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Research Article

The m-Path Cover Polynomial of a Graph and a Model for General Coefficient Linear Recurrences

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An m-path cover $\Gamma = \{P_{\ell_1}, P_{\ell_2}, \dots, P_{\ell_r}\}$ of a simple graph G is a set of vertex disjoint paths of G, each with $\ell_k \le m$ vertices, that span G. With every P_ℓ we associate a weight, $\omega(P_\ell)$, and define the weight of Γ to be $\omega(\Gamma) = \prod_{k=1}^{r}\omega(P_{\ell_k})$. The m-path cover polynomial of G is then defined as $\mathbb{P}_m(G) = \sum_\Gamma \omega(\Gamma)$, where the sum is taken over all m-path covers Γ of G. This polynomial is a specialization of the path-cover polynomial of Farrell. We consider the m-path cover polynomial of a weighted path $P(m-1, n)$ and find the $(m+1)$-term recurrence that it satisfies. The matrix form of this recurrence yields a formula equating the trace of the recurrence matrix with the m-path cover polynomial of a suitably weighted cycle $C(n)$. A directed graph, $T(m)$, the edge-weighted m-trellis, is introduced and so a third way to generate the solutions to the above $(m+1)$-term recurrence is presented. We also give a model for general-term linear recurrences and time-dependent Markov chains.

1. Introduction, m-Path Cover Polynomial, and Notation

Let G be a graph with no loops or multiple edges, with vertex set $V(G)$.

First we review some basic concepts to establish notation.

A *path* P_ℓ in G is a sequence of distinct vertices $P_\ell = [v_1, v_2, \dots, v_\ell]$ where each pair (v_i, v_{i+1}) for $1 \le i \le \ell - 1$ is an edge. The *length* of a path is the number of vertices in it. Thus a path of length 1 is a vertex and a path of length 2 is an edge, and P_ℓ has length ℓ. Path P_ℓ *begins* at vertex v_1, its *first* vertex, and *ends* at vertex v_ℓ, its *last* vertex. The path $[v_1, v_2, \dots, v_\ell]$ and its reverse $[v_\ell, v_{\ell-1}, \dots, v_1]$ are considered to be the same path. The set of vertices in P_ℓ is $V(P_\ell) = \{v_1, v_2, \dots, v_\ell\}$. Two paths P_ℓ and $P_{\ell'}$ in G are *disjoint* if $V(P_\ell) \cap V(P_{\ell'}) = \emptyset$. The *empty path* has 0 vertices. Finally, recall that a subgraph of G *spans* G if it has the same vertex set as G.

Now we introduce the central concept of this paper.

An *m-path* P_ℓ has $\ell \le m$; *that is*, it is a path of length at most m for some fixed m with $1 \le m \le |V(G)|$.

An *m-path cover* $\Gamma = \{P_{\ell_1}, P_{\ell_2}, \dots, P_{\ell_r}\}$ of G is a set of pairwise disjoint m-paths of G that span G. Thus each ℓ_k satisfies $1 \le \ell_k \le m$, and every vertex of G lies in exactly one m-path; that is, $V(G) = \cup_{k=1}^{r} V(P_{\ell_k})$ is a partition of $V(G)$.

With every m-path P_ℓ we associate a *weight*, $\omega(P_\ell)$, and then the weight of Γ is $\omega(\Gamma) = \prod_{k=1}^{r}\omega(P_{\ell_k})$.

Definition 1. The *m-path cover polynomial* of G, $\mathbb{P}_m(G)$, is the sum of the weights of all m-path covers of G; that is,

$$\mathbb{P}_m(G) = \sum_{\Gamma}\omega(\Gamma), \quad (1)$$

where Γ is an m-path cover of G.

The path-cover polynomial (or path polynomial) of a graph G is a specialization of the F-cover polynomial of Farrell [1] where F is restricted to be a path; see Farrell [2]. Thus our m-path cover polynomial $\mathbb{P}_m(G)$ is a further specialization to paths of length $\ell \le m$. See also Chow [3] and D'Antona and Munarini [4].

It seems that this research is the first direct consideration of the m-path cover polynomial of a graph. See McSorley et al. [5] for specialization to the case $m = 2$, where all classical orthogonal polynomials are generated as 2-path cover polynomials of suitably weighted paths. For related work see the theory of weighted linear species developed in Joyal [6] and Bergeron et al. [7]. In particular, Munarini [8] uses the m-filtered linear partitions of a linearly ordered set to achieve some similar results; see especially our Sections 7 and 8.

In Section 2 we introduce a weighted path $P(m-1,n)$ and find the $(m+1)$-term recurrence that its m-path polynomial satisfies. In Section 3 the matrix form of this recurrence is presented and yields a trace formula that, in Section 4, gives the m-path cover polynomial of a suitably weighted cycle $C(n)$. Section 5 interprets our results in terms of a model for time-dependent Markov chains. In Section 6 a directed graph, $T(m)$, the edge-weighted m-trellis, is introduced and so a third way to generate the solutions to the above recurrence and trace is found. In Section 7 we model general constant coefficient linear recurrences, and we derive various relevant formulas with both algebraic and combinatorial proofs. Finally, in Section 8, we obtain a relevant new integer sequence and relate this sequence to known sequences in the literature.

Notation. We write $\mathbb{P}_m[v_1, v_2, \ldots, v_\ell]$, instead of $\mathbb{P}_m([v_1, v_2, \ldots, v_\ell])$, for the m-cover polynomial of the path $[v_1, v_2, \ldots, v_\ell]$; similarly we write $\omega[v_1, v_2, \ldots, v_\ell]$ instead of $\omega([v_1, v_2, \ldots, v_\ell])$, and so forth.

Vertices in $P(m-1,n)$ (Section 2) and in subpaths of $P(s,n)$ will be labelled u_i; vertices in $C(n)$ (Section 4) will be labelled v_i; and vertices in $T(m)$ (Section 6) will be labelled w_i.

For $1 \le \ell \le m$ we use indeterminate $x_{\ell,i}$ as the weight of a path of length ℓ in G. Throughout the paper $m \ge 1$ is fixed. In all the examples we set $m = 3$, and many examples have $n = 4$.

2. Weighted Path $P(m-1,\ n)$

For $m \ge 1$ and $n \ge 0$ the path $P(m-1,n)$ has $m-1+n$ vertices $\{u_1, u_2, \ldots, u_{m-1+n}\}$. The first $m-1$ vertices are weighted with weight 1 and the remaining n vertices are weighted, one by one, with the indeterminates from the set $\{x_{1,1}, x_{1,2}, \ldots, x_{1,n}\}$. Thus all vertices, that is, all paths of length $\ell = 1$, in $P(m-1,n)$ are weighted. For $2 \le \ell \le m$ a path of length ℓ in $P(m-1,n)$ is weighted with 0 if its last vertex has weight 1 and with $x_{\ell,i}$ if its last vertex has weight $x_{1,i}$. The path $P(0,0)$ is the empty path with no vertices.

Definition 2. For $n \ge 1$ let $f_{m,n}$ be the m-path cover polynomial of the weighted $P(m-1,n)$.

Starting conditions are $f_{m,n} = 1$ for $-(m-1) \le n \le 0$.

As mentioned in Section 1, throughout this paper the path $[u_a, u_{a+1}, \ldots, u_b]$ is a subpath of the weighted $P(m-1,n)$.

We now derive our main $(m+1)$-term recurrence.

Theorem 3. *For a fixed $m \ge 1$ and any $n \ge -(m-1)$,*

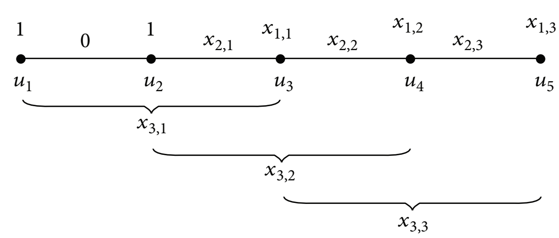
$$f_{m,n} = x_{1,n} f_{m,n-1} + x_{2,n} f_{m,n-2} + \cdots + x_{m,n} f_{m,n-m} = \sum_{\ell=1}^{m} x_{\ell,n} f_{m,n-\ell}. \quad (2)$$

Proof. The last vertex u_{m-1+n} of $P(m-1,n)$ lies in every m-path cover of $P(m-1,n)$. Suppose, in such an m-path cover, it is present as the last vertex in an m-path of length ℓ. Then this m-path has weight $x_{\ell,n}$ and begins at $u_{m+n-\ell}$. The sum of the weights of all such m-path covers is therefore

$$x_{\ell,n} \mathbb{P}_m [u_1, u_2, \ldots, u_{m-1+n-\ell}] = x_{\ell,n} f_{m,n-\ell}, \quad (3)$$

where $[u_1, u_2, \ldots, u_{m-1+n-\ell}]$ is a subpath of $P(m-1,n)$. Now summing over ℓ gives the result. The initial conditions $f_{m,n} = 1$ for $-(m-1) \le n \le 0$ ensure that this equation holds when $\ell \ge n$. □

Example 4. For $m = 3$ the weighted path $P(2,3)$ is

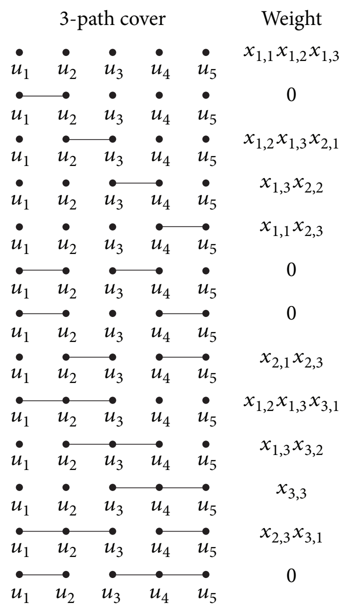


(4)

The weights of paths of lengths $\ell = 1$ and 2 (vertices and edges) are shown above the path. Vertex labels and weights of paths of length $\ell = 3$ are shown below the path. Let

$$\begin{aligned}
\ell = 1 : \omega[u_1] &= \omega[u_2] = 1, \\
\omega[u_3] &= x_{1,1}, \\
\omega[u_4] &= x_{1,2}, \\
\omega[u_5] &= x_{1,3}, \\
\ell = 2 : \omega[u_1, u_2] &= 0, \\
\omega[u_2, u_3] &= x_{2,1}, \\
\omega[u_3, u_4] &= x_{2,2}, \\
\omega[u_4, u_5] &= x_{2,3}, \\
\ell = 3 : \omega[u_1, u_2, u_3] &= x_{3,1}, \\
\omega[u_2, u_3, u_4] &= x_{3,2}, \\
\omega[u_3, u_4, u_5] &= x_{3,3}.
\end{aligned} \quad (5)$$

All 3-path covers of $P(2,3)$ and their weights are shown below:



(6)

So $f_{3,3} = x_{1,1}x_{1,2}x_{1,3} + x_{1,2}x_{1,3}x_{2,1} + x_{1,3}x_{2,2} + x_{1,1}x_{2,3} + x_{2,1}x_{2,3} + x_{1,2}x_{1,3}x_{3,1} + x_{1,3}x_{3,2} + x_{2,3}x_{3,1} + x_{3,3}$.

Example 5. Theorem 3 with $m = 3$ gives the 4-term recurrence for a fixed $n \geq 1$,

$$f_{3,n} = x_{1,n}f_{3,n-1} + x_{2,n}f_{3,n-2} + x_{3,n}f_{3,n-3}. \tag{7}$$

Then the starting conditions $f_{3,-2} = f_{3,-1} = f_{3,0} = 1$ give

$$\begin{aligned}
f_{3,1} &= x_{1,1} + x_{2,1} + x_{3,1},\\
f_{3,2} &= x_{1,1}x_{1,2} + x_{1,2}x_{2,1} + x_{1,2}x_{3,1} + x_{2,2} + x_{3,2},\\
f_{3,3} &= x_{1,1}x_{1,2}x_{1,3} + x_{1,2}x_{1,3}x_{2,1} + x_{1,3}x_{2,2} + x_{1,1}x_{2,3}\\
&\quad + x_{2,1}x_{2,3} + x_{1,2}x_{1,3}x_{3,1} + x_{1,3}x_{3,2} + x_{2,3}x_{3,1} + x_{3,3},\\
f_{3,4} &= x_{1,1}x_{1,2}x_{1,3}x_{1,4} + x_{1,2}x_{1,3}x_{1,4}x_{2,1} + x_{1,2}x_{1,3}x_{1,4}x_{3,1}\\
&\quad + x_{1,1}x_{1,2}x_{2,4} + x_{1,1}x_{1,4}x_{2,3} + x_{1,2}x_{2,1}x_{2,4}\\
&\quad + x_{1,2}x_{2,4}x_{3,1} + x_{1,3}x_{1,4}x_{2,2} + x_{1,3}x_{1,4}x_{3,2}\\
&\quad + x_{1,4}x_{2,1}x_{2,3} + x_{1,4}x_{2,3}x_{3,1} + x_{1,1}x_{3,4}\\
&\quad + x_{1,4}x_{3,3} + x_{2,1}x_{3,4} + x_{2,2}x_{2,4}\\
&\quad + x_{2,4}x_{3,2} + x_{3,1}x_{3,4}\\
&\ \ \vdots
\end{aligned} \tag{8}$$

We check $f_{3,3}$ from Example 4.

Definition 6. For $0 \leq r \leq m-1$ we define $P(r,n)$ as above for $P(m-1,n)$, except that we have r vertices instead of $m-1$ vertices of weight 1 at the beginning of the path. Thus $P(r,n)$ has $r+n$ vertices and is formed from $P(m-1,n)$ by truncating from the right. All m-paths in $P(r,n)$ are weighted as in $P(m-1,n)$. We let $\mathcal{P}_m(r,n)$ be the m-path cover polynomial of the weighted $P(r,n)$. We note that $f_{m,n} = \mathcal{P}_m(m-1,n)$.

Example 7. For $m = 3$ and $n = 4$,

$$\begin{aligned}
\mathcal{P}_3(0,4) &= x_{1,1}x_{1,2}x_{1,3}x_{1,4} + x_{1,1}x_{1,2}x_{2,4}\\
&\quad + x_{1,1}x_{1,4}x_{2,3} + x_{1,3}x_{1,4}x_{2,2}\\
&\quad + x_{1,1}x_{3,4} + x_{1,4}x_{3,3} + x_{2,2}x_{2,4},\\
\mathcal{P}_3(1,4) &= x_{1,1}x_{1,2}x_{1,3}x_{1,4} + x_{1,2}x_{1,3}x_{1,4}x_{2,1} + x_{1,3}x_{1,4}x_{2,2}\\
&\quad + x_{1,1}x_{1,4}x_{2,3} + x_{1,1}x_{1,2}x_{2,4} + x_{1,4}x_{2,1}x_{2,3}\\
&\quad + x_{1,2}x_{2,1}x_{2,4} + x_{1,3}x_{1,4}x_{3,2}\\
&\quad + x_{1,1}x_{3,4} + x_{1,4}x_{3,3} + x_{2,1}x_{3,4} + x_{2,2}x_{2,4}\\
&\quad + x_{2,4}x_{3,2},\\
\mathcal{P}_3(2,4) &= f_{3,4};
\end{aligned} \tag{9}$$

see Example 5.

For a fixed r with $0 \leq r \leq m-1$ we define the starting conditions,

$$\mathcal{P}_m(r,n) = \begin{cases} 0, & \text{if } -(m-1) \leq n \leq -r-1,\\ 1, & \text{if } -r \leq n \leq 0. \end{cases} \tag{10}$$

We then have the following recurrence; the proof is similar to the proof of Theorem 3, and setting $r = m-1$ recovers Theorem 3.

Theorem 8. *For a fixed r with $0 \leq r \leq m-1$ and any $n \geq 1$,*

$$\mathcal{P}_m(r,n) = \sum_{\ell=1}^{m} x_{\ell,n}\mathcal{P}_m(r,n-\ell). \tag{11}$$

We now work with the fundamental solutions to recurrence (2).

For $1 \leq j \leq m$ let $f_{m,n}^{(j)}$ denote the jth fundamental solution to (2). Thus the $f_{m,n}^{(j)}$ obey the recurrence

$$f_{m,n}^{(j)} = \sum_{\ell=1}^{m} x_{\ell,n} f_{m,n-\ell}^{(j)}, \tag{12}$$

with starting conditions

$$f_{m,-(m-1)+k}^{(j)} = \begin{cases} 1, & \text{if } k = m-j,\\ 0, & \text{if } k \neq m-j, \end{cases} \tag{13}$$

where $0 \leq k \leq m-1$.

We have

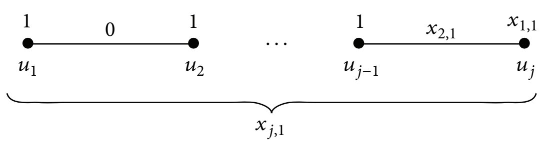
$$f_{m,n} = \sum_{j=1}^{m} f_{m,n}^{(j)}. \tag{14}$$

Our next result expresses $f_{m,n}^{(j)}$ as the difference of two m-path cover polynomials. Consistent with (10) we set $\mathcal{P}_m(-1,n) = 0$ for every $n \geq -(m-1)$.

Lemma 9. *For $n \geq 1$ and $1 \leq j \leq m$,*

$$f_{m,n}^{(j)} = \mathscr{P}_m(j-1,n) - \mathscr{P}_m(j-2,n). \tag{15}$$

Proof. By induction on n, first consider $n = 1$. Now $f_{m,1-\ell}^{(j)} = 1$ when $\ell = j$ and $f_{m,1-\ell}^{(j)} = 0$ otherwise. Each $f_{m,n}^{(j)}$ satisfies (12), so $f_{m,1}^{(j)} = \sum_{\ell=1}^{m} x_{\ell,1} f_{m,1-\ell}^{(j)} = x_{j,1}$. Now consider the path $P(j-1,1)$ shown below:



(16)

The first vertex u_1 lies in every m-path cover of $P(j-1,1)$ so, similar to the proof of Theorem 3, we have

$$\begin{aligned}\mathscr{P}_m(j-1,1) &= \omega[u_1]\mathscr{P}_m(j-2,1) \\ &\quad + \omega[u_1,u_2]\mathscr{P}_m(j-3,1) \\ &\quad + \cdots + \omega[u_1,u_2,\ldots,u_j] \\ &= 1\cdot\mathscr{P}_m(j-2,1) + 0\cdot\mathscr{P}_m(j-3,1) \\ &\quad + \cdots + x_{j,1}.\end{aligned} \tag{17}$$

Thus, from above, $f_{m,1}^{(j)} = x_{j,1} = \mathscr{P}_m(j-1,1) - \mathscr{P}_m(j-2,1)$; that is, (15) is true for $n = 1$.

Now we have

$$\begin{aligned}&f_{m,n+1}^{(j)} \\ &= \sum_{\ell=1}^{m} x_{\ell,n+1} f_{m,n+1-\ell}^{(j)} \\ &= \sum_{\ell=1}^{m} x_{\ell,n+1}\left\{\mathscr{P}_m(j-1,n+1-\ell) - \mathscr{P}_m(j-2,n+1-\ell)\right\} \\ &= \sum_{\ell=1}^{m} x_{\ell,n+1}\mathscr{P}_m(j-1,n+1-\ell) \\ &\quad - \sum_{\ell=1}^{m} x_{\ell,n+1}\mathscr{P}_m(j-2,n+1-\ell) \\ &= \mathscr{P}_m(j-1,n+1) - \mathscr{P}_m(j-2,n+1),\end{aligned} \tag{18}$$

using (12) again at the first line, the induction hypothesis at the second line and Theorem 8 at the last line. Hence the induction goes through and (15) is true for all $n \geq 1$. □

Example 10. Using (12) and the starting conditions following (12) for $m = 3$ and $n = 4$ the 3 fundamental solutions to recurrence (2) are

$$\begin{aligned}f_{3,4}^{(1)} &= x_{1,1}x_{1,2}x_{1,3}x_{1,4} + x_{1,1}x_{1,2}x_{2,4} + x_{1,1}x_{1,4}x_{2,3} \\ &\quad + x_{1,3}x_{1,4}x_{2,2} + x_{1,1}x_{3,4} + x_{1,4}x_{3,3} \\ &\quad + x_{2,2}x_{2,4}, \\ f_{3,4}^{(2)} &= x_{1,2}x_{1,3}x_{1,4}x_{2,1} + x_{1,3}x_{1,4}x_{3,2} + x_{1,4}x_{2,1}x_{2,3} \\ &\quad + x_{1,2}x_{2,1}x_{2,4} + x_{2,4}x_{3,2} + x_{2,1}x_{3,4}, \\ f_{3,4}^{(3)} &= x_{1,2}x_{1,3}x_{1,4}x_{3,1} + x_{1,4}x_{2,3}x_{3,1} + x_{1,2}x_{2,4}x_{3,1} \\ &\quad + x_{3,1}x_{3,4}.\end{aligned} \tag{19}$$

We check (14) using Example 5,

$$f_{3,4} = f_{3,4}^{(1)} + f_{3,4}^{(2)} + f_{3,4}^{(3)}. \tag{20}$$

We also check Lemma 9 using $\mathscr{P}_3(-1,4) = 0$ and Example 7,

$$\begin{aligned}f_{3,4}^{(1)} &= \mathscr{P}_3(0,4) - \mathscr{P}_3(-1,4) = \mathscr{P}_3(0,4), \\ f_{3,4}^{(2)} &= \mathscr{P}_3(1,4) - \mathscr{P}_3(0,4), \\ f_{3,4}^{(3)} &= \mathscr{P}_3(2,4) - \mathscr{P}_3(1,4).\end{aligned} \tag{21}$$

By iteration of such formulas, we have Corollary 11, where (ii) is a specialization of (i) with $r = 0$.

Corollary 11. *(i) For $1 \leq j \leq m$,*

$$\mathscr{P}_m(r,n) = \sum_{j=1}^{r+1} f_{m,n}^{(j)}, \tag{22}$$

(ii) the first fundamental solution to recurrence (2) *is given by*

$$f_{m,n}^{(1)} = \mathscr{P}_m(0,n). \tag{23}$$

Corollary 12 is a useful technical result.

Corollary 12. *For $n \geq 1$ and $1 \leq j \leq m$,*

$$f_{m,n+1-j}^{(j)} = \sum_{\ell=j}^{m} x_{\ell,\ell+1-j}\mathbb{P}_m\left[u_{m+\ell+1-j},\ldots,u_{m+n-j}\right]. \tag{24}$$

Proof. For $j = 1$ from Corollary 11(ii) we have $f_{m,n}^{(1)} = \mathscr{P}_m(0,n)$. Now in the weighted path $P(0,n)$ let vertex u_1 be covered by a path Q_ℓ of length ℓ where $1 \leq \ell \leq m$. Then Q_ℓ begins at vertex u_1 and ends at vertex u_ℓ, which has weight $x_{1,\ell}$; so $\omega(Q_\ell) = x_{\ell,\ell}$. Now in every m-path cover of $P(0,n)$ vertex u_1 must be covered by such a path Q_ℓ, so $f_{m,n}^{(1)} = \sum_{\ell=1}^{m} x_{\ell,\ell}\mathbb{P}_m[u_{\ell+1},\ldots,u_n]$, which is the above formula for $j = 1$.

For any $2 \leq j \leq m$ the path $[u_{m+1-j},\ldots,u_{m-1+n}]$ is a subpath of $P(m-1,n)$. In fact the weighted paths $P(j-$

$1, n)$ and $[u_{m+1-j}, \ldots, u_{m-1+n}]$ (except for vertex labels) are identical, so $\mathscr{P}_m(j-1, n) = \mathbb{P}_m[u_{m+1-j}, \ldots, u_{m-1+n}]$. From Lemma 9 we have

$$\begin{aligned} f^{(j)}_{m,n+1-j} &= \mathscr{P}_m\left(j-1, n+1-j\right) - \mathscr{P}_m\left(j-2, n+1-j\right) \\ &= \mathbb{P}_m\left[u_{m+1-j}, \ldots, u_{m+n-j}\right] \\ &\quad - 1 \cdot \mathbb{P}_m\left[u_{m+2-j}, \ldots, u_{m+n-j}\right] \\ &= \text{sum of terms of } \mathbb{P}_m\left[u_{m+1-j}, \ldots, u_{m+n-j}\right] \\ &\quad \text{in which vertex } u_{m+1-j} \text{ is covered} \\ &\quad \text{by a path whose weight is an indeterminate} \\ &\quad \text{as opposed to a path with weight 1.} \end{aligned} \tag{25}$$

So let vertex u_{m+1-j} be covered by a path Q_ℓ of length $\ell \geq 1$. Then Q_ℓ begins at vertex u_{m+1-j} and ends at vertex $u_{m+\ell-j}$, which has weight $x_{1,\ell+1-j}$. Hence $\omega(Q_\ell) = x_{\ell,\ell+1-j}$. Furthermore, because Q_ℓ ends at $u_{m+\ell-j}$ if $\ell < j$, then $m+\ell-j \leq m-1$; hence $w(Q_\ell) = 0$, a contradiction; so $\ell \geq j$.

Now, similar to the above, the sum of the terms of $\mathbb{P}_m[u_{m+1-j}, \ldots, u_{m+n-j}]$ that contain $x_{\ell,\ell+1-j}$ is $x_{\ell,\ell+1-j}\mathbb{P}_m[u_{m+\ell+1-j}, \ldots, u_{m+n-j}]$. Finally, summing over the lengths ℓ of all possible paths Q_ℓ, namely, summing over ℓ with $j \leq \ell \leq m$, gives the result. □

This completes study of the weighted path $P(m-1, n)$.

3. Matrix Formulation and Trace

We set up our $(m+1)$-term recurrence (2) in matrix form.

Let $X_{m,0} = I_m$ be the $m \times m$ identity matrix, and for $n \geq 1$ let $X_{m,n}$ be the $m \times m$ matrix

$$X_{m,n} = \begin{pmatrix} 0 & 1 & \cdot & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ x_{m,n} & x_{m-1,n} & x_{m-2,n} & \cdots & x_{1,n} \end{pmatrix}. \tag{26}$$

Let T denote transpose, and let $F_{m,n}$ be the vector $F_{m,n} = (f_{m,n-(m-1)}, \ldots, f_{m,n})^T$. Then recurrence (2) can be written as

$$F_{m,n} = X_{m,n} F_{m,n-1}, \tag{27}$$

where $F_{m,0} = (f_{m,-(m-1)}, \ldots, f_{m,0})^T = (1, \ldots, 1)^T$. By iterating this equation we have $F_{m,n} = Y_{m,n}\ F_{m,0}$, where

$$\begin{aligned} Y_{m,n} &= X_{m,n} X_{m,n-1} \cdots X_{m,0} \\ &= \begin{pmatrix} f^{(m)}_{m,n-(m-1)} & \cdot & \cdots & \cdot & f^{(1)}_{m,n-(m-1)} \\ f^{(m)}_{m,n-(m-2)} & \cdot & \cdots & \cdot & f^{(1)}_{m,n-(m-2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f^{(m)}_{m,n-1} & \cdot & \cdots & \cdot & f^{(1)}_{m,n-1} \\ f^{(m)}_{m,n} & \cdot & \cdots & \cdot & f^{(1)}_{m,n} \end{pmatrix}. \end{aligned} \tag{28}$$

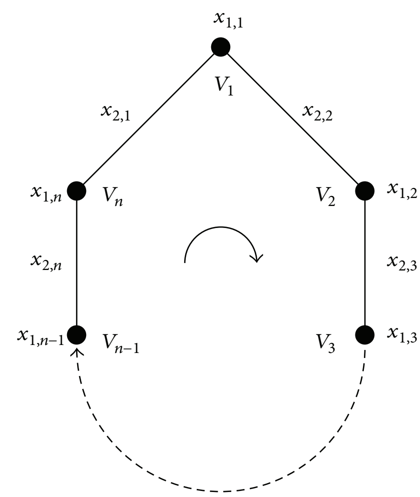


Figure 1: Weighted $C(n)$.

With tr denoting trace, we have the following.

Lemma 13. *For $n \geq 1$,*

$$\operatorname{tr}\left(Y_{m,n}\right) = \sum_{j=1}^{m} f^{(j)}_{m,n+1-j}. \tag{29}$$

We now apply these results to the weighted cycle $C(n)$.

4. Weighted Cycle $C(n)$ and Trace

We introduce the weighted cycle $C(n)$ for $n \geq 1$ shown in Figure 1. It has n vertices labelled $\{v_1, v_2, \ldots, v_n\}$ and n edges.

It is weighted as follows: for $1 \leq \ell \leq m$, let P_ℓ be a path of length ℓ that traverses $C(n)$ clockwise and ends at vertex v_i. We define $\omega(P_\ell) = x_{\ell,i}$.

Thus the weighted cycle $C(1)$ is an isolated vertex v_1 with weight $\omega(v_1) = x_{1,1}$; and the weighted cycle $C(2)$ has 2 vertices $\{v_1, v_2\}$ with $\omega(v_1) = x_{1,1}$ and $\omega(v_2) = x_{1,2}$ and 2 edges: edge (v_1, v_2) with $\omega(v_1, v_2) = x_{2,2}$ and edge (v_2, v_1) with $\omega(v_2, v_1) = x_{2,1}$.

In Figure 1 only the weights of paths of lengths $\ell = 1$ and 2 are shown.

Lemma 14. *For $1 \leq a \leq b \leq n$ the following m-path cover polynomials, the first which comes from $C(n)$ and the second from $P(m-1, n)$, are equal:*

$$\mathbb{P}_m\left[v_a, \ldots, v_b\right] = \mathbb{P}_m\left[u_{m-1+a}, \ldots, u_{m-1+b}\right]. \tag{30}$$

Proof. Except for vertex labels, the weighted paths $[v_a, \ldots, v_b]$ in $C(n)$ and $[u_{m-1+a}, \ldots, u_{m-1+b}]$ in $P(m-1, n)$ are identical. Hence the result is obtained. □

Definition 15. For $n \geq 1$ let $\mathscr{C}_m(n)$ be the m-path cover polynomial of the weighted $C(n)$.

In the following, when necessary, we reduce subscripts on u, v, and the second subscript on x, all modulo n. We write $u_{n+t} = u_t$, $v_{n+t} = v_t$, $x_{\ell,n+t} = x_{\ell,t}$, and so forth.

Theorem 16 is the main result of this section. Recall the matrix $Y_{m,n}$ from (28).

Theorem 16. *For* $n \geq 1$,

$$\mathscr{C}_m(n) = \operatorname{tr}(\mathrm{Y}_{\mathrm{m,n}}). \quad (31)$$

Proof. Consider the weighted $C(n)$. Vertex v_1 lies in every m-path cover of $C(n)$. Suppose, in such an m-path cover, it is covered by a path P_ℓ of length ℓ that begins at v_{n-p} and ends at $v_{n-p-1+\ell}$, for some $p \in \{-1, 0, 1, \ldots, \ell-2\}$. Now $1 \leq \ell \leq m$; that is, $p+2 \leq \ell \leq m$. The sum of the weights of all such paths is then

$$\sum_{\ell=p+2}^{m} x_{\ell,n-p-1+\ell}\mathbb{P}_m\left[v_{n-p+\ell}, \ldots, v_{n-p-1}\right]. \quad (32)$$

But $p \in \{-1, 0, 1, \ldots, m-2\}$, so

$$\begin{aligned}
\mathscr{C}_m(n) &= \sum_{p=-1}^{m-2}\sum_{\ell=p+2}^{m} x_{\ell,n-p-1+\ell}\mathbb{P}_m\left[v_{n-p+\ell}, \ldots, v_{n-p-1}\right] \\
&= \sum_{j=1}^{m}\sum_{\ell=j}^{m} x_{\ell,n+\ell+1-j}\mathbb{P}_m\left[v_{n+\ell+2-j}, \ldots, v_{n+1-j}\right] \\
&= \sum_{j=1}^{m}\sum_{\ell=j}^{m} x_{\ell,\ell+1-j}\mathbb{P}_m\left[u_{m+\ell+1-j}, \ldots, u_{m-j}\right] \quad (33)\\
&= \sum_{j=1}^{m} f_{m,n+1-j}^{(j)} \\
&= \operatorname{tr}(Y_{m,n}),
\end{aligned}$$

letting $j = p+2$ at the second line, and using subscript reduction modulo n and Lemma 14 at the third line, then Corollary 12 at the fourth line, and Lemma 13 at the last line. □

Example 17. For $m = 3$ and $n = 4$ consider the weighted $C(4)$ in Figure 2.

The 3-paths are weighted as follows:

$$\begin{aligned}
\ell = 1 : \omega[v_1] &= x_{1,1}, \\
\omega[v_2] &= x_{1,2}, \\
\omega[v_3] &= x_{1,3}, \\
\omega[v_4] &= x_{1,4}, \\
\ell = 2 : \omega[v_1, v_2] &= x_{2,2}, \\
\omega[v_2, v_3] &= x_{2,3}, \\
\omega[v_3, v_4] &= x_{2,4}, \\
\omega[v_4, v_1] &= x_{2,1}, \\
\ell = 3 : \omega[v_1, v_2, v_3] &= x_{3,3}, \\
\omega[v_2, v_3, v_4] &= x_{3,4}, \\
\omega[v_3, v_4, v_1] &= x_{3,1}, \\
\omega[v_4, v_1, v_2] &= x_{3,2}.
\end{aligned} \quad (34)$$

By considering all 3-path covers, the 3-path cover polynomial of the weighted $C(4)$ is

$$\begin{aligned}
\mathscr{C}_3(4) = {} & x_{1,1}x_{1,2}x_{1,3}x_{1,4} + x_{1,1}x_{1,2}x_{2,4} + x_{1,2}x_{1,3}x_{2,1} \\
& + x_{1,3}x_{1,4}x_{2,2} + x_{1,1}x_{1,4}x_{2,3} + x_{1,3}x_{3,2} + x_{1,4}x_{3,3} \\
& + x_{1,1}x_{3,4} + x_{1,2}x_{3,1} + x_{2,1}x_{2,3} + x_{2,2}x_{2,4}.
\end{aligned} \quad (35)$$

Similar to Example 10, the recurrence (12) and the starting conditions following (12) give

$$\begin{aligned}
f_{3,4}^{(1)} = {} & x_{1,1}x_{1,2}x_{1,3}x_{1,4} + x_{1,1}x_{1,2}x_{2,4} + x_{1,1}x_{1,4}x_{2,3} \\
& + x_{1,3}x_{1,4}x_{2,2} + x_{1,1}x_{3,4} + x_{1,4}x_{3,3} + x_{2,2}x_{2,4}, \\
f_{3,3}^{(2)} = {} & x_{1,2}x_{1,3}x_{2,1} + x_{1,3}x_{3,2} + x_{2,1}x_{2,3}, \\
f_{3,2}^{(3)} = {} & x_{1,2}x_{3,1}.
\end{aligned} \quad (36)$$

Together with the following matrices

$$Y_{3,4} = X_{3,4}X_{3,3}X_{3,2}X_{3,1}X_{3,0}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x_{3,4} & x_{2,4} & x_{1,4} \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x_{3,3} & x_{2,3} & x_{1,3} \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x_{3,2} & x_{2,2} & x_{1,2} \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x_{3,1} & x_{2,1} & x_{1,1} \end{pmatrix}$$

$$= \begin{pmatrix} x_{1,2}x_{3,1} & x_{1,2}x_{2,1}+x_{3,2} & x_{1,1}x_{1,2}+x_{2,2} \\ x_{1,2}x_{1,3}x_{3,1}+x_{2,3}x_{3,1} & \begin{matrix} x_{1,3}x_{3,2}+x_{2,1}x_{2,3} \\ +x_{1,2}x_{1,3}x_{2,1} \end{matrix} & \begin{matrix} x_{1,1}x_{2,3}+x_{1,3}x_{2,2} \\ +x_{1,1}x_{1,2}x_{1,3}+x_{3,3} \end{matrix} \\ \begin{matrix} x_{1,2}x_{2,4}x_{3,1}+x_{1,2}x_{1,3}x_{1,4}x_{3,1} \\ +x_{1,4}x_{2,3}x_{3,1}+x_{3,1}x_{3,4} \end{matrix} & \begin{matrix} x_{1,2}x_{2,4}x_{2,1}+x_{1,2}x_{1,3}x_{1,4}x_{2,1}+x_{1,3}x_{1,4}x_{3,2} \\ +x_{1,4}x_{2,1}x_{2,3}+x_{2,1}x_{3,4}+x_{2,4}x_{3,2} \end{matrix} & \begin{matrix} x_{1,1}x_{3,4}+x_{1,1}x_{1,2}x_{2,4}+x_{1,1}x_{1,4}x_{2,3} \\ +x_{1,1}x_{1,2}x_{1,3}x_{1,4}+x_{1,3}x_{1,4}x_{2,2} \\ +x_{1,4}x_{3,3}+x_{2,2}x_{2,4} \end{matrix} \end{pmatrix}, \tag{37}$$

we may check the results from Lemma 13 and Theorem 16,

$$\mathscr{C}_3(4) = \operatorname{tr}(Y_{3,4}) = \sum_{j=1}^{3} f^{(j)}_{3,5-j} = f^{(1)}_{3,4} + f^{(2)}_{3,3} + f^{(3)}_{3,2}. \tag{38}$$

5. Markov Chain Interpretation

In this section we consider an interesting special case, where in the matrix formulation of the recurrence we have stochastic matrices. A matrix of Form (26) can be considered a transition matrix for a Markov chain with m states under the conditions

$$\sum_j x_{j,n} = 1, \quad x_{j,n} \ge 0, \ \forall j. \tag{39}$$

Because the probabilities $x_{j,n}$ vary with n, these are the transition matrices for a nonhomogeneous Markov chain. Note also that, as transition matrices are multiplied from left to right, the process is effectively time reversed. In fact,

$$P[\text{jump at time } \nu \text{ from state } m \text{ to state } j] = x_{m-j+1,n-\nu+1}. \tag{40}$$

This process is often referred to as a *ladder process*. From any state j, with $j < m$, the process jumps with certainty to $j+1$, then to $j+2$, and so forth, up the ladder, till it reaches state m. At that point it jumps randomly back down the ladder to one of the intermediates states j, $1 \le j < m$, and the procedure repeats. Because all of the matrices are stochastic, the row sums of matrices such as $Y_{m,n}$, see (28), will all equal 1. Recall from Section 3 that

$$\begin{pmatrix} f_{m,n-m+1} \\ \vdots \\ f_{m,n} \end{pmatrix} = Y_{m,n} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{41}$$

Thus, we have the following.

Proposition 18. *In the stochastic case, all of the path polynomials $f_{m,n}$ evaluate to 1.*

5.1. Homogeneous Case. In the case of constant coefficients (see (26)), sending $x_{\ell,i} \to x_\ell$, forall i, we drop the dependence on n and write

$$X_m = \begin{pmatrix} 0 & 1 & \cdot & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ x_m & x_{m-1} & x_{m-2} & \cdots & x_1 \end{pmatrix}, \tag{42}$$

with $\sum x_j = 1$. Now

$$Y_{m,n} = (X_m)^n \tag{43}$$

is the n-step transition matrix. It is easy to see that a row vector (on the left) fixed by X_m is

$$(x_m, x_m + x_{m-1}, \ldots, x_m + x_{m-1} + \cdots + x_2, 1). \tag{44}$$

Furthermore, under the assumption $x_j > 0$, forall j, it is immediate that the chain is irreducible and aperiodic, hence ergodic. That is,

$$\lim_{n\to\infty} Y_{m,n} = \Omega \tag{45}$$

exists and has equal rows, and each row proportional to the left-invariant vector indicated above normalized to row sum 1.

Example 19. Take the uniform case $x_j = 1/m$, $1 \le j \le m$. Then we have the fixed vector $(1, 2, 3, \ldots, m)$ and the limits

$$\lim_{n\to\infty} f^{(j)}_{m,n} = \frac{2(m-j+1)}{m(m+1)}. \tag{46}$$

Thus, for large n, if we randomly choose an m-path cover of $P(m-1, n)$ then the probability that it belongs to the j-fundamental solution is $2(m-j+1)/m(m+1)$. In particular, the first fundamental solution satisfies

$$\lim_{n\to\infty} f^{(1)}_{m,n} = \frac{2}{m+1}. \tag{47}$$

So the m-path cover polynomial model provides a combinatorial model for nonhomogeneous Markov chains. A closely related model, the trellis, is discussed in detail below.

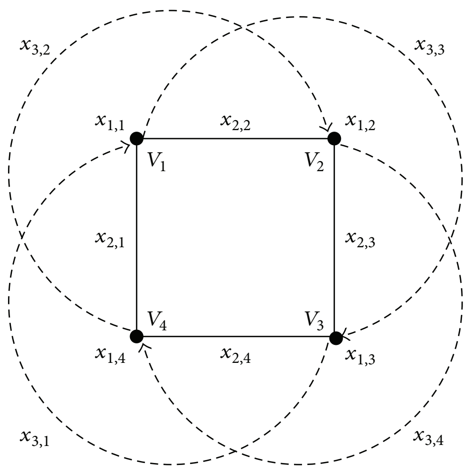


Figure 2: Weighted $C(4)$.

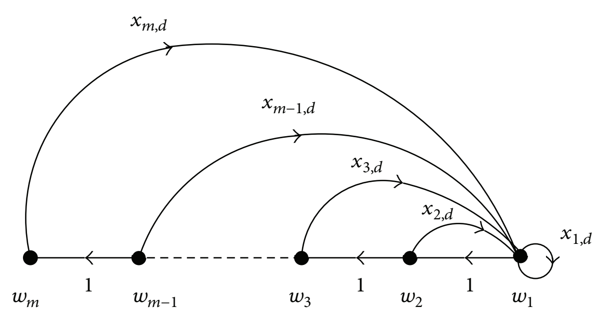


Figure 3: Edge-weighted m-trellis $T(m)$.

6. Edge-Weighted m-Trellis $T(m)$

In this section we deal with the *edge-weighted m-trellis*, $T(m)$, shown in Figure 3, and give another method of generating $f^{(j)}_{m,n}$ and $\mathscr{C}_m(n)$.

The vertices of $T(m)$ are labelled $\{w_1, w_2, \ldots, w_m\}$. All edges in $T(m)$ are *directed*, with arrows as shown. All circuits in $T(m)$ are directed and are traversed in the direction of the arrows. We use S to denote a directed circuit in $T(m)$, which we simply call a circuit. A circuit is *based* at vertex w_j if it begins and ends at vertex w_j. A circuit may pass through the same vertex more than once. The length of a circuit S is the number of edges in it.

The weights on the edges of $T(m)$ are taken from $\{1, x_{1,d}, \ldots, x_{m,d}\}$ where $d \geq 1$, as shown. The *weight* of circuit S, $w(S)$, is the product of the weights of all the edges in S. If the edge with weight $x_{j,d}$ is traversed as the kth edge in S, then $x_{j,k}$ is a factor in $w(S)$; thus the meaning of $x_{j,d}$ here is different from that in Sections 2 and 4. We allow empty circuits with length 0.

Definition 20. Let $\mathscr{T}_m(w_j, 0) = 1$ and, for $s \geq 1$, let $\mathscr{T}_m(w_j, s)$ be the sum of the weights of all circuits in $T(m)$ that are based at vertex w_j with length s.

Notation. We use standard multiset notation: $1^k = \underbrace{1 \cdot 1 \cdots 1 \cdot 1}_{k}$, and 1^0 means no occurrences of 1.

Theorem 21. *For $s \geq 0$,*

$$\mathscr{T}_m(w_1, s) = \mathscr{P}_m(0, s). \tag{48}$$

Proof. By strong induction on s. Now $\mathscr{T}_m(w_1, 0) = \mathscr{P}_m(0,0) = 1$; hence (48) is true for $s = 0$. We now assume that $\mathscr{T}_m(w_1, s') = \mathscr{P}_m(0, s')$ for all $0 \leq s' \leq s$. Consider any term in $\mathscr{T}_m(w_1, s+1)$ is the weight of some circuit S in $T(m)$ based at vertex w_1 with length $s+1$. Clearly S ends with a k-cycle based at vertex w_1, for some k with $1 \leq k \leq m$. Thus the last edge of S is (w_k, w_1), with weight $x_{k,s+1}$, and the previous $k-1$ edges are $(w_k, w_{k-1}), (w_{k-1}, w_{k-2}), \ldots, (w_2, w_1)$, each of weight 1. Hence $\omega(S) = \mathscr{T}_m(w_1, s+1-k) 1^{k-1} x_{k,s+1}$. Thus

$$\begin{aligned} \mathscr{T}_m(w_1, s+1) &= \sum_{k=1}^{m} x_{k,s+1} \mathscr{T}_m(w_1, s+1-k) \\ &= \sum_{k=1}^{m} x_{k,s+1} \mathscr{P}_m(0, s+1-k) \\ &= \mathscr{P}_m(0, s+1), \end{aligned} \tag{49}$$

using the strong induction hypothesis and then Theorem 8. So the induction goes through and (48) is true for all $s \geq 0$. □

Let $\mathscr{T}^{+c}_m(w_1, s)$ be the expression obtained when every indeterminate $x_{a,b}$ in $\mathscr{T}_m(w_1, s)$ is replaced by $x_{a,b+c}$; similarly for other expressions.

Recall that $[u_m, \ldots, u_{m-1+s}]$ is a subpath of $P(m-1, n)$ for $s \geq 0$; for $s = 0$ the path $[u_m, u_{m-1}]$ is the empty path $P(0,0)$, and $\mathscr{P}_m(0,0) = 1$.

Corollary 22. *For $s \geq 0$ and $0 \leq c \leq n - s$,*

$$\mathscr{T}^{+c}_m(w_1, s) = \mathbb{P}_m[u_{m+c}, \ldots, u_{m-1+s+c}]. \tag{50}$$

Proof. For $s = 0$ we have $\mathscr{T}^{+c}_m(w_1, 0) = \mathbb{P}_m[u_{m+c}, u_{m-1+c}] = 1$. For $s \geq 1$ then $[u_m, \ldots, u_{m-1+s}]$ is a subpath of $P(m-1, n)$ so, for every $n \geq s$, we have $\mathscr{P}_m(0, s) = \mathbb{P}_m[u_m, \ldots, u_{m-1+s}]$. Now, from Theorem 21, $\mathscr{T}_m(w_1, s) = \mathscr{P}_m(0, s)$, so $\mathscr{T}^{+c}_m(w_1, s) = \mathscr{P}^{+c}_m(0, s) = \mathbb{P}_m[u_{m+c}, \ldots, u_{m-1+s+c}]$, as required. □

We now connect $\mathscr{T}_m(w_j, n)$ and the fundamental solutions of the $(m+1)$-term recurrence (2).

Theorem 23. *For $n \geq 0$,*

$$\mathscr{T}_m(w_j, n) = f^{(j)}_{m,n+1-j}. \tag{51}$$

Proof. Consider a circuit S in $T(m)$ based at vertex w_j with n edges. Then, for some $0 \leq k \leq m - j$, the first k edges in this circuit are $(w_j, w_{j+1}), (w_{j+1}, w_{j+2}), \ldots, (w_{j+k-1}, w_{j+k})$, followed by edge (w_{j+k}, w_1) ending at vertex w_1. These edges contribute $1^k x_{j+k,k+1}$ to $w(S)$. Now, starting at vertex w_1, the last $j - 1$ edges traversed in S are

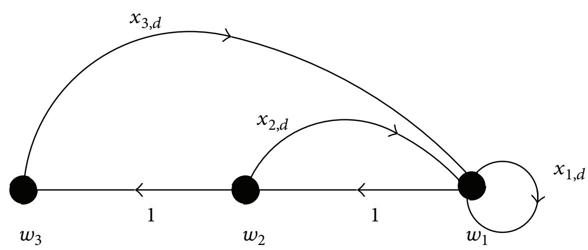


FIGURE 4: Edge-weighted 3-trellis $T(3)$.

$(w_1, w_2), (w_2, w_3), \dots, (w_{j-1}, w_j)$, contributing 1^{j-1} to $w(S)$. Hence $\omega(S) = x_{j+k,k+1}\mathcal{T}_m^{+(k+1)}(w_1, n-j-k)$.

Thus

$$\begin{aligned}\mathcal{T}_m\left(w_j, n\right) &= \sum_{k=0}^{m-j} x_{j+k,k+1}\mathcal{T}_m^{+(k+1)}\left(w_1, n-j-k\right)\\&= \sum_{\ell=j}^{m} x_{\ell,\ell+1-j}\mathcal{T}_m^{+(\ell+1-j)}\left(w_1, n-\ell\right)\\&= \sum_{\ell=j}^{m} x_{\ell,\ell+1-j}\mathbb{P}_m\left[u_{m+\ell+1-j}, \dots, u_{m+n-j}\right]\\&= f_{m,n+1-j}^{(j)},\end{aligned} \tag{52}$$

putting $\ell = j+k$ at the second line, then using Corollary 22 with $c = \ell+1-j$ and $s = n-\ell$ at the third line, and finally using Corollary 12 at the last line. □

Example 24. Consider $T(3)$, the edge-weighted 3-trellis; see Figure 4.

(a) $\mathcal{T}_3(w_2, 5)$ = sum of weights of circuits of $T(3)$ based at w_2 with length 5 $= x_{2,1}x_{1,2}x_{1,3}x_{1,4}1 + x_{2,1}x_{1,2}1x_{2,4}1 + x_{2,1}1 \cdot 1\, x_{3,4}1 + x_{2,1}1x_{2,3}x_{1,4}1 + 1\, x_{3,2}x_{1,3}x_{1,4}1 + 1\, x_{3,2}1x_{2,4}1 = f_{3,4}^{(2)}$, as in Example 10.

(b) $\mathcal{T}_3(w_3, 6)$ = sum of weights of circuits based at w_3 with length 6.

We observe that the first edge in such a circuit is edge (w_3, w_1) of weight $x_{3,1}$; hence $x_{3,1}$ is a factor of every term in $\mathcal{T}_3(w_3, 6) = f_{3,4}^{(3)}$, consistent with Example 10 again.

Finally, we bring the results from Lemma 13 and Theorems 16 and 23 together in Theorem 25.

Theorem 25. *For* $1 \le n \le m$,

$$\mathcal{C}_m(n) = \operatorname{tr}\left(\mathrm{Y}_{\mathrm{m,n}}\right) = \sum_{\mathrm{j}=1}^{\mathrm{m}} \mathcal{T}_{\mathrm{m}}\left(\mathrm{w_j}, \mathrm{n}\right). \tag{53}$$

Example 26. Again, from $T(3)$, we have $\mathcal{C}_3(4) = \sum_{j=1}^{3}\mathcal{T}_3(w_j, 4)$:

$$\begin{aligned}\mathcal{T}_3\left(w_1, 4\right) &= x_{1,1}x_{1,2}x_{1,3}x_{1,4} + x_{1,1}x_{1,2}1x_{2,4}\\&\quad + x_{1,1}1x_{2,3}x_{1,4} + x_{1,1}1 \cdot 1x_{3,4}\\&\quad + 1x_{2,2}x_{1,3}x_{1,4} + 1x_{2,2}1x_{2,4}\\&\quad + 1 \cdot 1x_{3,3}x_{1,4} = f_{3,4}^{(1)},\\\mathcal{T}_3\left(w_2, 4\right) &= x_{2,1}x_{1,2}x_{1,3}1 + x_{2,1}1\, x_{2,3}1 + 1x_{3,2}x_{1,3}1\\&= f_{3,3}^{(2)},\\\mathcal{T}_3\left(w_3, 4\right) &= x_{3,1}x_{1,2}1 \cdot 1 = f_{3,2}^{(3)},\end{aligned} \tag{54}$$

which are consistent with the above definitions and results and with Example 17.

7. Homogeneous Case, $x_{\ell,i} \rightarrow x_\ell$

In this section, we consider the case of constant coefficients, that is, where the indeterminates $x_{\ell,i}$ are independent of i.

Notation. We use $*$ to modify a path or expression or matrix in which weights or indeterminates $x_{\ell,i}$ are replaced with x_ℓ.

First we review some known properties of m-path polynomials using standard techniques. Then we show how our model recovers these results combinatorially.

7.1. Constant Coefficient Recurrences.

This subsection mainly establishes notation and recalls basic results of interest.

Consider the recurrence

$$y_n = \sum_{i=1}^{m} x_i y_{n-i}. \tag{55}$$

We begin with the first fundamental solution. The following is standard and readily derived via geometric series and multinomial expansion.

Proposition 27. *One has the generating function and formula*

$$\begin{aligned}\sum_{n\ge 0} h_n\, t^n &= \frac{1}{1-\sum_{i=1}^{m} x_i\, t^i}\\&= \sum_{n\ge 0}\sum_{\sum \ell s_\ell = n}\binom{s_1+s_2+\cdots+s_m}{s_1, s_2, \dots, s_m} x_1^{s_1}x_2^{s_2}\cdots x_m^{s_m}\, t^n\end{aligned} \tag{56}$$

giving the (first) fundamental solution, h_n, to the recurrence, that is, with initial values $h_i = 0$, $-(m-1) \le i < 0$, $h_0 = 1$.

The matrix X_m takes the form, confor Section 5.1,

$$X_m = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \ddots & 1\\ x_m & x_{m-1} & x_{m-2} & \cdots & x_1 \end{pmatrix} \tag{57}$$

so that $\det(I - tX_m) = 1 - \sum_{i=1}^{m} x_i t^i$. Define the $(r+1)$st fundamental solution to recurrence (55) to be the one with initial conditions

$$\begin{aligned} y_i &= 0, \quad \text{for } -(m-1) \le i \le 0,\ i \ne -r \\ y_{-r} &= 1, \end{aligned} \tag{58}$$

and denote this fundamental solution by $h_n^{(r+1)}$, with $h_n = h_n^{(1)}$. Then the entries in the bottom row of $(X_m)^n$ are exactly the values

$$\left((X_m)^n\right)_{(m,j)} = h_n^{(m-j+1)}. \tag{59}$$

In general,

$$\left((X_m)^n\right)_{(i,j)} = h_{n-m+i}^{(m-j+1)}. \tag{60}$$

The fundamental solutions for $r > 0$ can be expressed in terms of the first fundamental solution as follows.

Proposition 28. *The $(r+1)$st fundamental solution to the recurrence (55) is given by*

$$h_n^{(r+1)} = h_{n+r} - \sum_{k=0}^{r-1} h_{n+k} x_{r-k}, \tag{61}$$

where h_n denotes the first fundamental solution.

Proof. We will illustrate for $r \le 2$ that shows how the general case works. We have

$$\begin{aligned} h_n^{(1)} &= h_n, \\ h_n^{(2)} &= h_{n+1} - x_1 h_n, \\ h_n^{(3)} &= h_{n+2} - x_1 h_{n+1} - x_2 h_n. \end{aligned} \tag{62}$$

For $r = 1$, we obtain 0 for nonpositive n, except for $n = -1$, as required. Similarly, for $r = 2$, for nonpositive n we obtain 1 precisely for $n = -2$; otherwise we get 0. Note that the subtractions are necessary to cancel off terms when $0 \ge n > -r$. Since the coefficients are independent of n, these are indeed solutions to the recurrence. Thus the result is obtained. □

Now for the trace, we have the following.

Proposition 29. *The trace of $(X_m)^n$ is given by*

$$\operatorname{tr}(X_m)^n = \sum_{j=1}^{m} j h_{n-j} x_j. \tag{63}$$

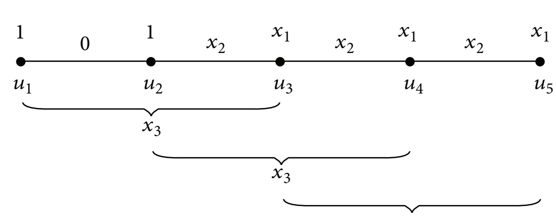
Proof. From (60), we have, using Proposition 28,

$$\begin{aligned} \operatorname{tr}(X_m)^n &= \sum_{i=1}^{m} h_{n-m+i}^{(m-i+1)} = \sum_{i=0}^{m-1} h_{n-i}^{(i+1)} \\ &= \sum_{i=0}^{m-1} \left[h_n - \sum_{k=0}^{i-1} h_{n-i+k} x_{i-k} \right] \\ &= \sum_{i=0}^{m-1} \left[h_n - \sum_{j=1}^{i} h_{n-j} x_j \right] \\ &= m h_n - \sum_{i=0}^{m-1} \sum_{j=1}^{i} h_{n-j} x_j \\ &\qquad (\text{next, interchanging the order of summation}) \\ &= m h_n - \sum_{j=1}^{m-1} \sum_{i=j}^{m-1} h_{n-j} x_j \\ &= m h_n - \sum_{j=1}^{m-1} (m-j) h_{n-j} x_j \\ &= m \left[h_n - \sum_{j=1}^{m-1} h_{n-j} x_j \right] + \sum_{j=1}^{m-1} j h_{n-j} x_j \\ &= \sum_{j=1}^{m} j h_{n-j} x_j \quad (\text{by the recurrence for } \{h_n\}). \end{aligned} \tag{64}$$

□

Remark 30. These are a variation on Newton's Identities relating power sum symmetric functions and elementary symmetric functions. Here, the homogeneous symmetric functions, h_n, play a role as well.

7.2. Combinatorial Proofs. We now show how these formulas may be derived combinatorially by our model with the specialization $x_{\ell,i} \to x_\ell$. The weighted path $P^*(2,3)$ looks like



(65)

Notation. Consistent with the above, we use h or $\mathscr{H}$ to represent expressions in which we have replaced $x_{\ell,i}$ with x_ℓ. Thus $\mathscr{H}_m(r,n) = \mathscr{P}_m^*(r,n)$, for $0 \le r \le m-1$; see Definition 6 of weighted path $P(r,n)$.

7.2.1. First Fundamental Solution. Proposition 27 is readily seen from the weighting of path $P^*(m-1,n)$. For the first fundamental solution, there are no vertices with weight 1, and no edges weighted 0. The first vertex has weight x_1, and so on. In an m-path cover the exponent s_ℓ is the number of paths of length ℓ, for each $1 \le \ell \le m$, and the multinomial coefficient counts the number of m-path covers obtained from any fixed set of m-paths. So this model gives a visual interpretation to the analytic formula.

7.2.2. Higher Fundamental Solutions. Start with the following.

Lemma 31. *For a fixed r with $1 \le r \le m-1$ and any $n \ge 1$,*

$$\mathscr{H}_m(r,n) - \mathscr{H}_m(r-1,n) = \sum_{\ell=r+1}^{m} x_\ell \mathscr{H}_m(0, n+r-\ell). \quad (66)$$

Proof. For $r \ge 1$, consider the weighted path $P^*(r,n)$. The first vertex u_1 must lie in every m-path cover of this path, say on a path Q_ℓ of length ℓ for $1 \le \ell \le m$, starting at u_1. If $\ell = 1$ then $\omega(Q_1) = \omega(u_1) = 1$, and the sum of all such m-path covers is thus $1 \cdot \mathscr{H}_m(r-1,n)$. If $2 \le \ell \le r$ then Q_ℓ finishes at vertex u_ℓ where $\omega(u_\ell) = 1$, so $\omega(Q_\ell) = 0$. And if $r+1 \le \ell \le m$ then Q_ℓ finishes at vertex u_ℓ where $\omega(u_\ell) = x_1$ and so $\omega(Q_\ell) = x_\ell$, and the sum of all such m-path covers is $x_\ell \mathscr{H}_m(0, n+r-\ell)$. Hence $\mathscr{H}_m(r,n) = \mathscr{H}_m(r-1,n) + \sum_{\ell=r+1}^{m} x_\ell \mathscr{H}_m(0,n+r-\ell)$, and so the result is obtained. □

Now for a combinatorial proof of Proposition 28.

Theorem 32. *For the fundamental solutions to the recurrence for the homogeneous path polynomials, one has*

$$h_n^{(r+1)} = h_{n+r} - \sum_{\ell=1}^{r} x_\ell h_{n+r-\ell}. \quad (67)$$

Proof. By our definitions and Corollary 11(ii) we have $h_n = f_{m,n}^{(1)*} = \mathscr{P}_m^*(0,n) = \mathscr{H}_m(0,n)$. And, from Lemmas 9 and 31, we have

$$h_n^{(r+1)} = \mathscr{H}_m(r,n) - \mathscr{H}_m(r-1,n) = \sum_{\ell=r+1}^{m} x_\ell h_{n+r-\ell}. \quad (68)$$

Now

$$\begin{aligned} h_{n+r} &= \mathscr{H}_m(0, n+r) \\ &= \sum_{\ell=1}^{m} x_\ell h_{n+r-\ell} \\ &= \sum_{\ell=1}^{r} x_\ell h_{n+r-\ell} + \sum_{\ell=r+1}^{m} x_\ell h_{n+r-\ell} \\ &= \sum_{\ell=1}^{r} x_\ell h_{n+r-\ell} + h_n^{(r+1)}, \end{aligned} \quad (69)$$

where, at the second line, we note that in every m-path cover of the weighted path $P^*(0,n+r)$ vertex u_{n+r} must lie on a path Q_ℓ of length ℓ and weight x_ℓ where $1 \le \ell \le m$, and at the last line we use (68). This gives the result. □

7.2.3. Trace Formula. We now give a combinatorial derivation of the trace formula in Proposition 29.

First let $\mathscr{T}_m(n)$ be the sum of the weights of all circuits of length n in $T^*(m)$ and the m-trellis with edge-weights $x_{\ell,i}$ replaced by x_ℓ; that is, $\mathscr{T}_m(n) = \sum_{j=1}^{m} \mathscr{T}_m^*(w_j, n)$; see Section 6.

Theorem 33. *For any $n \ge 1$,*

$$\operatorname{tr}(X_m)^n = \sum_{j=1}^{m} j x_j h_{n-j}. \quad (70)$$

Proof. We recall that the indeterminates in any term of $\mathscr{T}_m(n)$ are initially ordered according to the edges traversed in the corresponding circuit; see Example 26. Let $\mathscr{X} = x_j x_{\ell_1} x_{\ell_2} \cdots x_{\ell_r}$ be a typical ordered term in $\mathscr{T}_m(n)$ with all 1's removed and with first indeterminate x_j. We first show that term $\mathscr{X}$ occurs j times in $\mathscr{T}_m(n)$.

When there are two successive indeterminates x_ℓ and $x_{\ell'}$ in $\mathscr{X}$, then, in the corresponding circuit, the edges traversed are first (w_ℓ, w_1) of weight x_ℓ, followed by the $\ell'-1$ edges $(w_1,w_2),(w_2,w_3),\ldots,(w_{\ell'-1},w_{\ell'})$ each of weight 1, and then finishing with the edge $(w_{\ell'},w_1)$ of weight $x_{\ell'}$. Hence pair $x_\ell x_{\ell'}$ becomes $x_\ell 1^{\ell'-1} x_{\ell'}$ when the indeterminates are considered as weights on edges in a circuit in $T^*(m)$.

Now, because the first indeterminate in $\mathscr{X}$ is x_j, any circuit corresponding to $\mathscr{X}$ must be based at vertex $w_{j'}$ for some $j' \in \{1,2,\ldots,j\}$. Hence $\mathscr{X}$ will appear in $\mathscr{T}_m(n)$ as

$$1^{j-j'} x_j 1^{\ell_1-1} x_{\ell_1} 1^{\ell_2-1} x_{\ell_2} 1^{\ell_3-1} \cdots 1^{\ell_r-1} x_{\ell_r} 1^{j'-1}, \quad (71)$$

for each $j' \in \{1,2,\ldots,j\}$ in $\mathscr{T}_m(n)$. There are j such j', so there are j occurrences of term $\mathscr{X}$ in $\mathscr{T}_m(n)$.

Now consider an occurrence of $\mathscr{X}$ in which $j' = j$, namely,

$$x_j 1^{\ell_1-1} x_{\ell_1} 1^{\ell_2-1} x_{\ell_2} 1^{\ell_3-1} \cdots 1^{\ell_r-1} x_{\ell_r} 1^{j-1}. \quad (72)$$

Let

$$\frac{\mathscr{X}}{x_j 1^{j-1}} = 1^{\ell_1-1} x_{\ell_1} 1^{\ell_2-1} x_{\ell_2} 1^{\ell_3-1} \cdots 1^{\ell_r-1} x_{\ell_r} = \mathscr{Z}. \quad (73)$$

Then the sequence of edges traversed in $T^*(m)$ corresponding to $\mathscr{Z}$ begins at w_1 and ends at w_1, and so is a circuit based at w_1, with length $n-1-(j-1) = n-j$. Thus $\mathscr{Z} \in \mathscr{T}_m^*(w_1, n-j)$. Conversely given any $\mathscr{Z} \in \mathscr{T}_m^*(w_1, n-j)$ then $x_j \mathscr{Z}\, 1^{j-1}$ is an occurrence of term $\mathscr{X}$ starting with 1^0 and ending with 1^{j-1}. Thus $(\sum_{j'=j} \mathscr{X})/x_j = \mathscr{T}_m^*(w_1, n-j)$ and $\sum_{j'=j} \mathscr{X} = x_j \mathscr{T}_m^*(w_1, n-j)$.

Now we can partition the weighted circuits of $T^*(m)$ of length n by their first indeterminate x_j (ignoring the edges of weight 1 preceding this first indeterminate). That is, we can partition the terms of $\mathscr{T}_m(n)$ by their first indeterminate x_j. So, using the above arguments, we have

$$\mathscr{T}_m(n) = \sum_{j=1}^{m} j x_j \mathscr{T}_m^*(w_1, n-j). \quad (74)$$

Furthermore, $\mathcal{T}^*_m(w_1, n-j) = \mathcal{P}^*_m(0, n-j) = f^{(1)*}_{m,n-j} = h_{n-j}$; the first equality is Theorem 21, the second is Corollary 11(ii), and the third is by definition of h_n. So finally,

$$\operatorname{tr}\left(X^n_m\right) = \mathcal{T}_m(n) = \sum_{j=1}^{m} jx_j \mathcal{T}^*_m(w_1, n-j) = \sum_{j=1}^{m} jx_j h_{n-j}. \tag{75}$$

□

Example 34. See Examples 17 and 26. Here $m = 3$ and $n = 4$:

$$\begin{aligned}\operatorname{tr}\left(X^4_3\right) &= \mathcal{T}_3(4)\\ &= x_1^4 + 4x_1^2x_2 + 4x_1x_3 + 2x_2^2\\ &= x_1\left(x_1^3 + 2x_1x_2 + x_3\right) + 2x_2\left(x_1^2 + x_2\right) + 3x_3\left(x_1\right)\\ &= x_1\left(f^{(1)*}_{3,3}\right) + 2x_2\left(f^{(1)*}_{3,2}\right) + 3x_3\left(f^{(1)*}_{3,1}\right)\\ &= x_1h_3 + 2x_2h_2 + 3x_3h_1,\end{aligned} \tag{76}$$

where, at line 2, we have rearranged the terms according to their first indeterminate x_j, using Example 26, and combined like terms.

Remark 35. From Theorem 25 and our definitions of matrices $Y_{m,n}$ and X_m from Sections 3 and 5.1, respectively, we have the following equalities:

$$\begin{gathered}\mathcal{C}^*_m(n) = \operatorname{tr}\left(Y^*_{m,n}\right) = \sum_{j=1}^{m}\mathcal{T}^*_m\left(w_j, n\right),\\ \operatorname{tr}\left(Y^*_{m,n}\right) = \operatorname{tr}\left(X^n_m\right).\end{gathered} \tag{77}$$

Thus, from Theorem 33,

$$\sum_{j=1}^{m}\mathcal{T}^*_m\left(w_j, n\right) = \sum_{j=1}^{m} jx_j\mathcal{T}^*_m\left(w_1, n-j\right). \tag{78}$$

8. Sequences, $x_{\ell,i} \to 1$

In Section 7 we specialized by replacing weights $x_{\ell,i}$ with x_ℓ. In this section we specialize further by replacing all weights $x_{\ell,i}$ with 1. We denote this operation by #. We then use these # matrices to count m-path covers of the path and cycle.

Recall matrix $X_{m,n}$ from (26), we define matrix Z_m:

$$Z_m = X^{\#}_{m,n} = \begin{pmatrix} 0 & 1 & \cdot & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ 1 & 1 & 1 & \cdots & 1\end{pmatrix}. \tag{79}$$

Similarly, let $c_m(n) = \mathcal{C}^{\#}_m(n)$ be the expression $\mathcal{C}_m(n)$ evaluated when all $x_{\ell,i} = 1$. So $Y^{\#}_{m,n} = Z^n_m$ and $c_m(n) = \operatorname{tr}(Y^{\#}_{m,n}) = \operatorname{tr}(Z^n_m)$. Thus $c_m(n)$ counts the number of m-path covers of the weighted $C(n)$ or of an arbitrary n-cycle. (cf., Corollary 11.1, Section 8, Farrell [2].)

Theorem 36. *For $1 \le n \le m$, one has $c_m(n) = 2^n - 1$.*

Proof. Let $[n] = \{1, 2, \ldots, n\}$ and let $C[n]$ denote the cycle whose vertices are the elements of $[n]$ arranged clockwise in a circle. Now $n \le m$ so any path cover of $C[n]$ will be an m-path cover. We show that the number of path covers of $C[n]$ is $2^n - 1$.

Given a subset $\{i_1, i_2, \ldots, i_k\}$ of $[n]$ with $\{i_1 < i_2 < \cdots < i_k\}$ we define a path cover $[i_1, i_1+1, \ldots, i_2-1], [i_2, i_2+1, \ldots, i_3-1], \ldots, [i_k, i_k+1, \ldots, i_1-1]$, of $C[n]$. Conversely, given a path cover $[i_1, i_1+1, \ldots, i_2-1], [i_2, i_2+1, \ldots, i_3-1], \ldots, [i_k, i_k+1, \ldots, i_1-1]$ of $C[n]$ we take the first vertex from each path to form a subset $\{i_1, i_2, \ldots, i_k\}$ of $[n]$ and then rearrange its elements to form a subset of $[n]$ with increasing elements. These two operations illustrate a bijection from the set of non-empty subsets of $[n]$ to the set of m-path covers of $C[n]$. Hence $c_m(n) = 2^n - 1$. □

From recurrence (2), Lemma 13, and Theorems 16 and 36, for $n \ge m+1$ we see that $c_m(n)$ obeys the m-anacci recurrence,

$$\begin{aligned}c_m(n) &= c_m(n-1) + c_m(n-2) + \cdots + c_m(n-m)\\ &= \sum_{\ell=1}^{m} c_m(n-\ell),\end{aligned} \tag{80}$$

with starting conditions $c_m(n) = 2^n - 1$ for $1 \le n \le m$.

In the square array (see Table 1) $c_m(n)$ is the (n, m) entry, for $n, m \ge 1$. Column m is determined by the above m-anacci recurrence. We observe that the (m, m) main diagonal entry is $c_m(m) = 2^m - 1$.

Consider the triangle, in bold, where $c_m(n)$ is the (n, m) entry for all $n \ge 1$ and $1 \le m \le n$; it counts the number of m-path covers of a cycle with n vertices. We have entered the sequence obtained from reading this triangle row-by-row to the Online Encyclopedia of Integer Sequences [9]; it is sequence A185722.

Each of the 10 columns of the square array (see Table 1) appears as a sequence in [9]; for example, the second column ($m = 2$) gives sequence A000204 and the third column ($m = 3$) gives A001644. Thus we have a new combinatorial interpretation for each of these sequences and a connection between them.

A closely related sequence is A126198 (replace "k" by "m" in its description). Let $T(n, m)$ be the (n, m) entry of the triangle corresponding to A126198, then $T(n, m)$ counts the number of compositions of integer n into parts of size $\le m$. Now consider n vertices arranged in a path. A composition of n into parts of size $\le m$ corresponds naturally to an m-path cover of this path with n vertices by identifying a part of size ℓ in the composition with a path of length ℓ in the corresponding m-path cover. This correspondence can also be reversed. Thus in our terminology, $T(n, m)$ is the number of m-path covers of a path with n vertices, and, from Corollary 11(ii) and our operation #, we have $T(n, m) = f^{(1)\#}_{m,n} = \mathcal{P}^{\#}_m(0, n)$. The (m, m) main diagonal entry in this triangle is $T(m, m) = 2^{m-1}$ (as is well known, there are

Table 1

$n \setminus m$	1	2	3	4	5	6	7	8	9	10	$\cdots$
1	**1**	1	1	1	1	1	1	1	1	1	$\cdots$
2	**1**	**3**	3	3	3	3	3	3	3	3	$\cdots$
3	**1**	**4**	**7**	7	7	7	7	7	7	7	$\cdots$
4	**1**	**7**	**11**	**15**	15	15	15	15	15	15	$\cdots$
5	**1**	**11**	**21**	**26**	**31**	31	31	31	31	31	$\cdots$
6	**1**	**18**	**39**	**51**	**57**	**63**	63	63	63	63	$\cdots$
7	**1**	**29**	**71**	**99**	**113**	**120**	**127**	127	127	127	$\cdots$
8	**1**	**47**	**131**	**191**	**223**	**239**	**247**	**255**	255	255	$\cdots$
9	**1**	**76**	**241**	**367**	**439**	**475**	**493**	**502**	**511**	511	$\cdots$
10	**1**	**123**	**443**	**708**	**863**	**943**	**983**	**1003**	**1013**	**1023**	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

2^{m-1} compositions of m) and column m of this triangle is determined by the m-anacci recurrence,

$$T(n,m) = T(n-1,m) + T(n-2,m) + \cdots + T(n-m,m) = \sum_{\ell=1}^{m} T(n-\ell,m), \tag{81}$$

for $n \geq m+1$, with starting conditions $T(n,m) = 2^{n-1}$ for $1 \leq n \leq m$.

The (n,m) entry in our triangle, $c_m(n)$, counts the number of m-path covers of a cycle with n vertices. We have starting conditions $c_m(n) = 2^n - 1$ as opposed to $T(n,m) = 2^{n-1}$ above, for $1 \leq n \leq m$.

Furthermore, from above and the definition of matrix $Y_{m,n}$ from (28), we have $T(n,m) = f_{m,n}^{(1)\#}$ = the (m,m) entry of matrix $Y_{m,n}^{\#} = Z_m^n$. Thus both

$$c_m(n) = \operatorname{tr}(Z_m^n), \qquad T(n,m) = (Z_m^n)_{(m,m)}, \tag{82}$$

can be obtained from matrix Z_m^n. This gives a new derivation of $T(n,m)$, and so of sequence A126198.

Example 37. $m = 3$ and $n = 4$. We have

$$Z_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad Z_3^4 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \\ 4 & 6 & 7 \end{pmatrix}, \tag{83}$$

gives

$$c_3(4) = \operatorname{tr}(Z_3^4) = 11, \qquad T(4,3) = (Z_3^4)_{(3,3)} = 7; \tag{84}$$

see Examples 17, 26, and 7.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of the paper.

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References

[1] E. J. Farrell, "On a general class of graph polynomials," *Journal of Combinatorial Theory B*, vol. 26, no. 1, pp. 111–122, 1979.

[2] E. J. Farrell, "Path decompositions of chains and circuits," *International Journal of Mathematics and Mathematical Sciences*, vol. 6, no. 3, pp. 521–533, 1983.

[3] T. Y. Chow, "The path-cycle symmetric function of a digraph," *Advances in Mathematics*, vol. 118, no. 1, pp. 71–98, 1996.

[4] O. M. D'Antona and E. Munarini, "The cycle-path indicator polynomial of a digraph," *Advances in Applied Mathematics*, vol. 25, no. 1, pp. 41–56, 2000.

[5] J. P. McSorley, P. Feinsilver, and R. Schott, "Generating orthogonal polynomials and their derivatives using vertex|matching-partitions of graphs," *Ars Combinatoria*, vol. 87, pp. 75–95, 2008.

[6] A. Joyal, "Une théorie combinatoire des séries formelles," *Advances in Mathematics*, vol. 42, no. 1, pp. 1–82, 1981.

[7] F. Bergeron, G. Labelle, and P. Leroux, *Combinatorial Species and Tree-Like Structures*, vol. 67 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, Mass, USA, 1998.

[8] E. Munarini, "A combinatorial interpretation of the generalized Fibonacci numbers," *Advances in Applied Mathematics*, vol. 19, no. 3, pp. 306–318, 1997.

[9] Online Encyclopedia of Integer Sequences, published electronically, http://oeis.org.