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Research Article

Fundamental Solutions to Time-Fractional Advection Diffusion Equation in a Case of Two Space Variables

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The fundamental solutions to time-fractional advection diffusion equation in a plane and a half-plane are obtained using the Laplace integral transform with respect to time t and the Fourier transforms with respect to the space coordinates x and y. The Cauchy, source, and Dirichlet problems are investigated. The solutions are expressed in terms of integrals of Bessel functions combined with Mittag-Leffler functions. Numerical results are illustrated graphically.

1. Introduction

The classical advection diffusion equation

$$\frac{\partial c}{\partial t} = a\Delta c - \mathbf{v} \cdot \nabla c, \tag{1}$$

where a is the diffusivity coefficient, $\mathbf{v}$ is the velocity vector, has several physical interpretations in terms of Brownian motion, diffusion or heat transport with external force or with additional velocity field, diffusion of charge in the electrical field on comb structure, transport processes in porous media, groundwater hydrology, and so forth [1–7].

In the case of one spatial coordinate x, (1) has the following form:

$$\frac{\partial c}{\partial t} = a\frac{\partial^2 c}{\partial x^2} - v\frac{\partial c}{\partial x}. \tag{2}$$

Investigation of different physical phenomena in media with complex internal structure has led to considering differential equations with derivatives of fractional order. The space-fractional [8–19], time-fractional [20–31], and space-time-fractional [32–39] generalizations of the advection diffusion equation were studied by many authors. In the majority of the abovementioned papers, the fractional generalizations of one-dimensional equation (2) were considered. In the papers dealing with space-fractional or space-time-fractional equations, one term with space derivative was substituted by the corresponding term with the fractional derivative [8, 9, 11–19, 33, 39] or both terms with space derivatives had fractional order [32, 35–38]. Several numerical schemes were proposed: the implicit difference method based on the shifted Grünwald-Letnikov approximation [14, 37], the explicit difference scheme [37], transformation of fractional differential equation into a system of ordinary differential equations and using the method of lines [15], the random walk algorithms [16, 17], the spectral regularization method [28], the Crank-Nicholson difference scheme [29], Adomian's decomposition [26], a spatial and temporal discretization [30, 39], the fractional variational iteration method [31], and the homotopy perturbation method [27, 38].

In [24, 25], the analytical solution to one-dimensional time-fractional advection diffusion equation was obtained in terms of integrals of the H-function.

In this paper, we study the fundamental solutions to time-fractional advection diffusion equation

$$\frac{\partial^\alpha c}{\partial t^\alpha} = a\,\Delta c - \mathbf{v} \cdot \nabla c \tag{3}$$

in a plane and a half-plane. The Laplace transform with respect to time and the Fourier transform with respect to the space coordinates are used. The Cauchy and the source problems in a plane and the Dirichlet problem for a half-plane are solved. The analytical solutions are expressed in terms of integrals of the Mittag-Leffler functions. Numerical results are illustrated graphically.

In (3) we use the Caputo fractional derivative [40–42]:

$$\frac{d^{\alpha}c(t)}{dt^{\alpha}} = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t (t-\tau)^{n-\alpha-1}\frac{d^n c(\tau)}{d\tau^n}d\tau, & \\ \qquad n-1<\alpha<n, & \\ \dfrac{d^n c(t)}{dt^n}, \qquad \alpha = n, & \end{cases} \tag{4}$$

where $\Gamma(\alpha)$ is the gamma function. For its Laplace transform rule, the Caputo fractional derivative requires the knowledge of the initial values of the function $c(t)$ and its integer derivatives of order $k = 1, 2, \ldots, n-1$:

$$\mathscr{L}\left\{\frac{d^{\alpha}c(t)}{dt^{\alpha}}\right\} = s^{\alpha}\mathscr{L}\{c(t)\} - \sum_{k=0}^{n-1} c^{(k)}(0^+)\, s^{\alpha-1-k}, \qquad n-1<\alpha<n, \tag{5}$$

where s is the transform variable.

2. The Fundamental Solution to the Cauchy Problem

Consider the time-fractional advection diffusion equation

$$\frac{\partial^{\alpha}c}{\partial t^{\alpha}} = a\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\right) - v\frac{\partial c}{\partial x} - v\frac{\partial c}{\partial y}, \quad -\infty < x < \infty,\ -\infty < y < \infty, \quad 0 < t < \infty,\ 0 < \alpha \le 1, \tag{6}$$

under initial condition

$$t = 0: \quad c = p_0\delta(x)\delta(y). \tag{7}$$

In (7) we have introduced the constant multiplier p_0 to obtain the nondimensional quantity $\bar{c}$ (see (23)) displayed in Figures.

The zero conditions at infinity are also imposed:

$$\lim_{x\to\pm\infty} c(x,y,t) = 0, \qquad \lim_{y\to\pm\infty} c(x,y,t) = 0. \tag{8}$$

Introducing the new sought function

$$c(x,y,t) = \exp\left[\frac{v(x+y)}{2a}\right]u(x,y,t) \tag{9}$$

and taking into account that for the Dirac delta function, $f(x)\delta(x) = f(0)\delta(x)$, the initial-value problem (6)–(8) is reduced to the following ones:

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = a\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{v^2}{2a}u, \tag{10}$$

$$t = 0: \quad u = p_0\delta(x)\delta(y), \tag{11}$$

$$\lim_{x\to\pm\infty} u(x,y,t) = 0, \qquad \lim_{y\to\pm\infty} u(x,y,t) = 0. \tag{12}$$

Next, we use the Laplace transform with respect to time t (designated by the asterisk) and the double exponential Fourier transform with respect to the space coordinates x and y (marked by the tilde). In the transform domain, we get

$$\tilde{\tilde{u}}^{*} = \frac{p_0}{2\pi}\frac{s^{\alpha-1}}{s^{\alpha} + a(\xi^2+\eta^2) + v^2/2a}. \tag{13}$$

Here, s is the Laplace transform variable and ξ and η are the Fourier transform variables.

Inversion of the integral transforms gives

$$u(x,y,t) = \frac{p_0}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_{\alpha}\left\{-\left[a\left(\xi^2+\eta^2\right) + \frac{v^2}{2a}\right]t^{\alpha}\right\} \times \cos(x\xi)\cos(y\eta)\, d\xi\, d\eta, \tag{14}$$

where the formula [40–42]

$$\mathscr{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^{\alpha}+b}\right\} = E_{\alpha}(-bt^{\alpha}) \tag{15}$$

has been used with $E_{\alpha}(z)$ being the Mittag-Leffler function in one parameter α:

$$E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \quad \alpha > 0,\ z \in C. \tag{16}$$

Solution (14) is not convenient for numerical calculations. To obtain the solution amenable to numerical treatment, we introduce the polar coordinates in the (ξ, η)-plane:

$$\xi = \rho\cos\theta, \qquad \eta = \rho\sin\theta. \tag{17}$$

Hence,

$$u(x,y,t) = \frac{p_0}{4\pi^2}\int_0^{\infty}\int_0^{2\pi} E_{\alpha}\left[-\left(a\rho^2 + \frac{v^2}{2a}\right)t^{\alpha}\right] \times \cos(x\rho\cos\theta)\cos(y\rho\sin\theta)\,\rho\, d\rho\, d\theta. \tag{18}$$

Due to periodic properties of the integrand

$$\int_0^{2\pi}\cos(x\rho\cos\theta)\cos(y\rho\sin\theta)\,d\theta = 4\int_0^{\pi/2}\cos(x\rho\cos\theta)\cos(y\rho\sin\theta)\,d\theta. \tag{19}$$

Changing variable $w = \sin\theta$ and taking into account the following integral [43]:

$$\int_0^1 \frac{\cos\left(p\sqrt{1-x^2}\right)}{\sqrt{1-x^2}}\cos(qx)\,dx = \frac{\pi}{2}J_0\left(\sqrt{p^2+q^2}\right), \tag{20}$$

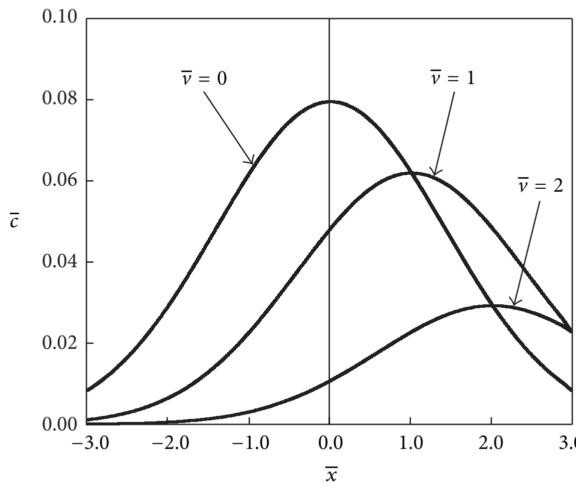

FIGURE 1: Dependence of the fundamental solution to the Cauchy problem on distance (the classical advection diffusion equation, $\alpha = 1$).

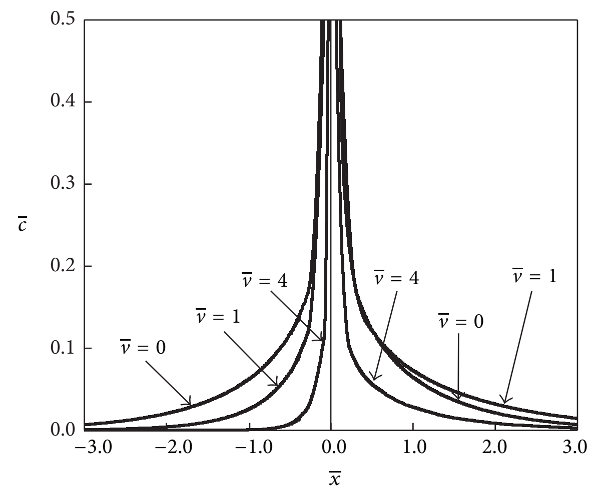

FIGURE 2: Dependence of the fundamental solution to the Cauchy problem on distance (the time-fractional advection diffusion equation, $\alpha = 0.5$).

where $J_n(z)$ is the Bessel function of the order n, we arrive at

$$u(x, y, t) = \frac{p_0}{2\pi} \int_0^\infty E_\alpha \left[-\left(a\rho^2 + \frac{v^2}{2a} \right) t^\alpha \right] \times J_0\left(\sqrt{x^2 + y^2}\rho\right) \rho \, d\rho \tag{21}$$

and, returning to the quantity $c(x, y, t)$ according to (9), we get

$$c(x, y, t) = \frac{p_0}{2\pi} \exp\left[\frac{v(x + y)}{2a} \right] \times \int_0^\infty E_\alpha \left[-\left(a\rho^2 + \frac{v^2}{2a} \right) t^\alpha \right] \times J_0\left(\sqrt{x^2 + y^2}\rho\right) \rho \, d\rho. \tag{22}$$

The particular case of solution (22) corresponding to the time-fractional diffusion equation ($v = 0$) was considered in [44, 45].

The results of numerical computations for $y = 0$ are presented in Figure 1 for $\alpha = 1$ and in Figure 2 for $\alpha = 0.5$.

The following nondimensional quantities:

$$\bar{c} = \frac{at^\alpha}{p_0} c, \qquad \bar{v} = \frac{t^{\alpha/2}}{\sqrt{a}} v \tag{23}$$

and the nondimensional coordinates (the similarity variables)

$$\bar{x} = \frac{x}{\sqrt{a} t^{\alpha/2}}, \qquad \bar{y} = \frac{y}{\sqrt{a} t^{\alpha/2}} \tag{24}$$

have been introduced.

To calculate the Mittag-Leffler function $E_\alpha(-x)$ in solution (22), we applied the algorithm suggested in [46].

3. The Fundamental Solution to the Source Problem

Consider the time-fractional advection diffusion equation with the source term

$$\begin{gathered} \frac{\partial^\alpha c}{\partial t^\alpha} = a\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} \right) - v\frac{\partial c}{\partial x} - v\frac{\partial c}{\partial y} \\ + q_0 \delta(x) \delta(y) \delta(t), \\ -\infty < x < \infty, \\ -\infty < y < \infty, \\ 0 < t < \infty, \; 0 < \alpha \le 1, \end{gathered} \tag{25}$$

under zero initial condition,

$$t = 0: \quad c = 0 \tag{26}$$

and conditions (8) at infinity.

The integral transform technique leads to

$$\tilde{\tilde{u}}^* = \frac{q_0}{2\pi} \frac{1}{s^\alpha + a(\xi^2 + \eta^2) + v^2/2a}, \tag{27}$$

$$c(x, y, t) = \frac{q_0 t^{\alpha-1}}{2\pi} \exp\left[\frac{v(x + y)}{2a} \right] \times \int_0^\infty E_{\alpha,\alpha} \left[-\left(a\rho^2 + \frac{v^2}{2a} \right) t^\alpha \right] \times J_0\left(\sqrt{x^2 + y^2}\rho\right) \rho \, d\rho. \tag{28}$$

Here, $E_{\alpha,\beta}(z)$ is the generalized Mittag-Leffler function in two parameters α and β:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \; \beta > 0, \; z \in C, \tag{29}$$

and the formula [40–42]

$$\mathscr{L}^{-1}\left\{\frac{s^{\alpha-\beta}}{s^{\alpha}+b}\right\}=t^{\beta-1}E_{\alpha,\beta}\left(-bt^{\alpha}\right) \tag{30}$$

for the inverse Laplace transform has been used.

The particular case of solution (28) corresponding to the time-fractional diffusion equation with $v=0$ was considered in [45, 47]. Solutions (22) and (28) coincide for $\alpha=1$.

The results of numerical computations for $y = 0$ are presented in Figure 3 for $\alpha=0.5$ with

$$\bar{c}=\frac{at}{q_0}c. \tag{31}$$

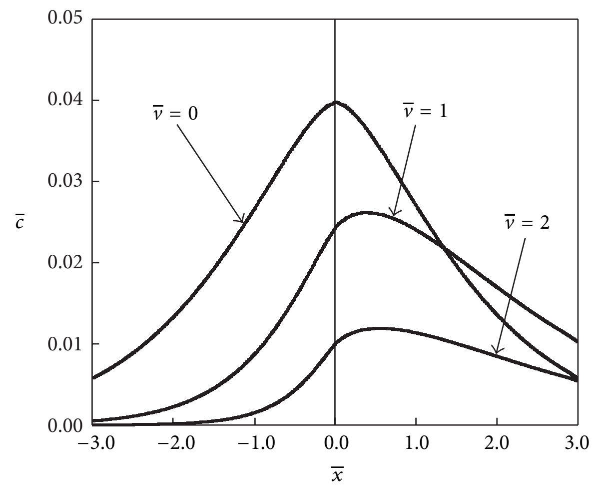

Figure 3: Dependence of the fundamental solution to the source problem on distance (the time-fractional advection diffusion equation, $\alpha=0.5$).

4. The Fundamental Solution to the Dirichlet Problem

In this case the time-fractional advection diffusion equation,

$$\frac{\partial^{\alpha}c}{\partial t^{\alpha}}=a\left(\frac{\partial^2 c}{\partial x^2}+\frac{\partial^2 c}{\partial y^2}\right)-v\frac{\partial c}{\partial x}-v\frac{\partial c}{\partial y},\quad 0<x<\infty,\ -\infty<y<\infty,\quad 0<t<\infty,\ 0<\alpha\le 1, \tag{32}$$

is considered under zero initial condition

$$t=0:\quad c=0 \tag{33}$$

and the Dirichlet boundary condition

$$x=0:\quad c=g_0\delta\left(y\right)\delta\left(t\right). \tag{34}$$

The zero conditions at infinity are imposed as follows:

$$\lim_{x\to\infty}c\left(x,y,t\right)=0,\qquad \lim_{y\to\pm\infty}c\left(x,y,t\right)=0. \tag{35}$$

As above, the new sought function u is introduced (see (9)), and, for (10) in the half-plane $x > 0$, the Laplace transform with respect to time t, the exponential Fourier transform with respect to the spatial coordinate y, and the sin-Fourier transform with respect to the spatial coordinate x are used. In the transform domain, we get

$$\tilde{\tilde{u}}^{*}=\frac{ag_0\xi}{\sqrt{2\pi}}\frac{1}{s^{\alpha}+a\left(\xi^2+\eta^2\right)+v^2/2a} \tag{36}$$

and, after inversion of the integral transforms,

$$u\left(x,y,t\right)=\frac{ag_0t^{\alpha-1}}{\pi^2}\int_{-\infty}^{\infty}\int_0^{\infty}E_{\alpha,\alpha}\left\{-\left[a\left(\xi^2+\eta^2\right)+\frac{v^2}{2a}\right]t^{\alpha}\right\}\times\sin\left(x\xi\right)\cos\left(y\eta\right)\xi\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{37}$$

Introducing the polar coordinates in the (ξ,η)-plane gives

$$u\left(x,y,t\right)=\frac{ag_0t^{\alpha-1}}{\pi^2}\int_0^{\infty}\int_0^{\pi}E_{\alpha}\left[-\left(a\rho^2+\frac{v^2}{2a}\right)t^{\alpha}\right]\times\sin\left(x\rho\cos\theta\right)\times\cos\left(y\rho\sin\theta\right)\rho^2\cos\theta\,\mathrm{d}\rho\,\mathrm{d}\theta. \tag{38}$$

Changing variables $w=\sin\theta$ and taking into account the following integral [43]:

$$\int_0^1\sin\left(p\sqrt{1-x^2}\right)\cos\left(qx\right)\mathrm{d}x=\frac{\pi}{2}\frac{p}{\sqrt{p^2+q^2}}J_1\left(\sqrt{p^2+q^2}\right), \tag{39}$$

we obtain

$$u\left(x,y,t\right)=\frac{ag_0t^{\alpha-1}x}{\pi\sqrt{x^2+y^2}}\int_0^{\infty}E_{\alpha,\alpha}\left[-\left(a\rho^2+\frac{v^2}{2a}\right)t^{\alpha}\right]\times J_1\left(\sqrt{x^2+y^2}\rho\right)\rho^2\mathrm{d}\rho, \tag{40}$$

$$c\left(x,y,t\right)=\frac{ag_0t^{\alpha-1}x}{\pi\sqrt{x^2+y^2}}\exp\left[\frac{v\left(x+y\right)}{2a}\right]\times\int_0^{\infty}E_{\alpha,\alpha}\left[-\left(a\rho^2+\frac{v^2}{2a}\right)t^{\alpha}\right]\times J_1\left(\sqrt{x^2+y^2}\rho\right)\rho^2\mathrm{d}\rho. \tag{41}$$

The particular case of solution (41) corresponding to the time-fractional diffusion equation ($v=0$) was considered in [48].

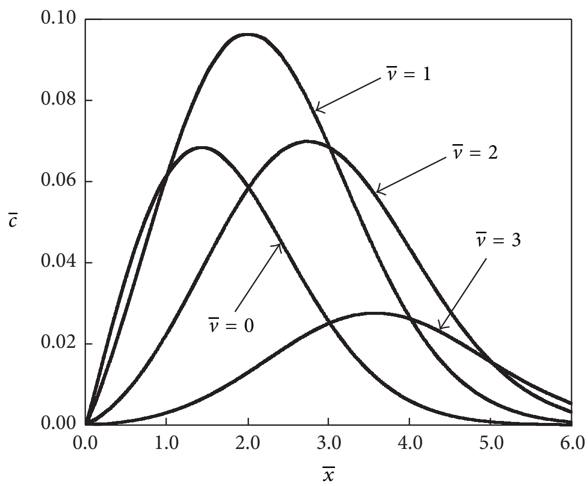

Figure 4: Dependence of the fundamental solution to the Dirichlet problem on distance (the classical advection diffusion equation, $\alpha = 1$).

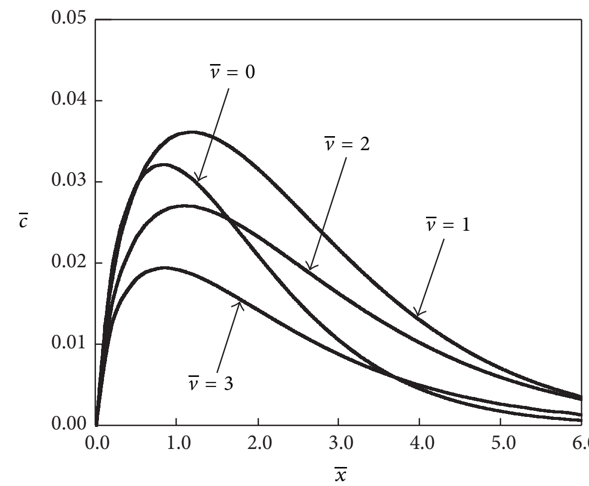

Figure 5: Dependence of the fundamental solution to the Dirichlet problem on distance (the time-fractional advection diffusion equation, $\alpha = 0.5$).

The results of numerical computations according to solution (41) for $y = 0$ are presented in Figure 4 for $\alpha = 1$ and in Figure 5 for $\alpha = 0.5$ with

$$\bar{c} = \frac{\sqrt{a}t^{1+\alpha/2}}{g_0}\, c. \tag{42}$$

Other nondimensional quantities are the same as in (23) and (24).

5. Conclusions

We have considered the time-fractional advection diffusion equation in a plane and in a half-plane. The fundamental solutions to the Cauchy problem and to the source problem in a plane have been obtained as well as to the Dirichlet problem in a half-plane. It should be emphasized that the fundamental solution to the Cauchy problem in the case $0 < \alpha < 1$ has the logarithmic singularity at the origin:

$$\begin{aligned} c(x, y, t) \sim & -\frac{p_0}{2\pi\Gamma(1-\alpha)\, at^{\alpha}}\; \exp\left[\frac{v(x+y)}{2a}\right] \\ & \times \ln\left(\sqrt{1+\frac{v^2t^{\alpha}}{2a}}\; \frac{\sqrt{x^2+y^2}}{\sqrt{a}t^{\alpha/2}}\right). \end{aligned} \tag{43}$$

This result is similar to the case of the time-fractional diffusion equation when $v = 0$ (see [44, 49]). Such a singularity disappears only for the classical advection diffusion equation ($\alpha = 1$). Due to singularity of the solution at the origin, in the case of $0 < \alpha < 1$, drift caused by the quantity v is less noticeable than in the case of $\alpha = 1$ (compare Figures 1 and 2).

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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