2 - k^2\right)\left[\frac{2(3-n)}{(n-1)^2}\left(v'\right)^2 + \frac{2}{(n-1)}vv''\right] + \alpha v^2 - \beta v^4 = 0. \quad (34)$$

Determining the balance number N of the new (34), we get $N = 1$. Consequently, we get

$$v(\xi) = a_0 + a_1\phi(\xi) + b_1\psi(\xi), \quad (35)$$

where a_0, a_1, and b_1 are constants to be determined later. There are three cases to be discussed as follows.

Case 1 (hyperbolic function solutions ($\lambda < 0$)) (Figure 2). If $\lambda < 0$, substituting (35) into (34) and using (4) and (6), the left-hand side of (34) becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be zero yields a system of algebraic equations in a_0, a_1, b_1, μ, λ, and C which are omitted here for simplicity. On using the Maple or Mathematica we have found the following results:

$$\begin{aligned} a_0 &= 0, \\ a_1 &= 0, \\ b_1 &= \pm\sqrt{-\frac{\alpha\sigma_1(n+1)}{2\beta}}, \\ C &= C, \\ \mu &= 0, \\ \lambda &= \frac{\alpha(n-1)^2}{4\left(C^2 - k^2\right)}, \end{aligned} \quad (36)$$

where $\alpha \neq 0$ and $\beta \neq 0$.

From (5), (35), and (36), we deduce the traveling wave solutions of (32) as follows:

$$u(\xi) = \left[\pm\sqrt{-\frac{\alpha\sigma_1(n+1)}{2\beta}} \times \left(A_1\sinh\left(\frac{(n-1)}{2}\sqrt{-\frac{\alpha}{(C^2-k^2)}}\xi\right) + A_2\cosh\left(\frac{(n-1)}{2}\sqrt{-\frac{\alpha}{(C^2-k^2)}}\xi\right)\right)^{-1}\right]^{2/(n-1)}, \quad (37)$$

where $\xi = x - Ct$.

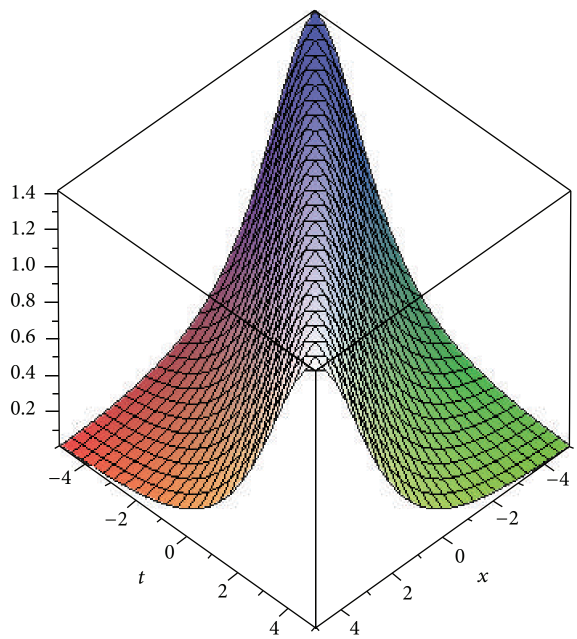

Figure 2: The plot of solution (38) when $\alpha = 1, C = 1, \beta = 1, n = 3$, and $k = 2$.

In particular, by setting $A_1 = 0$ and $A_2 \neq 0$ in (37), we have the solitary wave solutions

$$u(\xi) = \left[\pm\sqrt{\frac{\alpha(n+1)}{2\beta}}\operatorname{sech}\left(\frac{(n-1)}{2}\sqrt{-\frac{\alpha}{(C^2-k^2)}}\xi\right)\right]^{2/(n-1)}, \quad (38)$$

while if $A_1 \neq 0$ and $A_2 = 0$, then we have the solitary wave solutions

$$u(\xi) = \left[\pm\sqrt{-\frac{\alpha(n+1)}{2\beta}}\operatorname{csch}\left(\frac{(n-1)}{2}\sqrt{-\frac{\alpha}{(C^2-k^2)}}\xi\right)\right]^{2/(n-1)}. \quad (39)$$

Case 2 (trigonometric function solution ($\lambda > 0$)). If $\lambda > 0$, substituting (35) into (34) and using (4) and (8), the left-hand side of (34) becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be zero yields a system of algebraic equations in $a_0, a_1, b_1, \mu, \lambda$, and C which are omitted here for simplicity. On using the Maple or Mathematica we have found the following results:

$$a_0 = 0,$$
$$a_1 = 0,$$
$$b_1 = \pm\sqrt{\frac{\alpha\sigma_2(n+1)}{2\beta}},$$
$$C = C,$$
$$\mu = 0,$$
$$\lambda = \frac{\alpha(n-1)^2}{4(C^2-k^2)}, \quad (40)$$

where $\alpha \neq 0$ and $\beta \neq 0$.

From (7), (35), and (40), we deduce the traveling wave solutions of (32) as follows:

$$u(\xi) = \left[\pm\sqrt{\frac{\alpha\sigma_2(n+1)}{2\beta}} \times \left(A_1\sin\left(\frac{(n-1)}{2}\sqrt{\frac{\alpha}{(C^2-k^2)}}\xi\right) + A_2\cos\left(\frac{(n-1)}{2}\sqrt{\frac{\alpha}{(C^2-k^2)}}\xi\right)\right)^{-1}\right]^{2/(n-1)}, \quad (41)$$

where $\xi = x - Ct$.

In particular, by setting $A_1 = 0$ and $A_2 \neq 0$ in (41), we have the periodic solutions

$$u(\xi) = \left[\pm\sqrt{\frac{\alpha(n+1)}{2\beta}}\sec\left(\frac{(n-1)}{2}\sqrt{\frac{\alpha}{(C^2-k^2)}}\xi\right)\right]^{2/(n-1)}, \quad (42)$$

while if $A_1 \neq 0$ and $A_2 = 0$, then we have the periodic solutions

$$u(\xi) = \left[\pm\sqrt{\frac{\alpha(n+1)}{2\beta}}\csc\left(\frac{(n-1)}{2}\sqrt{\frac{\alpha}{(C^2-k^2)}}\xi\right)\right]^{2/(n-1)}. \quad (43)$$

Case 3 (rational function solutions ($\lambda = 0$)). If $\lambda = 0$, substituting (35) into (34) and using (4) and (10), the left-hand side of (34) becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be zero yields a system of algebraic equations in a_0, a_1, b_1, μ, and C which are omitted here for simplicity. On using the Maple or Mathematica we have found the following results:

$$a_0 = 0,$$
$$a_1 = \pm\frac{1}{n-1}\sqrt{\frac{(C^2-k^2)(n+1)}{2\beta}},$$
$$b_1 = b_1, \quad (44)$$
$$C = C,$$
$$\mu = \frac{A_1^2(C^2-k^2)(n+1) - 2\beta b_1^2(n-1)^2}{2A_1(n+1)(C^2-k^2)},$$

where $\beta \neq 0$.

From (9), (35), and (44), we deduce the traveling wave solution of (32) as follows:

$$u(\xi) = \left[\pm\frac{1}{n-1}\sqrt{\frac{\left(C^2-k^2\right)(n+1)}{2\beta}} \times\left(\frac{\mu\xi + A_1}{(\mu/2)\xi^2 + A_1\xi + A_2}\right) + \left(\frac{b_1}{(\mu/2)\xi^2 + A_1\xi + A_2}\right)\right]^{2/(n-1)}, \quad (45)$$

where $\xi = x - Ct$.

Example 6. In this example, we study the higher order nonlinear Pochhammer-Chree equation (2). To this end, we see that the traveling wave variable (12) permits us to convert (2) into the following ODE:

$$C^2u'' - C^2u'''' - \left(\alpha u + \beta u^{n+1} + \gamma u^{2n+1}\right)'' = 0. \quad (46)$$

Integrating (46) twice with respect to ξ and vanishing the constants of integration, we get

$$\left(C^2 - \alpha\right)u - C^2u'' - \beta u^{n+1} - \gamma u^{2n+1} = 0. \quad (47)$$

Let us discuss the following two possibilities:

(I) If $\gamma \neq 0$.

By balancing between u'' and u^{2n+1} in (47) we get $N = 1/n$. According to Step 3, we use the transformation

$$u(\xi) = v^{1/n}(\xi), \quad (48)$$

where $v(\xi)$ is a new function of ξ. Substituting (48) into (47), we get the new ODE

$$\left(C^2-\alpha\right)n^2v^2 - C^2nvv'' - C^2(1-n)\left(v'\right)^2 - \beta n^2v^3 - \gamma n^2v^4 = 0. \quad (49)$$

Balancing vv'' with v^4 in (49), we get $N = 1$. Consequently, we get

$$v(\xi) = a_0 + a_1\phi(\xi) + b_1\psi(\xi), \quad (50)$$

where a_0, a_1, and b_1 are constants to be determined later. There are three cases to be discussed as follows.

Case 1 (hyperbolic function solutions ($\lambda < 0$)). If $\lambda < 0$, substituting (50) into (49) and using (4) and (6), the left-hand side of (49) becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be zero yields a system of algebraic equations in a_0, a_1, b_1, μ, λ, and C as follows:

$$\phi^4: \frac{C^2\lambda b_1^2}{\sigma_1\lambda^2+\mu^2} - C^2na_1^2 - n^2\gamma a_1^4 - C^2a_1^2 + \frac{C^2n\lambda b_1^2}{\sigma_1\lambda^2+\mu^2} - \frac{n^2\lambda^2\gamma b_1^4}{\left(\sigma_1\lambda^2+\mu^2\right)^2} + \frac{6n^2\lambda\gamma a_1^2b_1^2}{\sigma_1\lambda^2+\mu^2} = 0,$$

$$\phi^3: \frac{3n^2\beta\lambda a_1b_1^2}{\sigma_1\lambda^2+\mu^2} - n^2\beta a_1^3 - 2C^2na_0a_1 - 4n^2\gamma a_0a_1^3 - \frac{2C^2\lambda\mu a_1b_1}{\sigma_1\lambda^2+\mu^2} + \frac{12n^2\lambda\gamma a_0a_1b_1^2}{\sigma_1\lambda^2+\mu^2} - \frac{2C^2n\lambda\mu a_1b_1}{\sigma_1\lambda^2+\mu^2} + \frac{8n^2\lambda^2\gamma\mu a_1b_1^3}{\left(\sigma_1\lambda^2+\mu^2\right)^2} = 0,$$

$$\phi^3\psi: \frac{4n^2\lambda\gamma a_1b_1^3}{\sigma_1\lambda^2+\mu^2} - 2C^2na_1b_1 - 4n^2\gamma a_1^3b_1 - 2C^2a_1b_1 = 0,$$

$$\phi^2: C^2n^2a_1^2 - 2C^2\lambda a_1^2 - n^2\alpha a_1^2 - 6n^2\gamma a_0^2a_1^2 - 3n^2\beta a_0a_1^2 + \frac{C^2\lambda^2b_1^2}{\sigma_1\lambda^2+\mu^2} - \frac{2n^2\lambda^3\gamma b_1^4}{\left(\sigma_1\lambda^2+\mu^2\right)^2} + \frac{n^2\alpha\lambda b_1^2}{\sigma_1\lambda^2+\mu^2} + \frac{2C^2n\lambda^2b_1^2}{\sigma_1\lambda^2+\mu^2} - \frac{C^2n^2\lambda b_1^2}{\sigma_1\lambda^2+\mu^2} + \frac{C^2\lambda\mu^2a_1^2}{\sigma_1\lambda^2+\mu^2} + \frac{6n^2\lambda\gamma a_0^2b_1^2}{\sigma_1\lambda^2+\mu^2} + \frac{4n^2\lambda^3\gamma\mu^2b_1^4}{\left(\sigma_1\lambda^2+\mu^2\right)^3} + \frac{2n^2\beta\lambda^2\mu b_1^3}{\left(\sigma_1\lambda^2+\mu^2\right)^2} + \frac{6n^2\lambda^2\gamma a_1^2b_1^2}{\sigma_1\lambda^2+\mu^2} + \frac{3n^2\beta\lambda a_0b_1^2}{\sigma_1\lambda^2+\mu^2} - \frac{2C^2n\lambda^2\mu^2b_1^2}{\left(\sigma_1\lambda^2+\mu^2\right)^2} - \frac{C^2n\lambda\mu^2a_1^2}{\sigma_1\lambda^2+\mu^2} - \frac{C^2n\lambda\mu a_0b_1}{\sigma_1\lambda^2+\mu^2} + \frac{8n^2\lambda^2\gamma\mu a_0b_1^3}{\left(\sigma_1\lambda^2+\mu^2\right)^2} = 0,$$

$$\phi^2\psi: 2C^2\mu a_1^2 - 2C^2na_0b_1 + C^2n\mu a_1^2 - 3n^2\beta a_1^2b_1 - 12n^2\gamma a_0a_1^2b_1 - \frac{2C^2\lambda\mu b_1^2}{\sigma_1\lambda^2+\mu^2} + \frac{n^2\beta\lambda b_1^3}{\sigma_1\lambda^2+\mu^2} - \frac{3C^2n\lambda\mu b_1^2}{\sigma_1\lambda^2+\mu^2} + \frac{4n^2\lambda^2\gamma\mu b_1^4}{\left(\sigma_1\lambda^2+\mu^2\right)^2} + \frac{4n^2\lambda\gamma a_0b_1^3}{\sigma_1\lambda^2+\mu^2} - \frac{12n^2\lambda\gamma\mu a_1^2b_1^2}{\sigma_1\lambda^2+\mu^2} = 0,$$

$$\phi: 2C^2n^2a_0a_1 - 2n^2\alpha a_0a_1 - 3n^2\beta a_0^2a_1 - 4n^2\gamma a_0^3a_1 - 2C^2n\lambda a_0a_1 + \frac{3n^2\beta\lambda^2a_1b_1^2}{\sigma_1\lambda^2+\mu^2} - \frac{2C^2\lambda^2\mu a_1b_1}{\sigma_1\lambda^2+\mu^2} + \frac{12n^2\lambda^2\gamma a_0a_1b_1^2}{\sigma_1\lambda^2+\mu^2} - \frac{2C^2n\lambda^2\mu a_1b_1}{\sigma_1\lambda^2+\mu^2} + \frac{8n^2\lambda^3\gamma\mu a_1b_1^3}{\left(\sigma_1\lambda^2+\mu^2\right)^2} = 0,$$

$$\phi\psi:\ 2C^2n^2a_1b_1 - 2n^2\alpha a_1b_1 - 2C^2\lambda a_1b_1 - 12n^2\gamma a_0^2a_1b_1 + 3C^2n\mu a_0a_1 - C^2n\lambda a_1b_1 - 6n^2\beta a_0a_1b_1 + \frac{4n^2\lambda^2\gamma a_1b_1^3}{\sigma_1\lambda^2+\mu^2} + \frac{4C^2\lambda\mu^2a_1b_1}{\sigma_1\lambda^2+\mu^2} - \frac{6n^2\beta\lambda\mu a_1b_1^2}{\sigma_1\lambda^2+\mu^2} - \frac{16n^2\lambda^2\gamma\mu^2a_1b_1^3}{(\sigma_1\lambda^2+\mu^2)^2} + \frac{4C^2n\lambda\mu^2a_1b_1}{\sigma_1\lambda^2+\mu^2} - \frac{24n^2\lambda\gamma\mu a_0a_1b_1^2}{\sigma_1\lambda^2+\mu^2} = 0,$$

$$\psi:\ 2C^2n^2a_0b_1 - 2n^2\alpha a_0b_1 + 2C^2\lambda\mu a_1^2 - 3n^2\beta a_0^2b_1 - 4n^2\gamma a_0^3b_1 - C^2n\lambda a_0b_1 - 2C^2n\lambda\mu a_1^2 - \frac{2C^2\lambda\mu^3a_1^2}{\sigma_1\lambda^2+\mu^2} + \frac{n^2\beta\lambda^2b_1^3}{\sigma_1\lambda^2+\mu^2} - \frac{4n^2\beta\lambda^2\mu^2b_1^3}{(\sigma_1\lambda^2+\mu^2)^2} + \frac{4n^2\lambda^2\gamma a_0b_1^3}{\sigma_1\lambda^2+\mu^2} - \frac{8n^2\lambda^3\gamma\mu^3b_1^4}{(\sigma_1\lambda^2+\mu^2)^3} + \frac{4n^2\lambda^3\gamma\mu b_1^4}{(\sigma_1\lambda^2+\mu^2)^2} - \frac{2n^2\alpha\lambda\mu b_1^2}{\sigma_1\lambda^2+\mu^2} + \frac{4C^2n\lambda^2\mu^3b_1^2}{(\sigma_1\lambda^2+\mu^2)^2} + \frac{2C^2n\lambda\mu^3a_1^2}{\sigma_1\lambda^2+\mu^2} - \frac{3C^2n\lambda^2\mu b_1^2}{\sigma_1\lambda^2+\mu^2} + \frac{2C^2n^2\lambda\mu b_1^2}{\sigma_1\lambda^2+\mu^2} - \frac{6n^2\beta\lambda\mu a_0b_1^2}{\sigma_1\lambda^2+\mu^2} - \frac{16n^2\lambda^2\gamma\mu^2a_0b_1^3}{(\sigma_1\lambda^2+\mu^2)^2} - \frac{12n^2\lambda\gamma\mu a_0^2b_1^2}{\sigma_1\lambda^2+\mu^2} + \frac{2C^2n\lambda\mu^2a_0b_1}{\sigma_1\lambda^2+\mu^2} = 0,$$

$$\phi^0:\ C^2n^2a_0^2 - n^2\alpha a_0^2 - n^2\beta a_0^3 - n^2\gamma a_0^4 - C^2\lambda^2a_1^2 + C^2n\lambda^2a_1^2 - \frac{C^2n^2\lambda^2b_1^2}{\sigma_1\lambda^2+\mu^2} - \frac{n^2\lambda^4\gamma b_1^4}{(\sigma_1\lambda^2+\mu^2)^2} + \frac{C^2\lambda^2\mu^2a_1^2}{\sigma_1\lambda^2+\mu^2} + \frac{C^2n\lambda^3b_1^2}{\sigma_1\lambda^2+\mu^2} + \frac{n^2\alpha\lambda^2b_1^2}{\sigma_1\lambda^2+\mu^2} + \frac{3n^2\beta\lambda^2a_0b_1^2}{\sigma_1\lambda^2+\mu^2} - \frac{C^2n\lambda^2\mu^2a_1^2}{\sigma_1\lambda^2+\mu^2} + \frac{4n^2\lambda^4\gamma\mu^2b_1^4}{(\sigma_1\lambda^2+\mu^2)^3} + \frac{2n^2\beta\lambda^3\mu b_1^3}{(\sigma_1\lambda^2+\mu^2)^2} + \frac{6n^2\lambda^2\gamma a_0^2b_1^2}{\sigma_1\lambda^2+\mu^2} - \frac{2C^2n\lambda^3\mu^2b_1^2}{(\sigma_1\lambda^2+\mu^2)^2} - \frac{C^2n\lambda^2\mu a_0b_1}{\sigma_1\lambda^2+\mu^2} + \frac{8n^2\lambda^3\gamma\mu a_0b_1^3}{(\sigma_1\lambda^2+\mu^2)^2} = 0. \tag{51}$$

On solving the above algebraic equations using the Maple or Mathematica, we get the following results.

Result 1. Consider the following:

$$a_0 = -\frac{(n+1)\beta}{2(n+2)\gamma},\quad a_1 = \pm\frac{\sqrt{(n+1)\left[\beta^2(n+1) - \alpha\gamma(n+2)^2\right]}}{n(n+2)\gamma},\quad b_1 = 0,\quad \mu = 0,\quad \lambda = \frac{n^2(n+1)\beta^2}{4\left[\alpha\gamma(n+2)^2 - \beta^2(n+1)\right]},\quad C = \pm\frac{\sqrt{\gamma\left[\alpha\gamma(n+2)^2 - \beta^2(n+1)\right]}}{(n+2)\gamma}. \tag{52}$$

From (5), (50), and (52), we deduce the traveling wave solution of (47) as follows:

$$u(\xi) = \left[-\frac{(n+1)\beta}{2(n+2)\gamma} \times \left(1 \pm \left(A_1\cosh\left(\frac{n\beta}{2}\sqrt{-\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right) + A_2\sinh\left(\frac{n\beta}{2}\sqrt{-\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right)\right) \times \left(A_1\sinh\left(\frac{n\beta}{2}\sqrt{-\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right) + A_2 \times \cosh\left(\frac{n\beta}{2} \times \sqrt{-\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right)\right)^{-1}\right)\right]^{1/n}, \tag{53}$$

where $\xi = x \pm (\sqrt{\gamma[\alpha\gamma(n+2)^2 - \beta^2(n+1)]}/(n+2)\gamma)t$, $\beta \neq 0$.

In particular, by setting $A_1 = 0$ and $A_2 \neq 0$ in (53), we have the solitary wave solutions

$$u(\xi) = \left[-\frac{(n+1)\beta}{2(n+2)\gamma} \times \left(1 \pm \tanh\left(\frac{n\beta}{2}\sqrt{-\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right)\right)\right]^{1/n}, \tag{54}$$

while if $A_1 \neq 0$ and $A_2 = 0$, then we have the solitary wave solutions

$$u(\xi) = \left[-\frac{(n+1)\beta}{2(n+2)\gamma} \times \left(1 \pm \coth\left(\frac{n\beta}{2}\sqrt{-\frac{(n+1)}{\alpha\gamma(n+2)^2 - \beta^2(n+1)}}\xi\right)\right)\right]^{1/n}. \quad (55)$$

Note that the solutions (53), (54), and (55) are in agreement with the solutions (16), (17), and (18) of [36], respectively.

Result 2. Consider the following:

$$\begin{aligned} a_0 &= 0, \\ a_1 &= 0, \\ b_1 &= \pm \lambda\alpha(n+2) \times \sqrt{\frac{\sigma_1(n+1)}{(\lambda+n^2)\left[\lambda\alpha\gamma(n+2)^2 - \beta^2(n+1)(\lambda+n^2)\right]}}, \\ \mu &= \pm\lambda\beta\sqrt{\frac{\sigma_1(n+1)(\lambda+n^2)}{\lambda\alpha\gamma(n+2)^2 - \beta^2(n+1)(\lambda+n^2)}}, \\ \lambda &= \lambda, \\ C &= \pm n\sqrt{\frac{\alpha}{\lambda+n^2}}. \end{aligned} \quad (56)$$

In this result, we deduce the traveling wave solution of (47) as follows:

$$u(\xi) = \left[\pm\sqrt{\frac{\sigma_1(n+1)}{(\lambda+n^2)\left[\lambda\alpha\gamma(n+2)^2 - \beta^2(n+1)(\lambda+n^2)\right]}} \times \left(\lambda\alpha(n+2) \times \left(A_1\sinh\left(\sqrt{-\lambda}\xi\right) + A_2\cosh\left(\sqrt{-\lambda}\xi\right) \pm \beta \times \sqrt{\frac{\sigma_1(n+1)(\lambda+n^2)}{\lambda\alpha\gamma(n+2)^2 - \beta^2(n+1)(\lambda+n^2)}}\right)^{-1}\right)\right]^{1/n}, \quad (57)$$

where $\xi = x \pm n\sqrt{(\alpha/\lambda+n^2)}t$.

In particular, by setting $A_1 = 0$, $A_2 \neq 0$, and $\beta = 0$ in (57), we have the solitary wave solutions

$$u(\xi) = \left[\pm\sqrt{-\frac{(n+1)\lambda\alpha}{(\lambda+n^2)\gamma}}\operatorname{sech}(\sqrt{-\lambda}\xi)\right]^{1/n}; \quad (58)$$

while if $A_1 \neq 0$, $A_2 = 0$, and $\beta = 0$, then we have the solitary wave solutions

$$u(\xi) = \left[\pm\sqrt{\frac{(n+1)\lambda\alpha}{(\lambda+n^2)\gamma}}\operatorname{csch}(\sqrt{-\lambda}\xi)\right]^{1/n}. \quad (59)$$

Result 3. Consider the following:

$$\begin{aligned} a_0 &= -\frac{(n+1)\beta}{2(n+2)\gamma}, \\ a_1 &= \pm\frac{\sqrt{(n+1)\left[\beta^2(n+1) - \alpha\gamma(n+2)^2\right]}}{2n(n+2)\gamma}, \\ b_1 &= \pm\frac{(n+1)\beta\sqrt{\sigma_1}}{2(n+2)\gamma}, \\ \mu &= 0, \\ \lambda &= \frac{n^2\beta^2(n+1)}{\alpha\gamma(n+2)^2 - \beta^2(n+1)}, \\ C &= \pm\sqrt{\alpha - \frac{\beta^2(n+1)}{\gamma(n+2)^2}}. \end{aligned} \quad (60)$$

In this result, we deduce the traveling wave solution of (47) as follows:

$$u(\xi) = \left[-\frac{(n+1)\beta}{2(n+2)\gamma} \times \left(1 \pm \left(A_1\cosh\left(n\beta\sqrt{-\frac{(n+1)}{\alpha\gamma(n+2)^2 - \beta^2(n+1)}}\xi\right) + A_2\sinh\left(n\beta\sqrt{-\frac{(n+1)}{\alpha\gamma(n+2)^2 - \beta^2(n+1)}}\xi\right)\right) \times \left(A_1\sinh\left(n\beta\sqrt{-\frac{(n+1)}{\alpha\gamma(n+2)^2 - \beta^2(n+1)}}\xi\right) + A_2\cosh\left(n\beta\sqrt{-\frac{(n+1)}{\alpha\gamma(n+2)^2 - \beta^2(n+1)}}\xi\right)\right)^{-1}\right.$$

$$\pm\sqrt{\sigma_1}\times\left(A_1\sinh\left(n\beta\sqrt{-\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right)+A_2\times\cosh\left(n\beta\sqrt{-\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right)\right)^{-1}\Bigg)\Bigg]^{1/n}, \quad (61)$$

where $\xi = x \pm \sqrt{\alpha-(\beta^2(n+1)/\gamma(n+2)^2)}t$, $\beta \neq 0$.

In particular, by setting $A_1 = 0$ and $A_2 \neq 0$ in (61), we have the solitary wave solutions

$$u(\xi) = \left[-\frac{(n+1)\beta}{2(n+2)\gamma}\times\left(1\pm\tanh\left(n\beta\sqrt{-\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right)\pm i\,\mathrm{sech}\left(n\beta\sqrt{-\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right)\right)\right]^{1/n} \quad (62)$$

while if $A_1 \neq 0$, $A_2 = 0$, then we have the solitary wave solutions

$$u(\xi) = \left[-\frac{(n+1)\beta}{2(n+2)\gamma}\times\left(1\pm\coth\left(n\beta\sqrt{-\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right)\pm\mathrm{csch}\left(n\beta\sqrt{-\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right)\right)\right]^{1/n}. \quad (63)$$

Result 4. Consider the following:

$$\begin{aligned}
a_0 &= -\frac{(n+1)\beta}{2(n+2)\gamma},\\
a_1 &= \pm\frac{\sqrt{(n+1)\left[\beta^2(n+1)-\alpha\gamma(n+2)^2\right]}}{2n(n+2)\gamma},\\
b_1 &= \pm\left((n+1)^2\beta^4\left[n^4\sigma_1+\mu^2\right]+\alpha\gamma\mu^2(n+2)^2\times\left[\alpha\gamma(n+2)^2-2\beta^2(n+1)\right]\right)^{1/2}\times\left(2n^2(n+2)\gamma\beta\right)^{-1},\\
\mu &= \mu,\\
\lambda &= \frac{n^2\beta^2(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)},\\
C &= \pm\sqrt{\alpha-\frac{\beta^2(n+1)}{\gamma(n+2)^2}}.
\end{aligned} \quad (64)$$

In this result, we deduce the traveling wave solution of (47) as follows:

$$\begin{aligned}
u(\xi) = \Bigg[&-\frac{(n+1)\beta}{2(n+2)\gamma}\times\Bigg(1\pm\left(A_1\cosh\left(n\beta\sqrt{-\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right)+A_2\sinh\left(n\beta\sqrt{-\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right)\right)\\
&\times\left(A_1\sinh\left(n\beta\sqrt{-\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right)+A_2\cosh\left(n\beta\sqrt{-\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right)+\frac{\mu}{\lambda}\right)^{-1}\Bigg)\pm\frac{1}{2n^2(n+2)\gamma\beta}\\
&\times\left(\left((n+1)^2\beta^4\left[n^4\sigma_1+\mu^2\right]+\alpha\gamma\mu^2(n+2)^2\times\left[\alpha\gamma(n+2)^2-2\beta^2(n+1)\right]\right)^{1/2}\right.\\
&\left.\times\left(A_1\sinh\left(n\beta\sqrt{-\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right)+A_2\cosh\left(n\beta\sqrt{-\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right)+\frac{\mu}{\lambda}\right)^{-1}\right)\Bigg]^{1/n},
\end{aligned} \quad (65)$$

where $\xi = x \pm \sqrt{\alpha-(\beta^2(n+1)/\gamma(n+2)^2)}t$, $\beta \neq 0$.

In particular, by setting $A_1 = 0$, $A_2 \neq 0$, and $\mu = 0$ in (65), we have the same solitary wave solutions (62), while if $A_1 \neq 0$, $A_2 = 0$, and $\mu = 0$, then we have the same solitary wave solutions (63).

Case 2 (trigonometric function solution ($\lambda > 0$)). If $\lambda > 0$, substituting (50) into (49) and using (4) and (8), the left-hand side of (49) becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be zero yields a system of algebraic equations in a_0, a_1, b_1, μ, λ, and C as follows:

$$\phi^4\colon \frac{C^2\lambda b_1^2}{\mu^2-\lambda^2\sigma_2} - C^2 n a_1^2 - n^2\gamma a_1^4 - C^2 a_1^2 + \frac{C^2 n\lambda b_1^2}{\mu^2-\lambda^2\sigma_2} - \frac{n^2\lambda^2\gamma b_1^4}{(\lambda^2\sigma_2-\mu^2)^2} + \frac{6n^2\lambda\gamma a_1^2 b_1^2}{\mu^2-\lambda^2\sigma_2} = 0,$$

$$\phi^3\colon \frac{3n^2\beta\lambda a_1 b_1^2}{\mu^2-\lambda^2\sigma_2} - n^2\beta a_1^3 - 2C^2 n a_0 a_1 - 4n^2\gamma a_0 a_1^3 - \frac{2C^2\lambda\mu a_1 b_1}{\mu^2-\lambda^2\sigma_2} + \frac{12n^2\lambda\gamma a_0 a_1 b_1^2}{\mu^2-\lambda^2\sigma_2} - \frac{2C^2 n\lambda\mu a_1 b_1}{\mu^2-\lambda^2\sigma_2} + \frac{8n^2\lambda^2\gamma\mu a_1 b_1^3}{(\lambda^2\sigma_2-\mu^2)^2} = 0,$$

$$\phi^3\psi\colon \frac{4n^2\lambda\gamma a_1 b_1^3}{\mu^2-\lambda^2\sigma_2} - 2C^2 n a_1 b_1 - 4n^2\gamma a_1^3 b_1 - 2C^2 a_1 b_1 = 0,$$

$$\phi^2\colon C^2n^2a_1^2 - 2C^2\lambda a_1^2 - n^2\alpha a_1^2 - 6n^2\gamma a_0^2 a_1^2 - 3n^2\beta a_0 a_1^2 + \frac{C^2\lambda^2 b_1^2}{\mu^2-\lambda^2\sigma_2} - \frac{2n^2\lambda^3\gamma b_1^4}{(\lambda^2\sigma_2-\mu^2)^2} + \frac{n^2\alpha\lambda b_1^2}{\mu^2-\lambda^2\sigma_2} + \frac{2C^2 n\lambda^2 b_1^2}{\mu^2-\lambda^2\sigma_2} - \frac{C^2n^2\lambda b_1^2}{\mu^2-\lambda^2\sigma_2} + \frac{C^2\lambda\mu^2 a_1^2}{\mu^2-\lambda^2\sigma_2} + \frac{6n^2\lambda\gamma a_0^2 b_1^2}{\mu^2-\lambda^2\sigma_2} + \frac{4n^2\lambda^3\gamma\mu^2 b_1^4}{(\mu^2-\lambda^2\sigma_2)^3} + \frac{2n^2\beta\lambda^2\mu b_1^3}{(\lambda^2\sigma_2-\mu^2)^2} + \frac{6n^2\lambda^2\gamma a_1^2 b_1^2}{\mu^2-\lambda^2\sigma_2} + \frac{3n^2\beta\lambda a_0 b_1^2}{\mu^2-\lambda^2\sigma_2} - \frac{2C^2 n\lambda^2\mu^2 b_1^2}{(\lambda^2\sigma_2-\mu^2)^2} - \frac{C^2 n\lambda\mu^2 a_1^2}{\mu^2-\lambda^2\sigma_2} - \frac{C^2 n\lambda\mu a_0 b_1}{\mu^2-\lambda^2\sigma_2} + \frac{8n^2\lambda^2\gamma\mu a_0 b_1^3}{(\lambda^2\sigma_2-\mu^2)^2} = 0,$$

$$\phi^2\psi\colon 2C^2\mu a_1^2 - 2C^2 n a_0 b_1 + C^2 n\mu a_1^2 - 3n^2\beta a_1^2 b_1 - 12n^2\gamma a_0 a_1^2 b_1 - \frac{2C^2\lambda\mu b_1^2}{\mu^2-\lambda^2\sigma_2} + \frac{n^2\beta\lambda b_1^3}{\mu^2-\lambda^2\sigma_2} - \frac{3C^2 n\lambda\mu b_1^2}{\mu^2-\lambda^2\sigma_2} + \frac{4n^2\lambda^2\gamma\mu b_1^4}{(\lambda^2\sigma_2-\mu^2)^2} + \frac{4n^2\lambda\gamma a_0 b_1^3}{\mu^2-\lambda^2\sigma_2} - \frac{12n^2\lambda\gamma\mu a_1^2 b_1^2}{\mu^2-\lambda^2\sigma_2} = 0,$$

$$\phi\colon 2C^2n^2a_0a_1 - 2n^2\alpha a_0 a_1 - 3n^2\beta a_0^2 a_1 - 4n^2\gamma a_0^3 a_1 - 2C^2 n\lambda a_0 a_1 + \frac{3n^2\beta\lambda^2 a_1 b_1^2}{\mu^2-\lambda^2\sigma_2} - \frac{2C^2\lambda^2\mu a_1 b_1}{\mu^2-\lambda^2\sigma_2} + \frac{12n^2\lambda^2\gamma a_0 a_1 b_1^2}{\mu^2-\lambda^2\sigma_2} - \frac{2C^2 n\lambda^2\mu a_1 b_1}{\mu^2-\lambda^2\sigma_2} + \frac{8n^2\lambda^3\gamma\mu a_1 b_1^3}{(\lambda^2\sigma_2-\mu^2)^2} = 0,$$

$$\phi\psi\colon 2C^2n^2a_1b_1 - 2n^2\alpha a_1 b_1 - 2C^2\lambda a_1 b_1 - 12n^2\gamma a_0^2 a_1 b_1 + 3C^2 n\mu a_0 a_1 - C^2 n\lambda a_1 b_1 - 6n^2\beta a_0 a_1 b_1 + \frac{4n^2\lambda^2\gamma a_1 b_1^3}{\mu^2-\lambda^2\sigma_2} + \frac{4C^2\lambda\mu^2 a_1 b_1}{\mu^2-\lambda^2\sigma_2} - \frac{6n^2\beta\lambda\mu a_1 b_1^2}{\mu^2-\lambda^2\sigma_2} - \frac{16n^2\lambda^2\gamma\mu^2 a_1 b_1^3}{(\lambda^2\sigma_2-\mu^2)^2} + \frac{4C^2 n\lambda\mu^2 a_1 b_1}{\mu^2-\lambda^2\sigma_2} - \frac{24n^2\lambda\gamma\mu a_0 a_1 b_1^2}{\mu^2-\lambda^2\sigma_2} = 0,$$

$$\psi\colon 2C^2n^2a_0b_1 - 2n^2\alpha a_0 b_1 + 2C^2\lambda\mu a_1^2 - 3n^2\beta a_0^2 b_1 - 4n^2\gamma a_0^3 b_1 - C^2 n\lambda a_0 b_1 - 2C^2 n\lambda\mu a_1^2 - \frac{2C^2\lambda\mu^3 a_1^2}{\mu^2-\lambda^2\sigma_2} + \frac{n^2\beta\lambda^2 b_1^3}{\mu^2-\lambda^2\sigma_2} - \frac{4n^2\beta\lambda^2\mu^2 b_1^3}{(\lambda^2\sigma_2-\mu^2)^2} + \frac{4n^2\lambda^2\gamma a_0 b_1^3}{\mu^2-\lambda^2\sigma_2} - \frac{8n^2\lambda^3\gamma\mu^3 b_1^4}{(\mu^2-\lambda^2\sigma_2)^3} + \frac{4n^2\lambda^3\gamma\mu b_1^4}{(\lambda^2\sigma_2-\mu^2)^2} - \frac{2n^2\alpha\lambda\mu b_1^2}{\mu^2-\lambda^2\sigma_2} + \frac{4C^2 n\lambda^2\mu^3 b_1^2}{(\lambda^2\sigma_2-\mu^2)^2} + \frac{2C^2 n\lambda\mu^3 a_1^2}{\mu^2-\lambda^2\sigma_2} - \frac{3C^2 n\lambda^2\mu b_1^2}{\mu^2-\lambda^2\sigma_2} + \frac{2C^2n^2\lambda\mu b_1^2}{\mu^2-\lambda^2\sigma_2} - \frac{6n^2\beta\lambda\mu a_0 b_1^2}{\mu^2-\lambda^2\sigma_2} - \frac{16n^2\lambda^2\gamma\mu^2 a_0 b_1^3}{(\lambda^2\sigma_2-\mu^2)^2} - \frac{12n^2\lambda\gamma\mu a_0^2 b_1^2}{\mu^2-\lambda^2\sigma_2} + \frac{2C^2 n\lambda\mu^2 a_0 b_1}{\mu^2-\lambda^2\sigma_2} = 0,$$

$$\phi^0\colon C^2n^2a_0^2 - n^2\alpha a_0^2 - n^2\beta a_0^3 - n^2\gamma a_0^4 - C^2\lambda^2 a_1^2 + C^2 n\lambda^2 a_1^2 - \frac{C^2n^2\lambda^2 b_1^2}{\mu^2-\lambda^2\sigma_2} - \frac{n^2\lambda^4\gamma b_1^4}{(\lambda^2\sigma_2-\mu^2)^2} + \frac{C^2\lambda^2\mu^2 a_1^2}{\mu^2-\lambda^2\sigma_2} + \frac{C^2 n\lambda^3 b_1^2}{\mu^2-\lambda^2\sigma_2} + \frac{n^2\alpha\lambda^2 b_1^2}{\mu^2-\lambda^2\sigma_2} + \frac{3n^2\beta\lambda^2 a_0 b_1^2}{\mu^2-\lambda^2\sigma_2} - \frac{C^2 n\lambda^2\mu^2 a_1^2}{\mu^2-\lambda^2\sigma_2} + \frac{4n^2\lambda^4\gamma\mu^2 b_1^4}{(\mu^2-\lambda^2\sigma_2)^3} + \frac{2n^2\beta\lambda^3\mu b_1^3}{(\lambda^2\sigma_2-\mu^2)^2} + \frac{6n^2\lambda^2\gamma a_0^2 b_1^2}{\mu^2-\lambda^2\sigma_2} - \frac{2C^2 n\lambda^3\mu^2 b_1^2}{(\lambda^2\sigma_2-\mu^2)^2} - \frac{C^2 n\lambda^2\mu a_0 b_1}{\mu^2-\lambda^2\sigma_2} + \frac{8n^2\lambda^3\gamma\mu a_0 b_1^3}{(\lambda^2\sigma_2-\mu^2)^2} = 0. \tag{66}$$

On solving the above algebraic equations using the Maple or Mathematica, we get the following results.

Result 1. Consider the following:

$$a_0 = -\frac{(n+1)\beta}{2(n+2)\gamma},$$
$$a_1 = \pm\frac{\sqrt{(n+1)\left[\beta^2(n+1)-\alpha\gamma(n+2)^2\right]}}{n(n+2)\gamma},$$
$$b_1 = 0,$$
$$\mu = 0, \quad (67)$$
$$\lambda = \frac{n^2(n+1)\beta^2}{4\left[\alpha\gamma(n+2)^2-\beta^2(n+1)\right]},$$
$$C = \pm\frac{\sqrt{\gamma\left[\alpha\gamma(n+2)^2-\beta^2(n+1)\right]}}{(n+2)\gamma}.$$

From (7), (50), and (67), we deduce the traveling wave solution of (47) as follows:

$$u(\xi) = \Bigg[-\frac{(n+1)\beta}{2(n+2)\gamma} \times\Bigg(1\pm i \times\Bigg(\Bigg(A_1\cos\left(\frac{n\beta}{2}\sqrt{\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right) - A_2\sin\left(\frac{n\beta}{2}\sqrt{\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right)\Bigg) \times\Bigg(A_1\sin\left(\frac{n\beta}{2}\sqrt{\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right)+A_2 \times\cos\left(\frac{n\beta}{2}\sqrt{\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right)\Bigg)^{-1}\Bigg)\Bigg)\Bigg]^{1/n}, \quad (68)$$

where $\xi = x \pm (\sqrt{\gamma[\alpha\gamma(n+2)^2-\beta^2(n+1)]}/(n+2)\gamma)t$, $\beta\neq 0$.

In particular, by setting $A_1 = 0$ and $A_2\neq 0$ in (68), we have the periodic solutions

$$u(\xi) = \left[-\frac{(n+1)\beta}{2(n+2)\gamma} \times\left(1\pm i\tan\left(\frac{n\beta}{2}\sqrt{\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right)\right)\right]^{1/n}, \quad (69)$$

while if $A_1\neq 0$ and $A_2 = 0$, then we have the periodic solutions

$$u(\xi) = \left[-\frac{(n+1)\beta}{2(n+2)\gamma} \times\left(1\pm i\cot\left(\frac{n\beta}{2}\sqrt{\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right)\right)\right]^{1/n}. \quad (70)$$

Result 2 (Figure 3). Consider the following:

$$a_0 = 0,$$
$$a_1 = 0,$$
$$b_1 = \pm\lambda\alpha(n+2) \times\sqrt{\frac{\sigma_2(n+1)}{(\lambda+n^2)\left[\beta^2(n+1)(\lambda+n^2)-\lambda\alpha\gamma(n+2)^2\right]}},$$
$$\mu = \pm\lambda\beta\sqrt{\frac{\sigma_2(n+1)(\lambda+n^2)}{\beta^2(n+1)(\lambda+n^2)-\lambda\alpha\gamma(n+2)^2}},$$
$$\lambda = \lambda,$$
$$C = \pm n\sqrt{\frac{\alpha}{\lambda+n^2}}. \quad (71)$$

In this result, we deduce the traveling wave solution of (47) as follows:

$$u(\xi) = \Bigg[\pm\sqrt{\frac{\sigma_2(n+1)}{(\lambda+n^2)\left[\beta^2(n+1)(\lambda+n^2)-\lambda\alpha\gamma(n+2)^2\right]}} \times\Bigg(\lambda\alpha(n+2) \times\left(A_1\sin\left(\sqrt{\lambda}\xi\right)+A_2\cos\left(\sqrt{\lambda}\xi\right) \pm\beta\sqrt{\frac{\sigma_2(n+1)(\lambda+n^2)}{\beta^2(n+1)(\lambda+n^2)-\lambda\alpha\gamma(n+2)^2}}\right)^{-1}\Bigg)\Bigg]^{1/n}, \quad (72)$$

where $\xi = x \pm n\sqrt{(\alpha/\lambda+n^2)}t$.

In particular, by setting $A_1 = 0$, $A_2\neq 0$, and $\beta = 0$ in (72), we have the periodic solutions

$$u(\xi) = \left[\pm\sqrt{-\frac{(n+1)\lambda\alpha}{(\lambda+n^2)\gamma}}\sec(\sqrt{\lambda}\xi)\right]^{1/n} \quad (73)$$

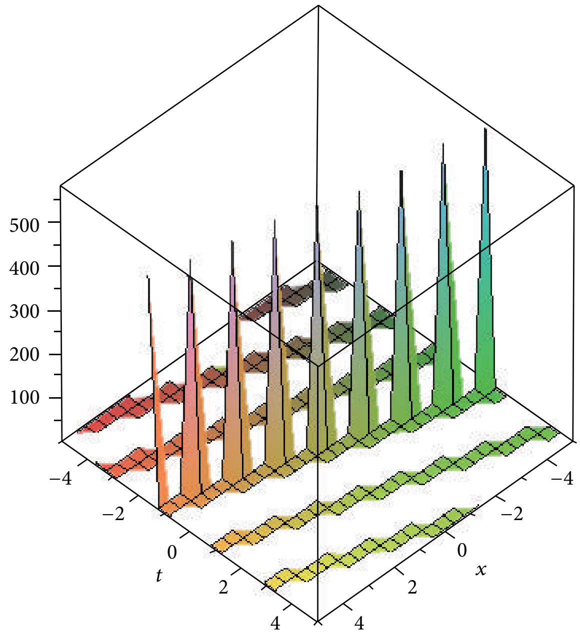

Figure 3: The plot of solution (74) when $\alpha = 10$, $\gamma = -1$, $\lambda = 1$, and $n = 3$.

while if $A_1 \neq 0$, $A_2 = 0$, and $\beta = 0$, then we have the periodic solutions

$$u(\xi) = \left[\pm\sqrt{-\frac{(n+1)\lambda\alpha}{(\lambda+n^2)\gamma}}\csc(\sqrt{\lambda}\xi)\right]^{1/n}. \tag{74}$$

Result 3. Consider the following:

$$\begin{aligned}
a_0 &= -\frac{(n+1)\beta}{2(n+2)\gamma}, \\
a_1 &= \pm\frac{\sqrt{(n+1)\left[\beta^2(n+1)-\alpha\gamma(n+2)^2\right]}}{2n(n+2)\gamma}, \\
b_1 &= \pm\frac{(n+1)\beta\sqrt{-\sigma_2}}{2(n+2)\gamma}, \\
\mu &= 0, \\
\lambda &= \frac{n^2\beta^2(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}, \\
C &= \pm\sqrt{\alpha-\frac{\beta^2(n+1)}{\gamma(n+2)^2}}.
\end{aligned} \tag{75}$$

In this result, we deduce the traveling wave solution of (47) as follows:

$$\begin{aligned}
u(\xi) &= \Bigg[-\frac{(n+1)\beta}{2(n+2)\gamma} \\
&\times\Bigg(1\pm i \\
&\times\Bigg(\Bigg(A_1\cos\left(n\beta\sqrt{\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right) \\
&\quad - A_2\sin\left(n\beta\sqrt{\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right)\Bigg) \\
&\quad\times\Bigg(A_1\sin\left(n\beta\sqrt{\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right)+A_2 \\
&\quad\times\cos\left(n\beta\sqrt{\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right)\Bigg)^{-1}\Bigg) \\
&\pm\left(\sqrt{-\sigma_2}\right) \\
&\times\Bigg(A_1\sin\left(n\beta\sqrt{\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right)+A_2 \\
&\quad\times\cos\left(n\beta\sqrt{\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right)\Bigg)^{-1}\Bigg)\Bigg]^{1/n},
\end{aligned} \tag{76}$$

where $\xi = x \pm \sqrt{\alpha-(\beta^2(n+1)/\gamma(n+2)^2)}t$, $\beta \neq 0$.

In particular, by setting $A_1 = 0$ and $A_2 \neq 0$ in (76), we have the periodic solutions

$$\begin{aligned}
u(\xi) &= \Bigg[-\frac{(n+1)\beta}{2(n+2)\gamma} \\
&\times\Bigg(1\pm i\tan\left(n\beta\sqrt{\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right) \\
&\quad\pm i\sec\left(n\beta\sqrt{\frac{(n+1)}{\alpha\gamma(n+2)^2-\beta^2(n+1)}}\xi\right)\Bigg)\Bigg]^{1/n}
\end{aligned} \tag{77}$$

while if $A_1 \neq 0$ and $A_2 = 0$, then we have the periodic solutions

$$u(\xi) = \left[-\frac{(n+1)\beta}{2(n+2)\gamma} \times \left(1 \pm i\cot\left(n\beta\sqrt{\frac{(n+1)}{\alpha\gamma(n+2)^2 - \beta^2(n+1)}}\xi\right) \pm i\csc\left(n\beta\sqrt{\frac{(n+1)}{\alpha\gamma(n+2)^2 - \beta^2(n+1)}}\xi\right)\right)\right]^{1/n}. \quad (78)$$

Result 4. Consider the following:

$$\begin{aligned}
a_0 &= -\frac{(n+1)\beta}{2(n+2)\gamma}, \\
a_1 &= \pm\frac{\sqrt{(n+1)\left[\beta^2(n+1) - \alpha\gamma(n+2)^2\right]}}{2n(n+2)\gamma}, \\
b_1 &= \pm\left((n+1)^2\beta^4\left[\mu^2 - n^4\sigma_2\right] + \alpha\gamma\mu^2(n+2)^2 \times\left[\alpha\gamma(n+2)^2 - 2\beta^2(n+1)\right]\right)^{1/2} \times\left(2n^2(n+2)\gamma\beta\right)^{-1}, \\
\mu &= \mu, \\
\lambda &= \frac{n^2\beta^2(n+1)}{\alpha\gamma(n+2)^2 - \beta^2(n+1)}, \\
C &= \pm\sqrt{\alpha - \frac{\beta^2(n+1)}{\gamma(n+2)^2}}.
\end{aligned} \quad (79)$$

In this result, we deduce the traveling wave solution of (47) as follows:

$$\begin{aligned}
u(\xi) = \Bigg[&-\frac{(n+1)\beta}{2(n+2)\gamma} \\
&\times\Bigg(1 \pm i \times\Bigg(\Bigg(A_1\cos\left(n\beta\sqrt{\frac{(n+1)}{\alpha\gamma(n+2)^2 - \beta^2(n+1)}}\xi\right) - A_2\sin\left(n\beta\sqrt{\frac{(n+1)}{\alpha\gamma(n+2)^2 - \beta^2(n+1)}}\xi\right)\Bigg) \\
&\times\Bigg(A_1\sin\left(n\beta\sqrt{\frac{(n+1)}{\alpha\gamma(n+2)^2 - \beta^2(n+1)}}\xi\right) + A_2\cos\left(n\beta\sqrt{\frac{(n+1)}{\alpha\gamma(n+2)^2 - \beta^2(n+1)}}\xi\right) + \frac{\mu}{\lambda}\Bigg)^{-1}\Bigg)\Bigg) \\
&\pm\frac{1}{2n^2(n+2)\gamma\beta} \times\left(\left((n+1)^2\beta^4\left[\mu^2 - n^4\sigma_2\right] + \alpha\gamma\mu^2(n+2)^2 \times\left[\alpha\gamma(n+2)^2 - 2\beta^2(n+1)\right]\right)^{1/2}\right. \\
&\left.\times\Bigg(A_1\sin\left(n\beta\sqrt{\frac{(n+1)}{\alpha\gamma(n+2)^2 - \beta^2(n+1)}}\xi\right) + A_2\cos\left(n\beta\sqrt{\frac{(n+1)}{\alpha\gamma(n+2)^2 - \beta^2(n+1)}}\xi\right) + \frac{\mu}{\lambda}\Bigg)^{-1}\right)\Bigg]^{1/n},
\end{aligned} \quad (80)$$

where $\xi = x \pm \sqrt{\alpha - (\beta^2(n+1)/\gamma(n+2)^2)}t$, $\beta \neq 0$.

In particular, by setting $A_1 = 0$, $A_2 \neq 0$, and $\mu = 0$ in (80), we have the same periodic solutions (77), while if $A_1 \neq 0$, $A_2 = 0$, and $\mu = 0$, then we have the same periodic solutions (78).

Case 3 (rational function solutions ($\lambda = 0$)). If $\lambda = 0$, substituting (50) into (49) and using (4) and (10), the left-hand side of (49) becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be zero yields a system of algebraic equations in a_0, a_1, b_1, and μ as follows:

$$\phi^4: \ -C^2a_1^2 - C^2na_1^2 - n^2\gamma a_1^4 - \frac{C^2b_1^2}{A_1^2 - 2\mu A_2} - \frac{n^2\gamma b_1^4}{(A_1^2 - 2\mu A_2)^2} - \frac{C^2nb_1^2}{A_1^2 - 2\mu A_2} - \frac{6n^2\gamma a_1^2b_1^2}{A_1^2 - 2\mu A_2} = 0,$$

$$\phi^3: \ \frac{2C^2\mu a_1b_1}{A_1^2 - 2\mu A_2} - n^2\beta a_1^3 - 2C^2na_0a_1 - 4n^2\gamma a_0a_1^3 - \frac{3n^2\beta a_1b_1^2}{A_1^2 - 2\mu A_2} + \frac{8n^2\gamma\mu a_1b_1^3}{(A_1^2 - 2\mu A_2)^2} - \frac{12n^2\gamma a_0a_1b_1^2}{A_1^2 - 2\mu A_2} + \frac{2C^2n\mu a_1b_1}{A_1^2 - 2\mu A_2} = 0,$$

$$\phi^3\psi: \ -2C^2a_1b_1 - 2C^2na_1b_1 - 4n^2\gamma a_1^3b_1 - \frac{4n^2\gamma a_1b_1^3}{A_1^2 - 2\mu A_2} = 0,$$

$$\phi^2:\ -n^2\alpha a_1^2 + C^2n^2a_1^2 - \frac{n^2\alpha b_1^2}{A_1^2-2\mu A_2} - 6n^2\gamma a_0^2a_1^2 + \frac{C^2n^2b_1^2}{A_1^2-2\mu A_2} - \frac{C^2\mu^2a_1^2}{A_1^2-2\mu A_2} - 3n^2\beta a_0a_1^2 - \frac{4n^2\gamma\mu^2b_1^4}{(A_1^2-2\mu A_2)^3} - \frac{2C^2n\mu^2b_1^2}{(A_1^2-2\mu A_2)^2} - \frac{3n^2\beta a_0b_1^2}{A_1^2-2\mu A_2} + \frac{C^2n\mu^2a_1^2}{A_1^2-2\mu A_2} - \frac{6n^2\gamma a_0^2b_1^2}{A_1^2-2\mu A_2} + \frac{2n^2\beta\mu b_1^3}{(A_1^2-2\mu A_2)^2} + \frac{8n^2\gamma\mu a_0b_1^3}{(A_1^2-2\mu A_2)^2} + \frac{C^2n\mu a_0b_1}{A_1^2-2\mu A_2} = 0,$$

$$\phi^2\psi:\ 2C^2\mu a_1^2 - 2C^2na_0b_1 + \frac{2C^2\mu b_1^2}{A_1^2-2\mu A_2} - \frac{n^2\beta b_1^3}{A_1^2-2\mu A_2} + C^2n\mu a_1^2 - 3n^2\beta a_1^2b_1 - 12n^2\gamma a_0a_1^2b_1 + \frac{3C^2n\mu b_1^2}{A_1^2-2\mu A_2} - \frac{4n^2\gamma a_0b_1^3}{A_1^2-2\mu A_2} + \frac{4n^2\gamma\mu b_1^4}{(A_1^2-2\mu A_2)^2} + \frac{12n^2\gamma\mu a_1^2b_1^2}{A_1^2-2\mu A_2} = 0,$$

$$\phi:\ 2C^2n^2a_0a_1 - 2n^2\alpha a_0a_1 - 3n^2\beta a_0^2a_1 - 4n^2\gamma a_0^3a_1 = 0,$$

$$\phi\psi:\ 2C^2n^2a_1b_1 - 2n^2\alpha a_1b_1 - 12n^2\gamma a_0^2a_1b_1 + 3C^2n\mu a_0a_1 - \frac{4C^2\mu^2a_1b_1}{A_1^2-2\mu A_2} - 6n^2\beta a_0a_1b_1 - \frac{4C^2n\mu^2a_1b_1}{A_1^2-2\mu A_2} + \frac{6n^2\beta\mu a_1b_1^2}{A_1^2-2\mu A_2} - \frac{16n^2\gamma\mu^2a_1b_1^3}{(A_1^2-2\mu A_2)^2} + \frac{24n^2\gamma\mu a_0a_1b_1^2}{A_1^2-2\mu A_2} = 0,$$

$$\psi:\ 2C^2n^2a_0b_1 - 2n^2\alpha a_0b_1 + \frac{2C^2\mu^3a_1^2}{A_1^2-2\mu A_2} - 3n^2\beta a_0^2b_1 - 4n^2\gamma a_0^3b_1 + \frac{8n^2\gamma\mu^3b_1^4}{(A_1^2-2\mu A_2)^3} + \frac{4C^2n\mu^3b_1^2}{(A_1^2-2\mu A_2)^2} + \frac{2n^2\alpha\mu b_1^2}{A_1^2-2\mu A_2} - \frac{4n^2\beta\mu^2b_1^3}{(A_1^2-2\mu A_2)^2} - \frac{2C^2n\mu^3a_1^2}{A_1^2-2\mu A_2} - \frac{2C^2n^2\mu b_1^2}{A_1^2-2\mu A_2} + \frac{12n^2\gamma\mu a_0^2b_1^2}{A_1^2-2\mu A_2} - \frac{2C^2n\mu^2a_0b_1}{A_1^2-2\mu A_2} + \frac{6n^2\beta\mu a_0b_1^2}{A_1^2-2\mu A_2} - \frac{16n^2\gamma\mu^2a_0b_1^3}{(A_1^2-2\mu A_2)^2} = 0,$$

$$\phi^0:\ C^2n^2a_0^2 - n^2\alpha a_0^2 - n^2\beta a_0^3 - n^2\gamma a_0^4 = 0. \tag{81}$$

On solving the above algebraic equations using the Maple or Mathematica, we get the following results:

$$a_0 = 0, \qquad a_1 = 0,$$
$$b_1 = \left(-2n\beta A_2(n+1) \pm 2\sqrt{n^2\beta^2A_2^2(n+1)^2 - \alpha\gamma A_1^2(n+2)^2(n+1)}\right) \times \left(2n\gamma(n+2)\right)^{-1},$$
$$C = \pm\sqrt{\alpha},$$
$$\mu = \left(n\beta\left[2n\beta A_2(n+1) \pm 2 \times \sqrt{n^2\beta^2A_2^2(n+1)^2 - \alpha\gamma A_1^2(n+2)^2(n+1)}\right]\right) \times \left(2\alpha\gamma(n+2)\right)^{-1}. \tag{82}$$

From (9), (50), and (82), we deduce the traveling wave solution of (47) as follows:

$$u(\xi) = \left[\left(\left(-2n\beta A_2(n+1) \pm 2 \times \sqrt{n^2\beta^2A_2^2(n+1)^2 - \alpha\gamma A_1^2(n+2)^2(n+1)}\right) \times \left(2n\gamma(n+2)\right)^{-1}\right) \times \left(\frac{1}{(\mu/2)\xi^2 + \xi A_1 + A_2}\right)\right]^{1/n}, \tag{83}$$

where $\xi = x \pm \sqrt{\alpha}t$ and $\mu = (n\beta[2n\beta A_2(n+1) \pm 2\sqrt{n^2\beta^2A_2^2(n+1)^2 - \alpha\gamma A_1^2(n+2)^2(n+1)}])/2\alpha\gamma(n+2)$.

In particular, by setting $\beta = 0$ in (83), we have the solutions

$$u(\xi) = \left[\pm\sqrt{\frac{-\alpha(n+1)}{n^2\gamma}}\left(\frac{A_1}{\xi A_1 + A_2}\right)\right]^{1/n}, \tag{84}$$

which are equivalent to the solutions (50) obtained in [36].

(II) If $\gamma = 0$, $\beta \neq 0$.

In this case, (47) converts to

$$\left(C^2 - \alpha\right)u - C^2u'' - \beta u^{n+1} = 0. \tag{85}$$

By balancing between u'' and u^{n+1} in (85) we get $N = 2/n$. According to Step 3, we use the transformation

$$u(\xi) = v^{2/n}(\xi), \tag{86}$$

where $v(\xi)$ is a new function of ξ. Substituting (86) into (85), we get the new ODE

$$\left(C^2 - \alpha\right)n^2v^2 - 2C^2nvv'' - 2C^2(2-n)\left(v'\right)^2 - \beta n^2v^4 = 0. \tag{87}$$

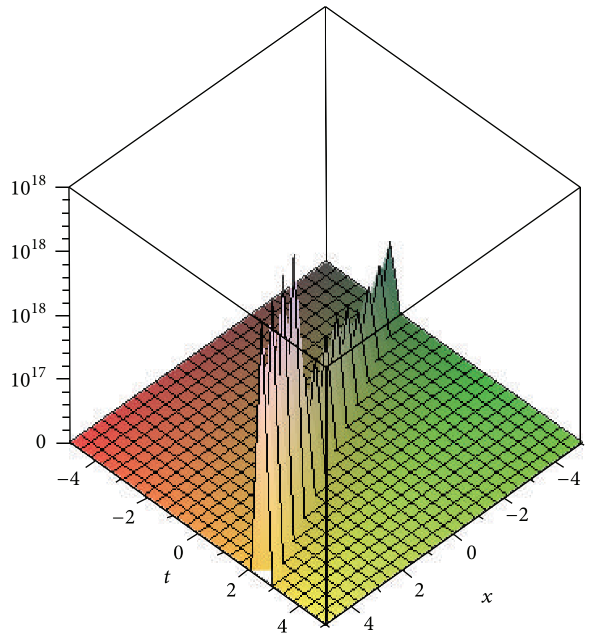

FIGURE 4: The plot of solution (92) when $\alpha = 1$, $\beta = -3$, $n = 1$, and $C = 2$.

Determining the balance number N of the new (87), we get $N = 1$. Consequently, we get

$$v(\xi) = a_0 + a_1\phi(\xi) + b_1\psi(\xi), \tag{88}$$

where a_0, a_1, and b_1 are constants to be determined later. There are three cases to be discussed as follows.

Case 1 (hyperbolic function solutions ($\lambda < 0$)) (Figure 4). If $\lambda < 0$, substituting (88) into (87) and using (4) and (6), the left-hand side of (87) becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be zero yields a system of algebraic equations in a_0, a_1, b_1, μ, λ, and C which are omitted here for simplicity. On using the Maple or Mathematica we have found the following results:

$$\begin{aligned} a_0 &= 0, \\ a_1 &= 0, \\ b_1 &= \pm\sqrt{\frac{\sigma_1(n+2)\left(\alpha - C^2\right)}{2\beta}}, \\ \mu &= 0, \\ \lambda &= -\frac{n^2\left(C^2-\alpha\right)}{4C^2}, \\ C &= C. \end{aligned} \tag{89}$$

From (5), (88), and (89), we deduce the traveling wave solution of (85) as follows:

$$\begin{aligned} u(\xi) &= \left[\pm\sqrt{\frac{\sigma_1(n+2)\left(\alpha - C^2\right)}{2\beta}} \times \left((1) \times \left(A_1 \sinh\left(\frac{n}{2C}\sqrt{C^2-\alpha\xi}\right) + A_2\cosh\left(\frac{n}{2C}\sqrt{C^2-\alpha\xi}\right)\right)^{-1}\right)\right]^{2/n}, \end{aligned} \tag{90}$$

where $\xi = x - Ct$.

In particular, by setting $A_1 = 0$ and $A_2 \neq 0$ in (90), we have the solitary wave solutions

$$u(\xi) = \left[\pm\sqrt{\frac{(n+2)(C^2-\alpha)}{2\beta}}\operatorname{sech}\left(\frac{n}{2C}\sqrt{C^2-\alpha\xi}\right)\right]^{2/n}, \tag{91}$$

while if $A_1 \neq 0$ and $A_2 = 0$, then we have the solitary wave solutions

$$u(\xi) = \left[\pm\sqrt{-\frac{(n+2)(C^2-\alpha)}{2\beta}}\operatorname{csch}\left(\frac{n}{2C}\sqrt{C^2-\alpha\xi}\right)\right]^{2/n}. \tag{92}$$

Note that the solutions (91) and (92) are in agreement with the solutions (33) and (34) of [36] when $\xi_0 = 0$, respectively.

Case 2 (trigonometric function solution ($\lambda > 0$)). If $\lambda > 0$, substituting (88) into (87) and using (4) and (8), the left-hand side of (87) becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be zero yields a system of algebraic equations in $a_0, a_1, b_1, \mu, \lambda$, and C which are omitted here for simplicity. On using the Maple or Mathematica we have found the following results:

$$\begin{aligned} a_0 &= 0, \\ a_1 &= 0, \\ b_1 &= \pm\sqrt{\frac{\sigma_2(n+2)\left(C^2-\alpha\right)}{2\beta}}, \\ \mu &= 0, \\ \lambda &= \frac{n^2\left(\alpha - C^2\right)}{4C^2}, \\ C &= C. \end{aligned} \tag{93}$$

From (7), (88), and (93), we deduce the traveling wave solution of (85) as follows:

$$u(\xi) = \left[\pm\sqrt{\frac{\sigma_2(n+2)\left(C^2-\alpha\right)}{2\beta}} \times \left((1) \times \left(A_1\sin\left(\frac{n}{2C}\sqrt{\alpha-C^2}\xi\right) + A_2\cos\left(\frac{n}{2C}\sqrt{\alpha-C^2}\xi\right)\right)^{-1}\right)\right]^{2/n}, \quad (94)$$

where $\xi = x - Ct$.

In particular, by setting $A_1 = 0$ and $A_2 \neq 0$ in (94), we have the periodic solutions

$$u(\xi) = \left[\pm\sqrt{\frac{(n+2)(C^2-\alpha)}{2\beta}}\sec\left(\frac{n}{2C}\sqrt{\alpha-C^2}\xi\right)\right]^{2/n}, \quad (95)$$

while if $A_1 \neq 0$ and $A_2 = 0$, then we have the solitary wave solutions

$$u(\xi) = \left[\pm\sqrt{\frac{(n+2)(C^2-\alpha)}{2\beta}}\csc\left(\frac{n}{2C}\sqrt{\alpha-C^2}\xi\right)\right]^{2/n}. \quad (96)$$

Note that the solutions (95) and (96) are in agreement with the solutions (37) and (38) of [36] when $\xi_0 = 0$, respectively.

Case 3 (rational function solutions ($\lambda = 0$)). If $\lambda = 0$, substituting (88) into (87) and using (4) and (10), the left-hand side of (87) becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to be zero yields a system of algebraic equations in a_0, a_1, b_1, μ, and C which are omitted here for simplicity. On using the Maple or Mathematica we have found the following results.

Result 1. Consider the following:

$$a_0 = 0, \quad a_1 = \pm\sqrt{-\frac{2\alpha(n+2)}{n^2\beta}}, \quad b_1 = 0, \quad C = \pm\sqrt{\alpha}, \quad \mu = 0. \quad (97)$$

From (9), (88), and (97), we deduce the traveling wave solution of (85) as follows:

$$u(\xi) = \left[\pm\sqrt{-\frac{2\alpha(n+2)}{n^2\beta}}\left(\frac{A_1}{A_1\xi + A_2}\right)\right]^{2/n}, \quad (98)$$

where $\xi = x \pm \sqrt{\alpha}t$.

Note that the solutions (98) are equivalent to the solutions (40) obtained in [36].

Result 2. Consider the following:

$$a_0 = 0, \quad a_1 = \pm\sqrt{-\frac{\alpha(n+2)}{2n^2\beta}}, \quad b_1 = \pm\sqrt{-\frac{\alpha(n+2)\left[A_1^2 - 2\mu A_2\right]}{2n^2\beta}}, \quad C = \pm\sqrt{\alpha}, \quad \mu = \mu. \quad (99)$$

In this result, we deduce the traveling wave solution of (85) as follows:

$$u(\xi) = \left[\pm\sqrt{-\frac{\alpha(n+2)}{2n^2\beta}}\left(\frac{\mu\xi + A_1}{(\mu/2)\xi^2 + A_1\xi + A_2}\right) \pm\sqrt{-\frac{\alpha(n+2)\left[A_1^2-2\mu A_2\right]}{2n^2\beta}} \times\left(\frac{1}{(\mu/2)\xi^2 + A_1\xi + A_2}\right)\right]^{2/n}, \quad (100)$$

where $\xi = x \pm \sqrt{\alpha}t$.

Finally, if we set $\mu = 0$ in (100) we get back to (98).

4. Conclusions

The two variable $(G'/G, 1/G)$-expansion method is used in this paper to obtain some new solutions of two higher order nonlinear evolution equations, namely, the nonlinear Klein-Gordon equations and the nonlinear Pochhammer-Chree equations. As the two parameters A_1 and A_2 take special values, we obtain the solitary wave solutions. When $\mu = 0$ and $b_i = 0$ in (3) and (14), the two variable $(G'/G, 1/G)$-expansion method reduces to the (G'/G)-expansion method. So the two variable $(G'/G, 1/G)$-expansion method is an extension of the (G'/G)-expansion method. The proposed method in this paper is more effective and more general than the (G'/G)-expansion method because it gives exact solutions in more general forms. In summary, the advantage of the two variable $(G'/G, 1/G)$-expansion method over the (G'/G)-expansion method is that the solutions obtained by using the first method recover the solutions obtained by using the second one. Finally, all solutions obtained in this paper have been checked with the Maple by putting them back into the original equations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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