Research Article

Multiplicity Results for the $p(x)$-Laplacian Equation with Singular Nonlinearities and Nonlinear Neumann Boundary Condition

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We investigate the singular Neumann problem involving the $p(x)$-Laplace operator: $(P_\lambda)\{-\Delta_{p(x)}u + |u|^{p(x)-2}u = 1/u^{\delta(x)} + f(x,u)$, in Ω; $u > 0$, in Ω; $|\nabla u|^{p(x)-2}\partial u/\partial \nu = \lambda u^{q(x)}$, on $\partial\Omega\}$, where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with C^2 boundary, λ is a positive parameter, and $p(x), q(x), \delta(x)$, and $f(x,u)$ are assumed to satisfy assumptions **(H0)**–**(H5)** in the Introduction. Using some variational techniques, we show the existence of a number $\Lambda \in (0,\infty)$ such that problem (P_λ) has two solutions for $\lambda \in (0,\Lambda)$, one solution for $\lambda = \Lambda$, and no solutions for $\lambda > \Lambda$.

1. Introduction

The purpose of this paper is to study the existence of multiple solutions for the following inhomogeneous singular Neumann problem involving the $p(x)$-Laplace operator:

$$(P_\lambda) \begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u = \dfrac{1}{u^{\delta(x)}} + f(x,u), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial \nu} = \lambda u^{q(x)}, & \text{on } \partial\Omega. \end{cases} \quad (1)$$

Here $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with C^2 boundary; λ is a positive parameter. For any continuous and bounded function a we define $a^+ := \sup a(x)$ and $a^- := \inf a(x)$. Associated with problem (P_λ) we have the singular functional $E_\lambda : W^{1,p(x)}(\Omega) \to \mathbb{R}$ given by

$$E_\lambda(u) \overset{\text{def}}{=} \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}dx + \int_\Omega \frac{|u|^{p(x)}}{p(x)}dx \\ - \int_\Omega \frac{|u|^{1-\delta(x)}}{1-\delta(x)}dx - \int_\Omega F(x,u)\,dx \\ - \lambda \int_{\partial\Omega} \frac{|u|^{q(x)+1}}{q(x)+1}dx, \quad (2)$$

where $F(x,t) = \int_0^t f(x,s)ds$.

Definition 1. $u \in W^{1,p(x)}(\Omega)$ is called a generalized solution of the equation

$$-\Delta_{p(x)}u + |u|^{p(x)-2}u = \frac{1}{u^{\delta(x)}} + f(x,u) \quad (3)$$

if for all $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\varphi\,dx + \int_\Omega |u|^{p(x)-2}u\varphi\,dx \\ = \int_\Omega |u|^{-\delta(x)}\varphi\,dx + \int_\Omega f(x,u)\varphi\,dx \quad (4) \\ + \lambda \int_{\partial\Omega} |u|^{q(x)}\varphi\,dx.$$

Obviously, every weak solution of problem (P_λ) is also a generalized solution of (3).

The operator $\Delta_{p(x)}u := \text{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$-Laplace where p is a continuous nonconstant function. This differential operator is a natural generalization of the p-Laplace operator $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$, where $p > 1$ is a real constant. However, the $p(x)$-Laplace operator possesses more complicated nonlinearity than p-Laplace operator, due to the fact that $\Delta_{p(x)}$ is not homogeneous. This fact implies some difficulties; for example, we cannot use

the Lagrange Multiplier Theorem in many problems involving this operator.

The study of differential and partial differential equations involving variable exponent is a new and an interesting topic. The interest in studying such problems was stimulated by their applications in elastic mechanics, fluid dynamics, electrorheological fluids, image processing, flow in porous media, calculus of variations, nonlinear elasticity theory, heterogeneous porous media models (see Acerbi and Mingione [1], Diening [2]), and so forth. These physical problems were facilitated by the development of Lebesgue and Sobolev spaces with variable exponent.

At this point, we briefly recall related existence and multiplicity results for elliptic equations with Neumann boundary conditions. Neumann type problems are studied in [3–6] and references therein. The multiplicity result for Neumann problem with Sobolev critical nonlinearity has been studied in [5] where authors considered the problem

$$-\Delta u + u = u^p, \quad u > 0 \text{ in } \Omega,$$
$$\frac{\partial u}{\partial \nu} = \lambda \psi u^q \quad \text{on } \partial\Omega. \tag{5}$$

Here $\Omega \subset \mathbb{R}^N$, $N \geq 3$, and $0 < q < 1 < p \leq 2N/(N-2)$. They proved the existence of $\widetilde{\Lambda}$ such that problem (5) admits at least two solutions for all $\lambda \in (0, \widetilde{\Lambda})$, one solution when $\lambda = \widetilde{\Lambda}$, and no solutions for $\lambda > \widetilde{\Lambda}$. The problem in two dimensions has been considered in [6] where the authors extended the results obtained by [5].

Results for $p(x)$-Laplacian problems Neumann boundary conditions are rare (see [7, 8]). In [7], Fan and Deng studied the Neumann problems with $p(x)$-Laplace operator and the nonlinear potential $f(x, u)$ under appropriate assumptions. By using the subsupersolution method and variation method, the authors get the multiplicity of positive solutions. In [8], Sreenadh and Tiwari extend previous works on nonlinear parametric problems with the $p(x)$-Laplace operator to the case where the Neumann boundary condition is nonlinear. Precisely, under supplementary hypotheses on p, q, the authors show that there exists a finite number Λ such that the posed problem has two solutions for $\lambda \in (0, \Lambda)$, one solution for $\lambda = \Lambda$, and no solutions for $\lambda > \Lambda$.

Before stating our main results, we make the following assumptions throughout this paper:

(H0) $p(\cdot) \in C(\overline{\Omega})$, $1 < p^- \leq p^+ < N$, and $p^+ \leq p_*^-$. As usual, $p^*(x) = Np(x)/(N - p(x))$.

(H1) $0 < q(x) \in C^\beta(\partial\Omega)$ for some $\beta \in (0, 1)$ satisfying $0 \leq q^+ + 1 < p^-(p^-/p^+)$.

(H2) $0 < \delta(x) \in C(\overline{\Omega})$ satisfying $0 < \delta^- \leq \delta^+ < 1$.

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a nondecreasing Carathéodory function satisfying the following:

(H3) $f(x, 0) = 0$ and $f(x, t)t \geq 0$ for all $(x, t) \in \Omega \times \mathbb{R}$.

(H4) There exist $C_1, C_2 > 0$ such that, for $\gamma > 0$, $a(x) \in C(\overline{\Omega})$ with $p^+ - 1 < \gamma$ and $p^+ < a^-$.

$$C_1 u^\gamma \leq f(x, t) \leq C_2 t^{a(x)-1}, \quad \forall x \in \Omega, \ t > 0. \tag{6}$$

(H5) There exist a constant $M > 0$ and $\theta > p^+$ such that

$$0 < \theta F(x, t) \leq f(x, t)t, \quad \forall x \in \Omega, \ |t| > M. \tag{7}$$

Next we describe in a more precise way our main results.

Theorem 2. *Assume that* **(H0)**–**(H5)** *hold and* $a(x) \leq p^*(x) = Np(x)/(N - p(x))$ *in* (6). *Then there exists* $\Lambda > 0$ *with the following properties:*

(1) *Problem* (P_λ) *has a solution* u_λ *for every* $\lambda \in (0, \Lambda)$.

(2) *Problem* (P_λ) *has a solution if* $\lambda = \Lambda$.

(3) *Problem* (P_λ) *does not have any solution if* $\lambda > \Lambda$.

Theorem 3. *Assume that* **(H0)**–**(H5)** *hold and* $a(x) < p^*(x)$ *in* (6). *Then, problem* (P_λ) *has at least two distinct solutions* $u_\lambda(x)$ *and* $v_\lambda(x)$ *for every* $\lambda \in (0, \Lambda)$.

This paper is organized as follows. In Section 2, we will recall some basic facts about the variable exponent Lebesgue and Sobolev spaces which we will use later. Proofs of our results will be presented in Sections 3 and 5.

2. Generalized Lebesgue-Sobolev Spaces Setting

To deal with the $p(x)$-Laplacian problem, we need to introduce some functional spaces $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$, $W_0^{1,p(\cdot)}(\Omega)$ and properties of the $p(x)$-Laplacian which we will use later. Denote by $S(\Omega)$ the set of all measurable real-valued functions defined in Ω. Note that two measurable functions are considered as the same element of $S(\Omega)$ when they are equal almost everywhere. Let

$$L^{p(\cdot)}(\Omega) = \left\{ u \in S(\Omega) : \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}, \tag{8}$$

with the norm

$$|u|_{p(\cdot)} = |u|_{L^{p(\cdot)}(\Omega)}$$
$$= \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}. \tag{9}$$

The space $(L^{p(\cdot)}(\Omega), | \cdot |_{p(\cdot)})$ becomes a Banach space. We call it variable exponent Lebesgue space. Moreover, this space is a separable, reflexive, and uniform convex Banach space; see [9, Theorems 1.6, 1.10, and 1.14].

The variable exponent Sobolev space

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\} \tag{10}$$

can be equipped with the norm

$$\|u\| = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)}, \quad \forall u \in W^{1,p(\cdot)}(\Omega). \tag{11}$$

Note that $W_0^{1,p(\cdot)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ under the norm $\|u\| = |\nabla u|_{p(\cdot)}$. The spaces $W^{1,p(\cdot)}(\Omega)$ and

$W_0^{1,p(\cdot)}(\Omega)$ are separable, reflexive, and uniform convex Banach spaces (see [9, Theorem 2.1]). The inclusion between Lebesgue spaces also generalizes naturally: if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents so that $p_1(x) \leq p_2(x)$ almost everywhere in Ω then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

We denote by $L^{q(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $1/q(x)+1/p(x) = 1$. For $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, the Hölder type inequality

$$\left| \int_\Omega u(x) v(x) dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(x)} |v|_{q(x)} \qquad (12)$$

holds true.

An important role in manipulating the generalized Lebesgue spaces is played by the modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)} dx. \qquad (13)$$

If $(u_n), u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$, then the following relations hold true.

Lemma 4. *Consider the following:*

$$\|u\|_{L^{p(x)}} > 1 \Longrightarrow$$

$$\|u\|_{L^{p(x)}}^{p^-} \leq \rho_{p(x)}(u) \leq \|u\|_{L^{p(x)}}^{p^+},$$

$$\|u\|_{L^{p(x)}} < 1 \Longrightarrow \qquad (14)$$

$$\|u\|_{L^{p(x)}}^{p^+} \leq \rho_{p(x)}(u) \leq \|u\|_{L^{p(x)}}^{p^-},$$

$$\|u_n - u\|_{L^{p(x)}} \longrightarrow 0 \quad \textit{iff } \rho_{p(x)}(u_n - u) \longrightarrow 0.$$

The following result generalizes the well-known Sobolev embedding theorem.

Theorem 5 (see [10, 11]). *Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with Lipschitz boundary and assume that $p \in C(\overline{\Omega})$ with $p(x) > 1$ for each $x \in \overline{\Omega}$. If $r \in C(\overline{\Omega})$ and $p(x) \leq r(x) \leq p^*(x)$ for all $x \in \overline{\Omega}$, then there exists a continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$. Also, the embedding is compact $r(x) < p^*(x)$ almost everywhere in $\overline{\Omega}$, where*

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)}, & \textit{if } p(x) < N, \\ +\infty, & \textit{if } p(x) \geq N. \end{cases} \qquad (15)$$

Now, we recall the following boundary trace embedding theorem from [12].

Theorem 6. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with Lipschitz boundary. If $q \in C(\partial\Omega)$ such that*

$$1 \leq q(x) \leq \frac{(N-1) p(x)}{N - p(x)}, \quad \forall x \in \partial\Omega, \qquad (16)$$

then $W^{1,p(x)} \hookrightarrow L^{q(x)}(\partial\Omega)$.

Next we give a comparison principle as follows.

Lemma 7 (see [8, Lemma 3.2]). *Let $u, v \in W^{1,p(\cdot)}(\Omega)$ be non-negative functions satisfying*

$$-\Delta_{p(x)}u + u^{p(x)-1} \geq -\Delta_{p(x)}v + v^{p(x)-1}, \quad (x \in \Omega),$$

$$|\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} \geq |\nabla v|^{p(x)-2} \frac{\partial v}{\partial \nu}. \qquad (17)$$

Then $u \geq v$ almost everywhere in Ω.

We recall the following strong maximum principle from [13].

Theorem 8. *Let $u, v \in C^{1,\beta}(\overline{\Omega})$, for some $0 < \beta < 1$, satisfy $0 \lneqq u$, $0 \lneqq v$ and*

$$-\Delta_{p(x)}u - \frac{\lambda}{u^{\delta x}} = g(x) \geq h(x) = -\Delta_{p(x)}v - \frac{\lambda}{v^{\delta(x)}}, \qquad (18)$$

with $u = v = 0$ on $\partial\Omega$, where $g, h \in L^\infty(\Omega)$ are such that $0 \leq g < h$ pointwise everywhere in Ω. If

$$\frac{\partial u}{\partial \mathbf{n}} > 0,$$

$$\frac{\partial v}{\partial \mathbf{n}} > 0, \qquad (19)$$

$$\textit{on } \partial\Omega,$$

where $\mathbf{n}$ is the inward unit normal on $\partial\Omega$, then, the following strong comparison principle holds:

$$u > v,$$

$$\textit{in } \Omega;$$

$$\textit{there is a positive } \epsilon \textit{ such that } \frac{\partial(u - v)}{\partial \mathbf{n}} \geq \epsilon, \qquad (20)$$

$$\textit{on } \partial\Omega.$$

3. Existence of a Solution

In this section, we show the existence of a local minimum for E_λ in a small neighborhood of the origin in $W^{1,p(x)}(\Omega)$. Firstly, let us define

$$\Lambda \overset{\text{def}}{=} \sup\{\lambda > 0 : (P_\lambda) \text{ has a weak solution}\}. \qquad (21)$$

Lemma 9. *There exists $\lambda_0 > 0$ such that (P_λ) admits a solution for $\lambda < \lambda_0$.*

Proof. Using (14) and the embeddings in Theorem 5, we estimate $E_\lambda(u)$ for $\|u\|_{W^{1,p(x)}(\Omega)} = r$ as follows:

$$E_\lambda(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx + \int_\Omega \frac{|u|^{p(x)}}{p(x)} dx$$

$$- \int_\Omega \frac{|u|^{1-\delta(x)}}{1 - \delta(x)} dx - \int_\Omega F(x, u) dx$$

$$- \lambda \int_{\partial\Omega} \frac{|u|^{q(x)+1}}{q(x+1)} dx$$

$$\geq \|\nabla u\|_{L^{p(x)}(\Omega)}^{p^+} + \|u\|_{L^{p(x)}(\Omega)}^{p^+} - \|u\|_{L^{p(x)}(\Omega)}^{1-\delta^+}$$

$$- \|u\|_{L^{p(x)}(\Omega)}^{a^-} - \lambda \|u\|_{L^{p(x)}(\partial\Omega)}^{q^-+1}$$

$$\geq \frac{1}{2^{p^+-1}} \left(\|\nabla u\|_{L^{p(x)}(\Omega)} + \|u\|_{L^{p(x)}(\Omega)} \right)^{p^+}$$

$$- \|u\|_{W^{1,p(x)}(\Omega)}^{1-\delta^+} - \|u\|_{W^{1,p(x)}(\Omega)}^{a^-} - \lambda \|u\|_{W^{1,p(x)}(\Omega)}^{q^-+1}$$

$$= \frac{1}{2^{p^+-1}} \|u\|_{W^{1,p(x)}(\Omega)}^{p^+} - \|u\|_{W^{1,p(x)}(\Omega)}^{1-\delta^+}$$

$$- \|u\|_{L^{p(x)}(\Omega)}^{a^-} - \lambda \|u\|_{W^{1,p(x)}(\Omega)}^{q^-+1}.$$

$$(22)$$

Hence, noting that $0 < \delta^+ < 1$ and $1 - \delta^+ < p^+ < a^-$, we can choose $r_0 > 0$ small enough, and there exists $\delta_0 > 0$ such that

$$\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx + \int_\Omega \frac{|u|^{p(x)}}{p(x)} dx - \int_\Omega F(x,u) dx$$

$$- \lambda \int_{\partial\Omega} \frac{|u|^{q(x)+1}}{q(x+1)} dx \geq 0, \quad \forall u \in B_{r_0},$$

$$\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx + \int \frac{|u|^{p(x)}}{p(x)} dx - \int_\Omega F(x,u) dx$$

$$(23)$$

$$- \lambda \int_{\partial\Omega} \frac{|u|^{q(x)+1}}{q(x+1)} dx \geq \delta_0, \quad \forall u \in \partial B_{r_0}.$$

Moreover, since $W^{1,p(x)}(\Omega) \hookrightarrow L^{1-\delta(x)}(\Omega)$, we have, for $\lambda > 0$, small enough,

$$E_\lambda \backslash_{\partial B_{r_0}} \geq \frac{\delta_0}{2} > 0. \tag{24}$$

Set

$$c_0 = \inf_{u \in B_{r_0}} E_\lambda(u). \tag{25}$$

Now, note that, for $t \to 0^+$ and $u \in W^{1,p(x)}(\Omega)$, we have

$$E_\lambda(tu) = \int_\Omega \frac{t^{p(x)} |\nabla u|^{p(x)}}{p(x)} dx + \int_\Omega \frac{|tu|^{p(x)}}{p(x)} dx$$

$$- \int_\Omega \frac{|tu|^{1-\delta(x)}}{1-\delta(x)} dx - \int_\Omega F(x,tu) dx$$

$$- \lambda \int_{\partial\Omega} \frac{|tu|^{q(x)+1}}{q(x)+1} dx$$

$$\leq \frac{t^{p^-}}{p^+} \int_\Omega |\nabla u|^{p(x)} dx + \frac{t^{p^-}}{p^+} \int_\Omega |u|^{p(x)} dx$$

$$- \frac{t^{1-\delta^+}}{1-\delta^+} \int_\Omega |u|^{1-\delta(x)} dx$$

$$- \int_{tu \geq t_*} F(x,tu) dx$$

$$- \frac{\lambda |t|^{q^-+1}}{q^-+1} \int_{\partial\Omega} |u|^{q(x)+1} dx$$

$$\leq \frac{t^{p^-}}{p^+} \int_\Omega |\nabla u|^{p(x)} dx + \frac{t^{p^-}}{p^+} \int_\Omega |u|^{p(x)} dx$$

$$- \frac{t^{1-\delta^+}}{1-\delta^+} \int_\Omega |u|^{1-\delta(x)} dx$$

$$- \frac{t^\theta}{t_*^\theta} \int_{tu \geq t_*} F(x,tu) dx$$

$$- \frac{\lambda |t|^{q^-+1}}{q^-+1} \int_{\partial\Omega} |u|^{q(x)+1} dx$$

$$\leq \frac{t^{p^-}}{p^+} \int_\Omega |\nabla u|^{p(x)} dx + \frac{t^{p^-}}{p^+} \int_\Omega |u|^{p(x)} dx$$

$$- \frac{t^{1-\delta^+}}{1-\delta^+} \int_\Omega |u|^{1-\delta(x)} dx$$

$$- t^\theta \int_{u \geq t_*} F(x,t_*) dx$$

$$- \frac{\lambda |t|^{q^-+1}}{q^-+1} \int_{\partial\Omega} |u|^{q(x)+1} dx.$$

$$(26)$$

Since $1 - \delta^+ < p^+ < \theta$, this implies that $E_\lambda(tu) \to -\infty$ as $t \to \infty$. Thus, $\inf_{\|u\|_{W^{1,p(x)}(\Omega)} \leq r} E_\lambda(u) = c_0 < 0$. Now, let $\{u_j\} \subset B_{r_0}$ be a minimizing sequence for c_0. Then $\|u_j\|_{W^{1,p(x)}(\Omega)} < r_0$ for $r_0 \in (0,r)$. Now by the Ekeland variational principle, there exists a sequence $\{v_j\}$ such that $\|v_j\|_{W^{1,p(x)}(\Omega)} < r$ and

$$\|u_j - v_j\|_{W^{1,p(x)}(\Omega)} \leq \frac{1}{n},$$

$$E_\lambda(v_j) \leq E_\lambda(u_j),$$

$$(27)$$

$$E_\lambda'(v_j) \longrightarrow 0 \quad \text{in } \left(W^{1,p(x)}(\Omega) \right)^*,$$

$$v_j \longrightarrow u_\lambda \quad \text{weakly in } W^{1,p(x)}(\Omega).$$

Moreover, using the Brézis-Lieb lemma in [14] combined with the $p(x)$ version of Theorem 2.1 in Boccardo and Murat [15], it follows that $\nabla v_j(x) \to \nabla u_\lambda(x)$ for almost every $x \in \Omega$,

$$\int_\Omega \frac{|\nabla v_j|^{p(x)}}{p(x)} dx = \int_\Omega \frac{|\nabla (v_j - u_\lambda)|^{p(x)}}{p(x)}$$

$$+ \int_\Omega \frac{|\nabla u_\lambda|^{p(x)}}{p(x)} + o_j(1),$$

$$\int_\Omega f\left(x, v_j\right) dx = \int_\Omega f\left(x, u_\lambda\right) dx$$

$$+ \int_\Omega f\left(x, v_j - u_\lambda\right) dx + o_j\left(1\right),$$

$$\int_\Omega \frac{\left|v_j\right|^{1-\delta(x)}}{1 - \delta\left(x\right)} dx = \int_\Omega \frac{\left|u_\lambda\right|^{1-\delta(x)}}{1 - \delta\left(x\right)} dx$$

$$+ \int_\Omega \frac{\left|v_j - u_\lambda\right|^{1-\delta(x)}}{1 - \delta\left(x\right)} dx + o_j\left(1\right).$$

$$(28)$$

Hence, from (28) and the compact boundary trace embedding in Theorem 5, we get $E_\lambda(v_j) \geq E_\lambda(u_\lambda) + o_j(1)$. Thus, it follows that $E_\lambda(u_\lambda) = c_0 < 0$. Hence $u_\lambda \not\equiv 0$ and it is a local minimizer of E_λ in $W^{1,p(x)}(\Omega)$. □

We prove now the existence of positive solution to (P_λ) for $0 < \lambda < \Lambda$. Precisely, we have the following result.

Lemma 10. *Problem (P_λ) possesses a solution u_λ for $0 < \lambda < \Lambda$.*

Proof. Fix $0 < \lambda < \lambda_2 < \Lambda$. λ_2 such that there exist solutions to (P_λ) for $\lambda = \lambda_2$, say u_{λ_2}. Note that u_{λ_2} is a supersolution for (P_λ). It is clear that 0 is not a local minimizer of E_λ on $[0, u_{\lambda_2}] \cap W^{1,p(x)}(\Omega)$ since $E_\lambda(0) = 0$ and $E_\lambda(tv) < 0$ for $t \to 0^+$. Now, we show the existence of a local minimizer of the functional energy. For this, we use the cut-off argument. Define

$$\widetilde{f}\left(x, s\right) = \begin{cases} f\left(x, 0\right), & \text{if } s < 0, \\ f\left(x, s\right), & \text{if } 0 \leq s \leq u_{\lambda_2}, \\ f\left(x, u_{\lambda_2}\right), & \text{if } s > u_{\lambda_2}\left(x\right), \end{cases}$$

$$\widetilde{g}\left(x, s\right) = \begin{cases} 0, & \text{if } s < 0, \\ s^{-\delta(x)}, & \text{if } 0 \leq s \leq u_{\lambda_2}, \\ u_{\lambda_2}^{-\delta(x)}, & \text{if } s > u_{\lambda_2}\left(x\right), \end{cases} \quad (29)$$

$$\widetilde{h}\left(x, s\right) = \begin{cases} 0, & \text{if } s < 0, \\ s^{q(x)}, & \text{if } 0 \leq s \leq u_{\lambda_2}, \\ u_{\lambda_2}^{q(x)}, & \text{if } s > u_{\lambda_2}\left(x\right). \end{cases}$$

Also define the functional $\widetilde{E}_\lambda : W^{1,p(x)}(\Omega) \to \mathbb{R}$ by

$$\widetilde{E}_\lambda\left(u\right) = \int_\Omega \frac{\left|\nabla u\right|^{p(x)}}{p\left(x\right)} dx + \int_\Omega \frac{\left|u\right|^{p(x)}}{p\left(x\right)} dx$$

$$- \int_\Omega \widetilde{G}\left(x, u\right) dx - \int_\Omega \widetilde{F}\left(x, u\right) dx \quad (30)$$

$$- \lambda \int_{\partial\Omega} \widetilde{H}\left(x, u\right) dx,$$

where $\widetilde{F}(x, t) = \int_0^t \widetilde{f}(x, s) ds$, $\widetilde{G}(x, t) = \int_0^t \widetilde{g}(x, s) ds$, and $\widetilde{H}(x, t) = \int_0^t \widetilde{h}(x, s) ds$. From the dominated convergence

and the compact boundary trace embedding in Theorem 5, it is easy to see that $\widetilde{E}_\lambda$ is bounded below and is weakly lower semicontinuous in $W^{1,p(x)}(\Omega)$. Then, there exists $\widetilde{u}_\lambda \in W^{1,p(x)}(\Omega)$ such that $\widetilde{E}_\lambda$ achieves its global minimum in $W^{1,p(x)}(\Omega)$. Moreover, since $\widetilde{E}_\lambda$ is $C^1(W^{1,p(x)}(\Omega), \mathbb{R})$ from Lemma A.4, $\widetilde{u}_\lambda$ solves the equation

$$-\Delta_{p(x)}\widetilde{u}_\lambda + \left|\widetilde{u}_\lambda\right|^{p(x)-2} \widetilde{u}_\lambda = \widetilde{g}\left(x, \widetilde{u}_\lambda\right) + \widetilde{f}\left(x, \widetilde{u}_\lambda\right),$$

$$\text{in } \Omega, \quad (31)$$

$$\left|\nabla\widetilde{u}_\lambda\right|^{p(x)-2} \frac{\partial\widetilde{u}_\lambda}{\partial\nu} = \lambda\widetilde{h}\left(x, \widetilde{u}_\lambda\right), \quad \text{on } \partial\Omega.$$

Now, using the strong maximum principle (see Theorem 8) $\widetilde{u}_\lambda \not\equiv 0$ and $\widetilde{u}_\lambda > 0$ since 0 is not a local minimizer of E_λ on $[0, u_{\lambda_2}] \cap W^{1,p(x)}(\Omega)$. By the definition of $\widetilde{f}$, $\widetilde{g}$, and $\widetilde{h}$ we have

$$\widetilde{g}\left(x, 0\right) \leq \widetilde{g}\left(x, \widetilde{u}_\lambda\right) \leq \widetilde{g}\left(x, u_{\lambda_2}\right), \quad \forall x \in \Omega,$$

$$\widetilde{f}\left(x, 0\right) \leq \widetilde{f}\left(x, \widetilde{u}_\lambda\right) \leq \widetilde{f}\left(x, u_{\lambda_2}\right), \quad \forall x \in \Omega, \quad (32)$$

$$\widetilde{h}\left(x, 0\right) \leq \widetilde{h}\left(x, \widetilde{u}_\lambda\right) \leq \widetilde{h}\left(x, u_{\lambda_2}\right), \quad \forall x \in \Omega.$$

Again, by the strong comparison principle (see Lemma 7), we conclude that $\widetilde{u}_\lambda \leq u_{\lambda_2}$ in Ω and $(\partial/\partial\nu)(u_{\lambda_2} - \widetilde{u}_\lambda) \leq 0$ on $\partial\Omega$. Hence $\widetilde{u}_\lambda$ is a solution to (P_λ). This completes the proof of Lemma 10. □

Now, we show the following result.

Lemma 11. *There exists at least one positive weak solution for $\lambda = \Lambda$ to problem (P_λ).*

Proof. Let $\{\lambda_k\}_{k\in\mathbb{N}} \in (0, \Lambda)$, $\lambda_k \uparrow \Lambda$ as $k \to \infty$, and u_{λ_k} be a solution of (P_{λ_k}) such that $u_{\lambda_{k-1}} \leq u_{\lambda_k}$ for all $x \in \Omega$. Now, taking u_{λ_k} as test function in (P_{λ_k}), we get

$$\int_\Omega \left|\nabla u_{\lambda_k}\right|^{p(x)} dx + \int_\Omega \left|u_{\lambda_k}\right|^{p(x)} dx$$

$$= \int_\Omega u_{\lambda_k}^{1-\delta(x)} dx + \int_\Omega f\left(x, u_{\lambda_k}\right) u_{\lambda_k} dx \quad (33)$$

$$+ \lambda_k \int_{\partial\Omega} u_{\lambda_k}^{q(x)+1} dx.$$

Moreover, as $E_{\lambda_k}(u_{\lambda_k}) \leq E_{\lambda_k}(0) = 0$, we have

$$\int_\Omega \frac{\left|\nabla u_{\lambda_k}\right|^{p(x)}}{p\left(x\right)} dx + \int_\Omega \frac{\left|u_{\lambda_k}\right|^{p(x)}}{p\left(x\right)} dx - \int_\Omega \frac{u_{\lambda_k}^{1-\delta(x)}}{1 - \delta\left(x\right)} dx$$

$$(34)$$

$$- \int_\Omega F\left(x, u_{\lambda_k}\right) dx - \lambda_k \int_{\partial\Omega} \frac{u_{\lambda_k}^{q(x)+1}}{q\left(x\right) + 1} dx \leq 0.$$

It follows that

$$\frac{1}{p^+}\left(\int_\Omega \left|\nabla u_{\lambda_k}\right|^{p(x)} dx + \int_\Omega \left|u_{\lambda_k}\right|^{p(x)} dx\right)$$

$$\leq \int_\Omega u_{\lambda_k}^{1-\delta(x)} dx + \int_\Omega F\left(x, u_{\lambda_k}\right) dx \qquad (35)$$

$$+ \lambda_k \int_{\partial\Omega} u_{\lambda_k}^{q(x)+1} dx.$$

Now, from (7) there exists $C > 0$ such that, for $\theta > p_+$ and for all $t > 0$,

$$\theta F(x,t) \leq t f(x,t) + C. \qquad (36)$$

Moreover, using Theorem 4.2 in [16], we get the existence of the constants $C_1 > 0$, $C_2 > 0$ such that

$$C_1 \left[d(x)\right]^{\theta_1} \leq \underline{u}_\lambda(x) \leq C_2 \left[d(x)\right]^{\theta_2}, \qquad (37)$$

where $\theta_1 = \max_{d(x)\leq\sigma}(p(x)/(p(x) - 1 + \delta(x)))$, $\theta_2 = \min_{d(x)\leq\sigma}(p(x)/(p(x) - 1 + \delta(x)))$, and $d(x) = \text{dist}(x, \partial\Omega)$. On the other hand, we recall the following inequality due to Lieberman [17]. There exists a constant $K(\Omega) > 0$ such that

$$\int_{\partial\Omega} |u| \leq \int_\Omega |\nabla u| + K(\Omega) \int_\Omega |u|, \quad \forall u \in W^{1,1}(\Omega). \qquad (38)$$

Inserting (36), (37), and (38) in (33), we get $\sup_{k\in\mathbb{N}} \|u_{\lambda_k}\|_{W^{1,p(x)}(\Omega)} < \infty$. It follows that $\{u_{\lambda_k}\}$ is bounded in $W^{1,p(x)}(\Omega)$ since $u_{\lambda_k}(x) \geq u_{\lambda_1}(x)$ for all $x \in \Omega$. Without loss of generality, $u_{\lambda_k} \rightharpoonup u_\Lambda$ in $W^{1,p(x)}(\Omega)$ and then by the Sobolev imbedding $u_{\lambda_k} \to u_\Lambda$ in $L^{q(x)}(\Omega)$ and $u_{\lambda_k(x)} \to u_\Lambda(x)$ for a.e. $x \in \Omega$. By the $L^\infty(\Omega)$-regularity results of [13], the boundedness of $\{\|u_{\lambda_k}\|\}$ implies the boundedness of $\{|u_{\lambda_k}|_{L^\infty(\Omega)}\}$. By the $C^{1,\alpha}(\Omega)$-regularity Theorem 16, the boundedness of $\{|u_{\lambda_k}|_{L^\infty}(\Omega)\}$ implies the boundedness of $\{\|u_{\lambda_k}\|_{C^{1,\alpha}(\overline\Omega)}\}$, where $\alpha \in (0,1)$ is a constant. Thus, we have $u_{\lambda_k} \to u_\Lambda$ in $C^1(\overline\Omega)$. For every $v \in W^{1,p(x)}(\Omega)$, since u_{λ_k} is a solution of problem (P_{λ_k}), we have that, for each k,

$$\int_\Omega \left|\nabla u_{\lambda_k}\right|^{p(x)-2} \nabla u_{\lambda_k} \nabla v \, dx + \int_\Omega \left|u_{\lambda_k}\right|^{p(x)-2} u_{\lambda_k} v \, dx$$

$$= \int_\Omega \left|u_{\lambda_k}\right|^{-\delta(x)} v \, dx + \int_\Omega f\left(x, u_{\lambda_k}\right) v \, dx \qquad (39)$$

$$+ \lambda_k \int_{\partial\Omega} \left|u_{\lambda_k}\right|^{q(x)} v \, dx.$$

Passing to the limit in (39) as $k \to \infty$ yields

$$\int_\Omega \left|\nabla u_\Lambda\right|^{p(x)-2} \nabla u_\Lambda \nabla v \, dx + \int_\Omega \left|u_\Lambda\right|^{p(x)-2} u_\Lambda v \, dx$$

$$= \int_\Omega u_\Lambda^{-\delta(x)} v \, dx + \int_\Omega f(x, u_\Lambda) v \, dx \qquad (40)$$

$$+ \Lambda \int_{\partial\Omega} u_\Lambda^{q(x)} v \, dx,$$

which shows that u_Λ is a solution of (P_Λ). Obviously $u_\Lambda \geq 0$ and $u_\Lambda \not\equiv 0$. Hence u_Λ is a positive solution of (P_Λ) in $W^{1,p(x)}(\Omega)$. This completes the proof of Lemma 11.   □

Then we prove the following nonexistence result.

Lemma 12. *Consider the following:*
  $\Lambda < \infty$.

Proof. Let u_λ be a solution of (P_λ). Taking $\varphi \equiv 1$ as a test function in the weak formulation of (P_λ), we get

$$\int_\Omega u_\lambda^{p(x)-1} dx = \int_\Omega u_\lambda^{-\delta(x)} dx + \int_\Omega f(x, u_\lambda) dx$$

$$+ \lambda \int_{\partial\Omega} u_\lambda^{q(x)} dx. \qquad (41)$$

On the other hand, we have

$$\int_\Omega u_\lambda^{p(x)-1} \leq \int_\Omega u_\lambda^{p^+-1} + C_1. \qquad (42)$$

Using assumption **(H2)** we have $C_2 \int_\Omega u_\lambda^{-\delta^+} \leq \int_\Omega u^{-\delta(x)}$, and using (6) we have $C_3 \int_\Omega u_\lambda^\gamma \leq \int_\Omega f(x, u) dx$. Therefore, from (41) and (42) we get

$$\int_\Omega u_\lambda^\gamma \leq C_3 \int_\Omega u_\lambda^{p^+} - C_4 \int_\Omega u_\lambda^{-\delta^+} + C_5. \qquad (43)$$

Now, since $p^+ - 1 < \gamma$ and $-\delta^+ < \gamma$, by the embedding of $L^\gamma(\Omega)$ into $L^{p^+-1}(\Omega)$ and by the embedding of $L^\gamma(\Omega)$ into $L^{-\delta^+}(\Omega)$ we obtain

$$\int_\Omega u_\lambda^{p^+-1} \leq C_6 \left(\int_\Omega u_\lambda^\gamma\right)^{(p^+-1)/\gamma},$$

$$\int_\Omega u_\lambda^{-\delta^+} \leq C_7 \left(\int_\Omega u_\lambda^\gamma\right)^{-\delta^+/\gamma}. \qquad (44)$$

Substituting (44) in (43) we get

$$\|u_\lambda\|_{L^\gamma(\Omega)}^\gamma \leq C_8 + C_9 \|u_\lambda\|_{L^\gamma(\Omega)}^{p^+-1} + C_{10} \|u_\lambda\|_{L^\gamma(\Omega)}^{-\delta^+}. \qquad (45)$$

Hence, $\|u_\lambda\|_{L^\gamma(\Omega)}$ is bounded by a constant independent of λ. Now, taking $\varphi = u_\lambda^{-q^-}$ as a test function in the weak formulation of (P_λ) and using **(H2)** we get

$$\int_\Omega |u_\lambda|^{p(x)-1-q^-} dx \geq \int_\Omega u_\lambda^{-\delta(x)-q^-} dx$$

$$+ \int_\Omega f(x, u_\lambda) u_\lambda^{-q^-} dx$$

$$+ \lambda \int_{\partial\Omega} u_\lambda^{q(x)-q^-} dx \qquad (46)$$

$$\geq \lambda \int_{\partial\Omega} u_\lambda^{q(x)-q^-} dx$$

$$\geq \lambda \int_\Gamma u_\lambda^{q^--q^-} dx = \lambda |\Gamma|_{N-1}.$$

Now since

$$\int_\Omega |u_\lambda|^{p(x)-1-q^-} dx \leq C + \|u_\lambda\|_{L^\gamma(\Omega)}^\gamma \qquad (47)$$

it follows that Λ is finite. The proof of Lemma 12 is now completed.   □

Proof of Theorem 2. Theorem 2 follows from Lemmas 10, 11, and 12.   □

Now, we prove that the solution of problem (P_λ) obtained in Lemma 10 is a local minimum for the functional energy associated to problem (P_λ). Precisely, we have the following result.

Lemma 13. *Let u_λ be the weak solution of problem (P_λ) obtained in Lemma 10. Then, u_λ is a local minimum for $E_\lambda\backslash_{C^1(\overline{\Omega})}$.*

Proof. Fix $0 < \lambda_1 < \lambda < \lambda_2 < \Lambda$ and let $u_{\lambda_1}, u_{\lambda_2}$ be solutions to (P_λ) for $\lambda = \lambda_1$ and $\lambda = \lambda_2$, respectively, such that $u_{\lambda_1} \leq u_\lambda \leq u_{\lambda_2}$. By Lemma 7, $u_{\lambda_1} < u_\lambda < u_{\lambda_2}$ on $\overline{\Omega}$. Define the following cut-off functions:

$$\widehat{f}(x,t) = \begin{cases} f\left(x, u_{\lambda_1}(x)\right), & t < u_{\lambda_1}(x), \\ f(x,t), & u_{\lambda_1}(x) \leq t \leq u_{\lambda_2}(x), \\ f\left(x, u_{\lambda_2}(x)\right), & t > u_{\lambda_2}(x), \end{cases}$$
$$(x \in \Omega, \ t \in \mathbb{R}),$$

$$\widehat{g}(x,t) = \begin{cases} u_{\lambda_1(x)}^{-\delta(x)}, & t < u_{\lambda_1}(x), \\ t^{-\delta(x)}, & u_{\lambda_1}(x) \leq t \leq u_{\lambda_2}(x), \\ u_{\lambda_2}^{-\delta(x)}, & t > u_{\lambda_2}(x), \end{cases} \quad (48)$$
$$(x \in \Omega, \ t \in \mathbb{R}),$$

$$\widehat{h}(x,t) = \begin{cases} \psi(x) u_{\lambda_1}^{q(x)}, & t < u_{\lambda_1}(x), \\ \psi(x) t^{q(x)}, & u_{\lambda_1}(x) \leq t \leq u_{\lambda_2}(x), \\ \psi(x) u_{\lambda_2}^{q(x)}, & t > u_{\lambda_2}(x), \end{cases}$$
$$(x \in \partial\Omega, \ t \in \mathbb{R}).$$

Then the corresponding functional $\widehat{E}_\lambda : W^{1,p(x)}(\Omega) \to \mathbb{R}$ given by

$$\widehat{E}_\lambda(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} + \int_\Omega \frac{|u|^{p(x)}}{p(x)} - \int_\Omega \widehat{G}(x,u)$$
$$- \int_\Omega \widehat{F}(x,u) - \lambda \int_{\partial\Omega} \widehat{H}(x,u), \quad (49)$$

where $\widehat{F}(x,u) = \int_0^u \widehat{f}(x,t)dt$ $(x \in \Omega)$, $\widehat{G}(x,u) = \int_0^u \widehat{g}(x,t)dt$ $(x \in \Omega)$, and $\widehat{H}(x,u) = \int_0^u \widehat{h}(x,t)dt(x \in \partial\Omega)$. Firstly, from Lemma A.4 $\widehat{E}_\lambda$ is $C^1(W^{1,p(x)}(\Omega), \mathbb{R})$. Then, it is simple to see that $\widehat{E}_\lambda$ is coercive and bounded below. Let u_λ denote the global minimum of $\widehat{E}_\lambda$ on $W^{1,p(x)}(\Omega)$ which satisfies the equation

$$- \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + |u|^{p(x)-2}u$$
$$= \widehat{g}(x,u) + \widehat{f}(x,u), \quad u > 0 \text{ in } \Omega, \quad (50)$$
$$|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = \lambda \widehat{h}(x,u), \quad \text{on } \partial\Omega.$$

Therefore, from the regularity results Theorem 16, we conclude that u_λ is in $C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$. Now, using **(H2)** and since $f(x,t)$ is nondecreasing, by the definition of $\widehat{f}, \widehat{g}$, and $\widehat{h}$ we have

$$\widehat{g}\left(x, u_{\lambda_1}\right) \leq \widehat{g}(x, u_\lambda) \leq \widehat{g}\left(x, u_{\lambda_2}\right), \quad \forall x \in \Omega,$$
$$\widehat{f}\left(x, u_{\lambda_1}\right) \leq \widehat{f}(x, u_\lambda) \leq \widehat{f}\left(x, u_{\lambda_2}\right), \quad \forall x \in \Omega, \quad (51)$$
$$\widehat{h}\left(x, u_{\lambda_1}\right) \leq \widehat{h}(x, u_\lambda) \leq \widehat{h}\left(x, u_{\lambda_2}\right), \quad \forall x \in \partial\Omega.$$

Again, by the strong comparison principle (see Lemma 7), we conclude that $u_{\lambda_1} \leq u_\lambda \leq u_{\lambda_2}$ in Ω. Let $\beta = \inf_{x \in \overline{\Omega}}\{|u_{\lambda_1} - u_{\lambda_2}| + |\nabla u_{\lambda_1} - \nabla u_{\lambda_2}|\}$. We claim that $\beta > 0$. If not, then there exists $x \in \partial\Omega$ such that $u_{\lambda_1} = u_{\lambda_2}$ and $\nabla u_{\lambda_1} = \nabla u_{\lambda_2}$. But this contradicts the boundary data as $\lambda_1 \neq \lambda_2$. Therefore, if $v \in \mathscr{C} = \{u \in C^1(\overline{\Omega}) \backslash \|u - u_\lambda\|_{C^1(\overline{\Omega})} < \beta/2\}$ then $u_{\lambda_1} < u_\lambda < u_{\lambda_2}$ in $\overline{\Omega}$ since $E_\lambda \equiv \widehat{E}_\lambda$ on the set $\mathscr{C}$. Hence u_λ is a local minimum for $E_\lambda\backslash_{C^1(\overline{\Omega})}$. This completes the proof of Lemma 13. □

4. C^1 versus $W^{1,p(x)}$ Local Minimizers of the Energy

The following lemma is crucial in showing multiplicity of solutions. It has been shown in the case $p = 2$ in [18] for the case of critical growth functionals $E_\lambda : H_0^1(\Omega) \to \mathbb{R}$, $\Omega \subset \mathbb{R}^N$, $N \geq 3$ and later for critical growth functionals $E_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$, $1 < p < N$, $\Omega \subset \mathbb{R}^N$, $N \geq 3$ in [19]. A key feature of these latter works is the uniform $C^{1,\alpha}$ estimate they obtain for equations like (P_ϵ) but involving two p-Laplace operators. Using constraints based on L^p-norms rather than Sobolev norms as in [19], the equations for which uniform estimates are required can be simplified to a standard type involving only one p-Laplace operator. This approach was followed in [20] in the subcritical case, in [21] in the critical case, in [22–24], and also in this work to deal with the boundary value problem involving the nonlinear $p(x)$-Laplacian case. More precisely, we have the following result.

Lemma 14. *Suppose that conditions **(H0)**–**(H5)** are satisfied. Let $u_0 \in C^1(\overline{\Omega})$ satisfying*

$$u_0 \geq \eta d(x, \partial\Omega) \quad \text{for some } \eta > 0 \quad (52)$$

be a local minimizer of E_λ in $C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$ topology. Then, u_0 is a local minimum of E_λ in $W^{1,p(x)}(\Omega)$ also.

For proving Lemma 14, we will need the following uniform L^∞-estimates for a family of solutions to (P_ϵ).

Proposition 15. *Let $\{u_\epsilon\}_{\epsilon \in (0,1)}$ be a family of solutions to (P_ϵ), where u_0 satisfies (52) and solves (P_λ). Let $\gamma > 1$ be such that*

$$\sup_{\epsilon \in (0,1)} \left(\|f(x, u_\epsilon)\|_{L^\gamma} + \|u_\epsilon\|_{W^{1,p(x)}(\Omega)}\right) < \infty. \quad (53)$$

Then,

$$\|u_\epsilon\|_{L^\infty(\Omega)} < C. \quad (54)$$

The proof of Proposition 15 is a consequence of the results proved in Appendix A in [13]. Hence, the regularity results of Saoudi and Ghanmi [13] give the following regularity result for weak solutions to problem (P_λ).

Theorem 16. *Let $u \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ be a solution to problem (P_λ). Then, there exists $0 < K = K(\|u\|_{L^\infty(\Omega)}, p^+, p^-, \delta^+, \delta^-, \Omega, N)$ such that any weak solution to problem (P_λ) belongs to $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$.*

Proof of Lemma 14. Assume that the conclusion of Lemma 14 is not true. We define the following constraint for each $\epsilon > 0$:

$$\mathscr{S}_\epsilon \overset{\text{def}}{=} \left\{ u \in W^{1,p(x)}(\Omega) : \rho(u) \overset{\text{def}}{=} \int_\Omega \frac{u^{\alpha(x)}}{\alpha(x)} \right.$$
$$\left. + \frac{1}{p^-} \int_{\partial\Omega} u^{p^-} \le \epsilon \right\}. \tag{55}$$

We consider the following constraint minimization problem:

$$I_\epsilon = \inf_{u \in \mathscr{S}_\epsilon} E_\lambda(u). \tag{56}$$

Firstly, clearly $I_\epsilon > -\infty$. Moreover, we note that $\mathscr{S}_\epsilon$ is a convex set. Using the trace embeddings we see that $\mathscr{S}_\epsilon$ is also a closed set in $W^{1,p(x)}(\Omega)$ which implies that $\mathscr{S}_\epsilon$ is weakly closed in $W^{1,p(x)}(\Omega)$; with the fact that E_λ is weakly low semicontinuous in $W^{1,p(x)}(\Omega)$, it follows that for $\epsilon \in (0,1)$ I_ϵ is achieved on some $u_\epsilon \in \mathscr{S}_\epsilon$; that is

$$E_\lambda(u_\epsilon) = I_\epsilon,$$
$$E_\lambda(u_\epsilon) < E_\lambda(u_0) \tag{57}$$
$$\forall \epsilon \in (0,1).$$

Moreover, since $E_\lambda(u_\epsilon^+) \le E_\lambda(u_\epsilon)$ and $u_\epsilon^+ \in \mathscr{S}_\epsilon$, we may assume that $u_\epsilon \ge 0$.

We now consider the following two cases.

(1) Let $\rho(u_\epsilon) < \epsilon$. Then u_ϵ is also a local minimizer of E_λ in $W^{1,p(x)}(\Omega)$. We now show that E_λ admits Gâteaux derivatives on u_ϵ to derive that u_ϵ satisfies the Euler-Lagrange equation associated with E_λ. For this, according to Lemma A.2, in the Appendix, we need to prove that $\exists \widetilde{\eta} > 0$ such that

$$u_\epsilon \ge \widetilde{\eta} d(x) \operatorname{dist}(x, \partial\Omega), \tag{58}$$

where $d(x) \overset{\text{def}}{=} \operatorname{dist}(x, \partial\Omega)$. To prove (58), we argue by contradiction: $\forall \eta > 0$ let

$$\Omega_\eta = \operatorname{Supp}\left\{ (\eta d(x) - u_\epsilon)^+ \right\}, \tag{59}$$

and suppose that Ω_η has a nonzero measure.

Let $u_\eta = (\eta d(x) - u_\epsilon)^+$ and for $0 < t \le 1$ set $\xi(t) = E_\lambda(u_\epsilon + tu_\eta)$. Then, there exists $c(t)$ satisfying $c(t) > \eta t$ such that $\inf((u_\epsilon + tu_\eta)/d(x)) \ge c(t)$ for $t > 0$. Then, from Lemma A.4 ξ is differentiable for $0 < t \le 1$ and $\xi'(t) = \langle E_\lambda'(u_\epsilon + tu_\eta), u_\eta \rangle$. Thus,

$$\xi'(t) = \int_\Omega \left| \nabla(u_\epsilon + tu_\eta) \right|^{p(x)-2} \nabla(u_\epsilon + tu_\eta) \nabla u_\eta$$

$$+ \int_\Omega \left| u_\epsilon + tu_\eta \right|^{p(x)-2} u_\eta$$
$$- \int_\Omega (u_\epsilon + tu_\eta)^{-\delta(x)} u_\eta$$
$$- \int_\Omega f(x, u_\epsilon + tu_\eta) u_\eta$$
$$- \lambda \int_\Omega \left| u_\epsilon + tu_\eta \right|^{q(x)-1} (u_\epsilon + tu_\eta) u_\eta. \tag{60}$$

From **(H1)**–**(H3)**, we see that

$$\xi'(1) = \int_\Omega \left| \nabla \eta d(x) \right|^{p(x)-2} \nabla(\eta d(x)) \nabla u_\eta$$
$$+ \int_\Omega \left| \eta d(x) \right|^{p(x)-2} u_\eta - \int_\Omega (\eta d(x))^{-\delta(x)} u_\eta$$
$$- \int_\Omega f(x, \eta d(x)) u_\eta$$
$$- \lambda \int_\Omega \left| \eta d(x) \right|^{q(x)-1} (\eta d(x)) u_\eta < 0, \tag{61}$$

for $\eta > 0$ small enough.

Now, since $s^{-\delta(x)} + f(x,s) + \lambda s^{q(x)}$ is nonincreasing for $0 < s$ small enough uniformly to $x \in \Omega$ (by **(H1)**–**(H3)**) and from the monotonicity of the operator $-\Delta_{p(x)} u + |u|^{p(x)-1} u$, we have that for $0 < \eta$ small enough $0 \le \xi'(1) - \xi'(t)$. Therefore, from Taylor's expansion and since $\rho(u_\epsilon) \le \epsilon$, there exists $0 < \gamma < 1$ such that

$$0 \le E_\lambda(u_\epsilon + u_\eta) - E_\lambda(u_\epsilon) = \langle E_\lambda'(u_\epsilon + \gamma u_\eta), u_\eta \rangle$$
$$= \xi'(\gamma). \tag{62}$$

Letting $t = \gamma$ we have $\xi'(\gamma) \le \xi'(1) < 0$. We obtain a contradiction with (62) and then $u_\epsilon \ge \eta d(x)$ for some $\eta > 0$ (which depends a priori on ϵ). Since u_ϵ is a local minimizer of E_λ and E_λ is Gâteaux differentiable in u_ϵ, we get that $E_\lambda'(u_\epsilon)$ is defined and $E_\lambda'(u_\epsilon) = 0$. Recalling that $\underline{u}$ is the solution to the pure singular problem given by Theorem 4.2 in [16] and from the weak comparison principle, there exist $C_1 > 0, C_2 > 0$ such that

$$C_1 [d(x)]^{\theta_1} \le \underline{u} \le u_\epsilon, \tag{63}$$

where $\theta_1 = \max_{d(x) \le \sigma} (p(x)/(p(x) - 1 + \delta(x)))$, for some $\eta > 0$ (independent of ϵ). Since $u_\epsilon \in \mathscr{S}_\epsilon$ and from the fact that u_ϵ satisfies (P_λ), we get that $\{u_\epsilon\}_{\epsilon \ge 0}$ is uniformly bounded in $W^{1,p(x)}(\Omega)$. Now, using Proposition 15 and Theorem 16, we get

$$|u_\epsilon|_{C^{1,\alpha}(\overline{\Omega})} \le C \quad \text{for some } \alpha \in (0,1) \tag{64}$$

and as $\epsilon \to 0^+$

$$u_\epsilon \longrightarrow u_0 \quad \text{in } C^1(\overline{\Omega}) \tag{65}$$

which contradicts the fact that u_0 is a local minimizer in $C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$.

Now, we deal with the second case.

(2) $\rho(u_\epsilon) = \epsilon$: we again show that $u_\epsilon \geq \eta d(x)$ in Ω for some $\eta > 0$. Taking $u_\eta = (\eta d(x) - u_\epsilon)^+$, $\xi(t) = E_\lambda(u_\epsilon + tu_\eta)$, we obtain as above that $\xi'(t) \leq \xi'(1) < 0$ for $0 < t < 1$ and $0 < \eta$ small enough.

Then $\xi(t) = E_\lambda(u_\epsilon + tu_\eta)$ is decreasing. This implies that $E_\lambda(u_\epsilon) > E_\lambda(u_\epsilon + tu_\eta)$ for $t > 0$ and using (52)

$$\rho(u_\epsilon + tu_\eta) < \rho(u_\epsilon) = \epsilon. \tag{66}$$

This yields a contradiction with the fact that u_ϵ is a global minimizer of E_λ on $\mathcal{S}_\epsilon$. In this case, using Lemma A.4 and from the Lagrange multiplier rule we have

$$E'_\lambda(u_\epsilon) = \mu_\epsilon \rho'(u_\epsilon), \quad \text{for some } \mu_\epsilon \in \mathbb{R}. \tag{67}$$

$$(P_\epsilon) \begin{cases} -\Delta_{p(x)}u_\epsilon + |u_\epsilon|^{p(x)-1}u_\epsilon = \dfrac{1}{|u_\epsilon|^{\delta(x)}} + f(x, u_\epsilon) + \mu_\epsilon |u_\epsilon|^{\alpha(x)-2} u_\epsilon, & u_\epsilon > 0 \text{ in } \Omega, \\[2mm] |\nabla u_\epsilon|^{p(x)-2} \dfrac{\partial u_\epsilon}{\partial \nu} = \lambda u_\epsilon^{q(x)} + \mu_\epsilon |u_\epsilon|^{p^- -2} u_\epsilon, & \text{on } \partial\Omega. \end{cases} \tag{70}$$

In this case, from (57), we have that $\sup_{\epsilon \in (0,1)} \|u_\epsilon\|_{W^{1,p(x)}(\Omega)} < \infty$. Hence, we can apply Proposition 15 to conclude that $\sup_{\epsilon \in (0,1)} \|u_\epsilon\|_{L^\infty(\Omega)} < K$ for some constant $K > 0$ independent of ϵ. Therefore, using Theorem 16 we conclude that $|u_\epsilon|_{C^{1,\alpha}(\overline{\Omega})} \leq C$ for some constant $C > 0$ independent of ϵ and as $\epsilon \to 0^+$

$$u_\epsilon \longrightarrow u_0 \quad \text{in } C^1(\overline{\Omega}) \tag{71}$$

which contradicts the fact that u_0 is a local minimizer in $C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$.

Now, we deal with the second case.

Case 2 ($\inf_{\epsilon \in (0,1)} \mu_\epsilon = -\infty$). From above, we can assume that $\mu_\epsilon \leq -1$ for $0 < \epsilon$ small enough. Furthermore, we can find a number $M > 0$ independent of $\epsilon > 0$ and $x \in \overline{\Omega}$, such that $(1/s^{\delta(x)} + f(x,s) + \mu_\epsilon |s|^{\alpha(x)-2}s)$ and $(|s|^{q(x)} + \mu_\epsilon |s|^{p^- -2}s)s$ are negative for all $|s| \geq M$. Then, from the weak comparison principle (see Lemma 7 and using $(u_\epsilon - M)^+$ as test function) we have that $\sup_{\epsilon \in (0,1)} \|u_\epsilon\|_{L^\infty(\Omega)} \leq M$ for $\epsilon > 0$ small enough. Now, since $u_0 \in W^{1,p(x)}(\Omega)$ is a C^1 local minimizer, u_0 is a weak solution to (P_λ); that is, satisfies ess $\inf_K u_0 > 0$ over every compact set $K \subset \Omega$ and

$$\int_\Omega |\nabla u_0|^{p(x)-2} \nabla u_0 \nabla \phi \, dx + \int_\Omega |u_0|^{p(x)-2} \phi \, dx$$

$$= \int_\Omega \frac{1}{u_0^{\delta(x)}} \phi \, dx - \int_\Omega f(x, u_0) \phi \, dx \tag{72}$$

$$- \lambda \int_{\partial\Omega} |u_0|^{q(x)-1} u_0 \phi \, dx,$$

We first show that $\mu_\epsilon \leq 0$. We argue by contradiction. Suppose that $\mu_\epsilon > 0$; then there exists $\varphi \in W^{1,p(x)}(\Omega)$ such that

$$\left\langle E'_\lambda(u_\epsilon), \varphi \right\rangle < 0,$$
$$\left\langle \rho'(u_\epsilon), \varphi \right\rangle < 0 \tag{68}$$

and then for t small we have

$$E_\lambda(u_\epsilon + t\varphi) < E_\lambda(u_\epsilon),$$
$$\rho(u_\epsilon + t\varphi) < \rho(u_\epsilon) \leq \epsilon. \tag{69}$$

This contradicts the fact that u_ϵ is a global minimizer of E_λ in $\mathcal{S}_\epsilon$.

We deal now with the two following cases.

Case 1 ($\inf_{\epsilon \in (0,1)} \mu_\epsilon \stackrel{\text{def}}{=} l > -\infty$). In this case, we write (67) in its PDE form as

for all $\phi \in C_c^\infty(\Omega)$. Therefore, for every function $w \in W^{1,p(x)}(\Omega)$, u_0 satisfies

$$\int_\Omega |\nabla u_0|^{p(x)-2} \nabla u_0 \nabla w \, dx + \int_\Omega |u_0|^{p(x)-2} w \, dx$$

$$= \int_\Omega \frac{1}{u_0^{\delta(x)}} w \, dx - \int_\Omega f(x, u_0) w \, dx \tag{73}$$

$$- \lambda \int_{\partial\Omega} |u_0|^{q(x)-1} u_0 w \, dx.$$

Similarly,

$$\int_\Omega |\nabla u_\epsilon|^{p(x)-2} \nabla u_\epsilon \nabla w \, dx + \int_\Omega |u_\epsilon|^{p(x)-2} w \, dx$$

$$= \int_\Omega \frac{1}{u_\epsilon^{\delta(x)}} w \, dx - \int_\Omega f(x, u_\epsilon) w \, dx \tag{74}$$

$$- \lambda \int_{\partial\Omega} |u_\epsilon|^{q(x)-1} u_\epsilon w \, dx.$$

Now, substracting the above relations with $w = (u_\epsilon - u_0)|u_\epsilon - u_0|^{\beta-1}$, with $\beta > 1$, as a test function in (P_ϵ), integrate by parts and use the fact that $u \mapsto -\Delta_{p(x)}u + |u|^{p(x)-1}u$ is a monotone operator to obtain

$$- \mu_\epsilon \left[\int_\Omega |u_\epsilon - u_0|^{\alpha(x)} |u_\epsilon - u_0|^{\beta-1} (u_\epsilon - u_0) \, dx \right.$$

$$\left. + \int_{\partial\Omega} |u_\epsilon - u_0|^{p^- +\beta-1} \, dx \right] \leq \int_\Omega \left(\frac{1}{u_\epsilon^{\delta(x)}} - \frac{1}{u_0^{\delta(x)}} \right)$$

$$\cdot |u_\epsilon - u_0|^{\beta-1} (u_\epsilon - u_0) \, dx$$

$$+ \int_\Omega (f(x, u_\epsilon) - f(x, u_0)) |u_\epsilon - u_0|^{\beta-1}$$

$$\cdot (u_\epsilon - u_0) \, dx + \lambda \int_{\partial\Omega} \left(|u_\epsilon|^{q-1} u_\epsilon |u_0|^{q-1} u_0 \right)$$

$$\cdot |u_\epsilon - u_0|^{\beta-1} (u_\epsilon - u_0) \, dx. \tag{75}$$

Using the bounds of u_ϵ, u_0 we get

$$-\mu_\epsilon \left[\int_\Omega |u_\epsilon - u_0|^{\alpha(x)} |u_\epsilon - u_0|^{\beta-1} (u_\epsilon - u_0) \, dx \right.$$

$$\left. + \lambda \int_{\partial\Omega} |u_\epsilon - u_0|^{p^- + \beta - 1} \, dx \right] \le C \left[\int_\Omega |u_\epsilon - u_0|^\beta \, dx \right. \tag{76}$$

$$\left. + \lambda \int_{\partial\Omega} |u_\epsilon - u_0|^\beta \, dx \right],$$

where C does not depend on β and ϵ. Now, using Hölder's inequality and the bounds of u_ϵ combined with Lemma 4 we obtain

$$\int_\Omega |u_\epsilon|^\beta \le C \|u_\epsilon\|_{L^{\alpha(x)+\beta-1}}^\beta$$

$$\le \left(\int_\Omega |u_\epsilon|^{\alpha(x)+\beta-1} \right)^{\beta/(\alpha_- + \beta - 1)}. \tag{77}$$

Therefore

$$-\mu_\epsilon \left[\int_\Omega |u_\epsilon - u_0|^{\alpha(x)} |u_\epsilon - u_0|^{\beta-1} (u_\epsilon - u_0) \, dx \right.$$

$$\left. + \lambda \int_{\partial\Omega} |u_\epsilon - u_0|^{p^- + \beta - 1} \, dx \right] \le C (|\Omega|)$$

$$\cdot \left[\int_\Omega |u_\epsilon - u_0|^{\alpha(x)+\beta-1} \, dx \right. \tag{78}$$

$$\left. + \lambda \int_{\partial\Omega} |u_\epsilon - u_0|^{\alpha(x)+\beta-1} \, dx \right]^{\beta/(\alpha_- + \beta - 1)}.$$

Thus for any $\beta > 1$

$$-\mu_\epsilon \left[\|u_\epsilon - u_0\|_{L^{\alpha(x)+\beta-1}(\Omega)}^{\alpha_- - 1} + \|u_\epsilon - u_0\|_{L^{\alpha(x)+\beta-1}(\partial\Omega)}^{\alpha_- - 1} \right]$$

$$\le C (|\Omega|). \tag{79}$$

Passing to the limit in (79) $\beta \to +\infty$ we get

$$\sup (-\mu_\epsilon) \left[\|u_\epsilon - u_0\|_{L^\infty(\Omega)}^{\alpha_- - 1} + \|u_\epsilon - u_0\|_{L^\infty(\partial\Omega)}^{\alpha_- - 1} \right]$$

$$\le C (|\Omega|). \tag{80}$$

Then, using (80) combined with Proposition 15, the uniform L^∞ bounds for $\{u_\epsilon\}_{\epsilon\in(0,1)}$ in Ω as well as $\partial\Omega$, we get that the right-hand side terms in (P_ϵ) are uniformly bounded in $L^\infty(\Omega)$ and in $L^\infty(\partial\Omega)$ from which as in the first case we obtain that u_ϵ ($0 < \epsilon \le 1$) is bounded in $C^{1,\alpha}(\overline{\Omega})$

independently of ϵ. Finally, using Ascoli-Arzela Theorem we find a sequence $\epsilon_n \to 0^+$ such that

$$u_{\epsilon_n} \longrightarrow u_0 \quad \text{in } C^1(\overline{\Omega}). \tag{81}$$

It follows that, for $\epsilon > 0$ sufficiently small,

$$E_\lambda (u_{\epsilon_n}) < E_\lambda (u_0), \tag{82}$$

which contradicts the fact that u_0 is a local minimizer of E_λ for the $C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$ topology. The proof of Lemma 14 is now completed. □

5. Existence of a Second Weak Solution for $0 < \lambda < \Lambda$

In this section, we fix $\lambda \in (0, \Lambda)$ and let $0 < \lambda_1 < \lambda < \lambda_2 < \Lambda$, u_{λ_1} and u_{λ_2} be as in Section 3. Now, we are able to show the existence of a second solution using the generalized Mountain Pass Theorem. Since the functional is not C^1, we use the cut-off functional $\overline{E}_\lambda$ defined in (86). Define the cut-off functions
$\overline{g} : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\overline{g}(x, s) = \begin{cases} \dfrac{1}{s^{\delta(x)}}, & \text{if } s > u_\lambda(x), \\ \dfrac{1}{u(x)_\lambda^{\delta(x)}}, & \text{if } s \le u_\lambda(x), \end{cases} \tag{83}$$

$\overline{f} : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\overline{f}(x, s) = \begin{cases} f(x, s), & \text{if } s > u_\lambda(x), \\ f(x, u_\lambda(x)), & \text{if } s \le u_\lambda(x), \end{cases} \tag{84}$$

$\overline{h} : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\overline{h}(x, s) = \begin{cases} \psi(x) s^{q(x)}, & \text{if } s > u_\lambda(x), \\ \psi(x) u_\lambda^{q(x)}(x), & \text{if } s \le u_\lambda(x), \end{cases} \tag{85}$$

and define now the corresponding functional $\overline{E}_\lambda : W^{1,p(x)}(\Omega) \to \mathbb{R}$ given by

$$\overline{E}_\lambda(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} + \int_\Omega \frac{|u|^{p(x)}}{p(x)} - \int_\Omega \overline{G}(x, u) \, dx$$

$$- \int_\Omega \overline{F}(x, u) \, dx - \lambda \int_{\partial\Omega} \overline{H}(x, u) \, dx, \tag{86}$$

where $\overline{G}(x, u) = \int_0^u \overline{g}(x, t) dt$, $\overline{F}(x, u) = \int_0^u \overline{f}(x, t) dt$, and $\overline{H}(x, u) = \int_0^u \overline{h}(x, t) dt$. Firstly, we prove the following lemma on compactness of Palais-Smale sequences.

Lemma 17. *The functional $\overline{E}_\lambda$ satisfies the Palais-Smale condition.*

Proof. Let $\{u_n\}$ be a (PS) sequence; namely, $\overline{E}_\lambda(u_n)$ is bounded and $\overline{E}'_\lambda(u_n) \to 0$ when $n \to \infty$. Then,

$$c \geq E_\lambda(u_n) - \frac{1}{\theta}\left\langle E'_\lambda(u_n), u_n\right\rangle + \frac{1}{\theta}\left\langle E'_\lambda(u_n), u_n\right\rangle$$

$$\geq \int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)} + \int_\Omega \frac{|u_n|^{p(x)}}{p(x)} - \int_\Omega \overline{G}(x, u_n)$$

$$- \int_\Omega \overline{F}(x, u_n) - \frac{1}{\theta}\left[\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)} + \int_\Omega \frac{|u_n|^{p(x)}}{p(x)}\right] \quad (87)$$

$$- \int_\Omega \overline{G}(x, u_n) - \int_\Omega \overline{F}(x, u_n) - \lambda \int_{\partial\Omega} \overline{H}(x, u_n)\right]$$

$$+ \frac{1}{\theta}\left\langle E'_\lambda(u_n), u_n\right\rangle.$$

Now, we estimate the boundary term from above as follows:

$$\int_{\partial\Omega} \overline{H}(x, u_n) = \int_{\partial\Omega\cap\{u_n \leq 1\}} \overline{H}(x, u_n)$$

$$+ \int_{\partial\Omega\cap\{u_n > 1\}} \overline{H}(x, u_n)$$

$$\leq K_1 + \int_{\partial\Omega\cap\{u_n \leq 1\}} \frac{|u_n|^{q(x)+1}}{q(x)+1} \quad (88)$$

$$\leq K_1 + \frac{1}{q_- + 1}\int_{\partial\Omega} |u_n|^{q^+ + 1}$$

$$\leq K_1 + K_2 \left(\int_\Omega |\nabla u_n|^{p^-}\right)^{(q^+ + 1)/p^-}.$$

Hence, taking (88) in (87) and using **(H2)** combined with (7) we get

$$\geq \left(\frac{1}{p^+} - \frac{1}{\theta}\right)\int_\Omega \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right)$$

$$- K_1 \left(\int_\Omega |\nabla u_n|^{p^-}\right)^{(q_+ + 1)/p_-} + \frac{1}{\theta}\left\langle \overline{E}'_\lambda(u_n), u_n\right\rangle$$

$$- K_2$$

$$\geq \left(\frac{1}{p^+} - \frac{1}{\theta}\right)\int_\Omega \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right) \quad (89)$$

$$- K_1 \left(\int_\Omega |\nabla u_n|^{p^-}\right)^{(q_+ + 1)/p_-}$$

$$- \frac{1}{\theta}\left\|\overline{E}'_\lambda(u_n)\right\| \|u_n\|_{W^{1,p(x)}(\Omega)} - K_2.$$

Now, using Lemma 4 and the fact that $L^{p(x)} \subset L^{p^-}(\Omega)$, we get

$$\|\nabla u_n\|_{L^{p(x)}}^{q^+ + 1} \geq \left(\frac{1}{p^+} - \frac{1}{\theta}\right)\int_\Omega \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right)$$

$$- \frac{1}{\theta}\left\|\overline{E}'_\lambda(u_n)\right\| \|u_n\|_{W^{1,p(x)}(\Omega)} - K_2$$

$$\geq \left(\frac{1}{p^+} - \frac{1}{\theta}\right)\left(\|\nabla u_n\|_{L^{p(x)}}^{p^-} + \|u_n\|_{L^{p(x)}}^{p^-}\right)$$

$$- \frac{1}{\theta}\left(\|\nabla u_n\|_{L^{p(x)}} + \|u_n\|_{L^{p(x)}}\right) - K_2. \quad (90)$$

Hence, $\|u_n\|_{W^{1,p(x)}(\Omega)}$ is bounded. Without loss of generality, we assume that there exists a subsequence of $\{u_n\}$ such that $u_n \rightharpoonup u_0$. Therefore, using Theorems 5 and 6 we get

$$\lim_{n\to\infty} \int_\Omega \overline{g}(x, u_n)(u_n - u_0) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,$$

$$\lim_{n\to\infty} \int_\Omega \overline{f}(x, u_n)(u_n - u_0) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \quad (91)$$

$$\lim_{n\to\infty} \int_{\partial\Omega} \overline{h}(x, u_n)(u_n - u_0) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Observe that

$$\left\langle \overline{E}_\lambda(u_n), u_n - u_0\right\rangle = \int_\Omega \left(|\nabla u_n|^{p(x)-2}u_n\nabla(u_n - u_0)\right.$$

$$\left. + |u_n|^{p(x)-2}u_n(u_n - u_0)\right) - \int_\Omega \overline{g}(x, u_n)(u_n - u_0) \quad (92)$$

$$- \int_\Omega \overline{f}(x, u_n) - \int_{\partial\Omega} \overline{h}(x, u_n)(u_n - u_0).$$

We already know that

$$\left\langle \overline{E}_\lambda(u_n), u_n - u_0\right\rangle \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \quad (93)$$

Using (91), we obtain

$$\int_\Omega \left(|\nabla u_n|^{p(x)-2}\nabla u_n\nabla(u_n - u_0)\right.$$

$$\left. + |u_n|^{p(x)-2}u_n(u_n - u_0)\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \quad (94)$$

This together with the convergence of $u_n \rightharpoonup u_0$ in $W^{1,p(x)}(\Omega)$ implies that $u_n \to u_0$ strongly in $W^{1,p(x)}(\Omega)$; that is, $\overline{E}_\lambda$ satisfies the (PS) condition. The proof of Lemma 17 is now completed. □

Proof of Theorem 3. Firstly, note that $\overline{E}_\lambda(u) \geq \overline{E}_\lambda(u_\lambda)$ for any solution u of (50). Hence, as in Section 3 we can conclude that u_λ is a local minimum for $\overline{E}_\lambda$ in $W^{1,p(x)}(\Omega)$. By the strong comparison principle and Hopf lemma, we can conclude that any critical point of $\overline{E}_\lambda$ is also a critical point of E_λ and hence u also solves (P_λ). It is easy to see that u_{λ_1} and u_{λ_2} are a subsolution and a supersolution to the problem associated with the functional energy $\overline{E}_\lambda$. Therefore, using the approach as in Theorem 2, we prove that this problem has a solution $v_\lambda \in [u_{\lambda_1}, u_{\lambda_2}] \cap C^1(\overline{\Omega})$ such that v_λ is a local minimizer of $\overline{E}_\lambda$ in the C^1 topology. Now, by the comparison principle we can see that $v_\lambda \geq u_\lambda$ and also v_λ solves (P_λ). If $v_\lambda \not\equiv u_\lambda$ the conclusion of Theorem 3 holds. That is, we can assume $v_\lambda \equiv u_\lambda$ and u_λ is a strict local minimum of $\overline{E}_\lambda$ in

the $W^{1,p(x)}(\Omega)$ topology. Then, from Lemma A.4, the functional $\overline{E}_\lambda \in C^1(W^{1,p(x)}(\Omega), \mathbb{R})$ and note that $\overline{E}_\lambda(t\varphi) \to -\infty$ as $t \to \infty$. Thus, we can apply Lemma 17 combined with the Mountain Pass Theorem to conclude that problem (P_λ) has a solution v_λ such that $v_\lambda \neq u_\lambda$. Therefore the proof of Theorem 3 is now completed. $\qquad\square$

Appendix

We start with an important technical tool which enables us to estimate the singularity in the Gâteaux derivative of the energy functional $E_\lambda : W^{1,p(x)}(\Omega) \to \mathbb{R}$ defined in (2).

Lemma A.1. *Let $0 < \delta < 1$. Then there exists a constant $C_\delta > 0$ such that the inequality*

$$\int_0^1 |\mathbf{a} + s\mathbf{b}|^{-\delta} ds \leq C_\delta \left(\max_{0 \leq s \leq 1} |\mathbf{a} + s\mathbf{b}| \right)^{-\delta} \quad \text{(A.1)}$$

holds true for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$ with $|\mathbf{a}| + |\mathbf{b}| > 0$.

An elementary proof of this lemma can be found in Takáč [25, Lemma A.1, p. 233]. We continue by showing the Gâteaux differentiability of the energy functional E_λ at a point $u \in W^{1,p(x)}(\Omega)$ satisfying $u \geq C_1[\text{dist}(x)]^{\theta_1}$ in Ω with $\theta_1 = \max_{\text{dist}(x) \leq \sigma}(p(x)/(p(x) - 1 + \delta(x)))$ (for details see Theorem 4.2 in [16]).

Lemma A.2. *Let assumptions* (H0)–(H5) *be satisfied. Assume that $u, v \in W^{1,p(x)}(\Omega)$ and u satisfies $u \geq C_1[\text{dist}(x)]^{\theta_1}$ in Ω. Then we have*

$$\lim_{t \to 0} \frac{1}{t} \left(E_\lambda(u + tv) - E_\lambda(u) \right)$$
$$= \int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla v \, dx + \int_\Omega |u|^{p(x)-2} uv \, dx$$
$$\quad - \int_\Omega \frac{1}{u^{\delta(x)}} v \, dx - \int_\Omega f(x, u) v \, dx \quad \text{(A.2)}$$
$$\quad - \lambda \int_{\partial\Omega} |u|^{q(x)} v \, dx.$$

Proof. We show the result only for the singular term $\int_\Omega (v/u^{\delta(x)}) dx$; the other two terms are treated in a standard way. So let

$$H(u) = \int_\Omega \frac{1}{u^{\delta(x)}} dx, \quad \text{for } u \in W^{1,p(x)}(\Omega). \quad \text{(A.3)}$$

For $\xi \in \mathbb{R} \setminus \{0\}$ we define

$$z(\xi) = \frac{d}{d\xi} H(\xi^+) = \begin{cases} \dfrac{1}{u^{\delta(x)}}, & \text{if } \xi > 0; \\ 0, & \text{if } \xi < 0. \end{cases} \quad \text{(A.4)}$$

Consequently,

$$\frac{1}{t} \left(H(u + tv) - H(u) \right)$$
$$= \int_\Omega \left(\int_0^1 z(u + stv) \, ds \right) v \, dx. \quad \text{(A.5)}$$

Notice that for almost every $x \in \Omega$ we have $u(x) > 0$ and

$$\int_0^1 z(u(x) + stv(x)) \, ds \longrightarrow z(u(x)) = \frac{1}{u^{\delta(x)}}, \quad \text{(A.6)}$$
$$\text{as } t \longrightarrow 0.$$

Moreover, the integral on the left-hand side (with nonnegative integrand) is dominated by

$$\int_0^1 z(u(x) + stv(x)) \, ds$$
$$\leq C \int_0^1 |u(x) + stv(x)|^{-\delta(x)} \, ds$$
$$\leq C_{\delta^-} \int_0^1 |u(x) + stv(x)|^{-\delta^+} \, ds \quad \text{(A.7)}$$
$$\leq C_{\delta^-} \left(\max_{0 \leq s \leq 1} |u(x) + stv(x)| \right)^{-\delta^+} \leq C_{\delta^-} u^{-\delta^+}$$
$$\leq C_{\delta^-} \left(C_1 [d(x)]^{\theta_1} \right)^{-\delta^+} = C_{\delta^-, C_1} d(x)^{-\delta^+\theta_1}$$

with constants $C_{\delta^-, C_1} > 0$ independent of $x \in \Omega$. Here, we have used the estimate (A.1) from Lemma A.1 above. Finally, we have $vd(x)^{-\delta^+\theta_1} \in L^1(\Omega)$, by $v \in W^{1,p(x)}(\Omega)$ and Hardy's inequality. Hence, we are allowed to invoke the Lebesgue dominated convergence theorem in (A.5) from which the lemma follows by letting $t \to 0$. $\qquad\square$

Corollary A.3. *Let assumptions* (H0)–(H5) *be satisfied. Then the energy functional $E_\lambda : W^{1,p(x)}(\Omega) \to \mathbb{R}$ is Gâteaux differentiable at every point $u \in W^{1,p(x)}(\Omega)$ that satisfies $u \geq C_1[d(x)]^{\theta_1}$ in Ω. Its Gâteaux derivative $E'_\lambda(u)$ at u is given by*

$$\langle E'_\lambda(u), v \rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla v \, dx$$
$$\quad + \int_\Omega |u|^{p(x)-2} uv \, dx - \int_\Omega u^{-\delta(x)} v \, dx \quad \text{(A.8)}$$
$$\quad - \int_\Omega f(x, u) v \, dx - \lambda \int_{\partial\Omega} |u|^{q(x)} v \, dx$$

for $v \in W^{1,p(x)}(\Omega)$.

We continue by proving the C^1-differentiability of the cut-off energy functional $\overline{E}_\lambda$ defined below.

Lemma A.4. *Let assumptions* (H0)–(H5) *be satisfied, and $w \in W^{1,p(x)}(\Omega)$ such that $w \geq C_1[d(x)]^{\theta_1}$ in Ω. Setting for $x \in \Omega$*

$$\overline{h}(x, s)$$
$$= \begin{cases} s^{-\delta(x)} + f(x, s) + \lambda |s|^{q(x)}, & \text{if } s \geq w(x), \\ w^{-\delta(x)}(x) + f(x, w(x)) + \lambda |w(x)|^{q(x)}, & \text{if } s < w(x), \end{cases} \quad \text{(A.9)}$$

$\overline{H}(x, s) = \int_0^s h(x, t) dt$ and for $u \in W^{1, p(x)}(\Omega)$

$$\overline{E}_\lambda (u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx - \int_\Omega \overline{H}(x, u) dx, \qquad \text{(A.10)}$$

we have that $\overline{E}_\lambda$ belongs to $C^1(W^{1, p(x)}(\Omega), \mathbb{R})$.

Proof. As in Lemma A.2, we concentrate on the singular term, the others being standard. Let

$$g(x, s) = \begin{cases} s^{-\delta(x)}, & \text{if } s \geq w(x), \\ w^{-\delta(x)}(x), & \text{if } s < w(x), \end{cases} \qquad \text{(A.11)}$$

$G(x, s) = \int_0^s g(x, t) dt$, and $S(u) = \int_\Omega G(x, u) dx$. Proceeding as in Lemma A.2, we obtain that, for all $u \in W_0^{1, p(x)}(\Omega)$, $S(u)$ has a Gâteaux derivative $S'(u)$ given by

$$\langle S'(u), v \rangle = \int_\Omega (\max\{u(x), w(x)\})^{-\delta(x)} v(x) dx. \quad \text{(A.12)}$$

Let $u_k \in W_0^{1, p(x)}(\Omega)$, $u_k \to u_0$. Then

$$\left| \langle S'(u_k) - S'(u_0), v \rangle \right|$$
$$= \left| \int_\Omega \left((\max\{u_k(x), w(x)\})^{-\delta(x)} v(x) \right.\right.$$
$$\left.\left. - (\max\{u_0(x), w(x)\})^{-\delta(x)} v(x) \right) dx \right| \qquad \text{(A.13)}$$
$$\leq 2C \int_\Omega w^{-\delta(x)} |v| dx \leq 2C \int_\Omega w^{-\delta^+} |v| dx$$
$$\leq 2C C_1^{-\delta^+} \int_\Omega [d(x)]^{-\delta^+ \theta_1} |v| dx$$

for all $v \in W^{1, p(x)}(\Omega)$. Again, as in Lemma A.2, we use Hardy's inequality to deduce that $[d(x)]^{-\delta^+ \theta_1} v \in L^1(\Omega)$, so that by Lebesgue's dominated convergence theorem we conclude that the Gâteaux derivative of S is continuous which implies that $S \in C^1(W^{1, p(x)}(\Omega), \mathbb{R})$. □

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

Acknowledgments

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