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Research Article

Stochastic $\mathscr{H}_\infty$ Finite-Time Control of Discrete-Time Systems with Packet Loss

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This paper investigates the stochastic finite-time stabilization and $\mathscr{H}_\infty$ control problem for one family of linear discrete-time systems over networks with packet loss, parametric uncertainties, and time-varying norm-bounded disturbance. Firstly, the dynamic model description studied is given, which, if the packet dropout is assumed to be a discrete-time homogenous Markov process, the class of discrete-time linear systems with packet loss can be regarded as Markovian jump systems. Based on Lyapunov function approach, sufficient conditions are established for the resulting closed-loop discrete-time system with Markovian jumps to be stochastic $\mathscr{H}_\infty$ finite-time boundedness and then state feedback controllers are designed to guarantee stochastic $\mathscr{H}_\infty$ finite-time stabilization of the class of stochastic systems. The stochastic $\mathscr{H}_\infty$ finite-time boundedness criteria can be tackled in the form of linear matrix inequalities with a fixed parameter. As an auxiliary result, we also give sufficient conditions on the robust stochastic stabilization of the class of linear systems with packet loss. Finally, simulation examples are presented to illustrate the validity of the developed scheme.

1. Introduction

Networked control systems (NCSs) are feedback control systems with control closed loops via digital communication channel. Compared with traditional point-to-point controller architectures, the advantages of NCSs include low cost, high reliability, less wiring, and easy maintenance [1]. In recent years, NCSs have found successful applications in broad range of modern scientific areas such as internet-based control, distributed communication, and industrial automation [2]. However, the insertion of the communication channels creates discrepancies between the data records to be transmitted and their associated remotely transmitted images, which hence makes the traditional control theory confronts new challenges. Among these challenges, random communication delay, data packet dropout, and signal quantization are known to be three main interesting problems for the stability and performance degradation of the controlled networked system. In view of this, many researchers have made to study how to design control systems by packet loss, delay, and quantization, see [3–6] and the references cited therein. Among a number of issues arising from such a framework, packet losses of NCSs are an important issue to be addressed and have received great attention, see [7–15]. Meanwhile, Markovian jumps systems are regarded to be as a special family of hybrid systems and stochastic systems, which are very appropriate to model plants whose structure is subject to random abrupt changes, see [16–22] and references therein.

It is well known that classical Lyapunov theory focuses mainly on the state convergence property of the systems in infinite time interval, which, just as was mentioned above, does not usually specify bounds on the trajectories in finite interval. However, the main attention in many practical applications is the behavior of the dynamic systems over a specified time interval, for instance, large values of the state are not acceptable in the presence of saturations [23]. To discuss this transient performance of control dynamics, finite-time stability or short-time stability was presented in [24]. Then, some appealing results were found in [25–32]. However, to date and to the best of our knowledge, the problems of stochastic finite-time stability and stabilization of network control systems with packet loss have not fully investigated and still remain challenging, although results related to systems over networks with packet loss are reported in the existing literature, see [6–15, 33–36].

Motivated by the above discussion, in this paper, we address the stochastic $\mathscr{H}_\infty$ finite-time boundedness (S$\mathscr{H}_\infty$FTB) problems for linear discrete-time systems over networks with packet dropout, parametric uncertainties, and time-varying norm-bounded disturbance. Firstly, we present dynamic model description studied, which, if the data packet loss is assumed to be a time homogenous Markov process, the class of linear discrete-time systems with packet loss can be referred as Markovian jump systems. Thus, the class of linear systems investigated could be studied by the theoretical framework of Markov jumps systems. Then, the concepts of stochastic finite-time stability, stochastic finite-time boundedness, and S$\mathscr{H}_\infty$FTB and problem formulation are given. The main contribution of this paper is to design a state feedback controller which guarantees the resulting closed-loop discrete-time system with Markovian jumps S$\mathscr{H}_\infty$FTB. As an auxiliary result, we also give sufficient conditions on the robust stochastic stabilization of the class of linear systems with packet loss. The S$\mathscr{H}_\infty$FTB criteria of the class of Markovian jump systems can be addressed in the form of linear matrix inequalities (LMIs) with a fixed parameter.

The rest of this paper is organized as follows. Section 2 is devoted to the dynamic model description and problem formulation. The results on the S$\mathscr{H}_\infty$FTB are presented in Section 3. Section 4 presents numerical examples to demonstrate the validity of the proposed methodology. Finally, in Section 5, the conclusions are given.

Notation 1. The notation used throughout the paper is fairly standard, $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$, and $\mathbb{Z}_{k\geq 0}$ denote the sets of n component real vectors, $n \times m$ real matrices, and the set of nonnegative integers, respectively. The superscript T stands for matrix transposition or vector, and $\mathbf{E}\{\cdot\}$ denotes the expectation operator with respective to some probability measure $\mathcal{P}$. In addition, the symbol $*$ denotes the transposed elements in the symmetric positions of a matrix, and diag$\{\cdots\}$ stands for a block-diagonal matrix. $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denote the smallest and the largest eigenvalue of matrix P, respectively. Notations sup and inf denote the supremum and infimum, respectively. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2. Problem Formulation and Preliminaries

Let us consider a linear discrete-time system (LDS) as follows:

$$\begin{aligned} x(k+1) &= [A+\Delta A(k)]x(k) + [B+\Delta B(k)]u(k) + Gw(k), \\ z(k) &= Cx(k) + D_1u(k) + D_2w(k), \end{aligned} \tag{2.1}$$

where $x(k) \in \mathbb{R}^n$ is the state, $z(k) \in \mathbb{R}^{l_1}$ is the measure output, and $u(k) \in \mathbb{R}^m$ is the control input. The noise signal $w(k) \in \mathbb{R}^{l_2}$ satisfies

$$\sum_{k=0}^{\infty} w^T(k)w(k) < d^2, \quad d > 0. \tag{2.2}$$

The matrices $\Delta A(k)$ and $\Delta B(k)$ are uncertain matrices and satisfy

$$[\Delta A(k), \Delta B(k)] = F\Delta(k)[E_1, E_2], \tag{2.3}$$

where $\Delta(k)$ is an unknown, time-varying matrix function, and satisfies

$$\Delta^T(k)\Delta(k) \leq I, \quad \forall k \in \mathbb{Z}_{k\geq 0}. \tag{2.4}$$

Due to the existence of the packet dropout of the communication during the transmission, the packet dropout process of the network can be regarded as a time-homogenous Markov process $\{\gamma(k), k \geq 0\}$. Let $\gamma(k) = 1$ mean that the packet has been successfully delivered to the decoder, while $\gamma(k) = 0$ corresponds to the dropout of the packet. The Markov chain has a transition probability matrix defined by

$$\mathcal{P}\{\gamma(k+1) = j \mid \gamma(k) = i\} = \begin{bmatrix} 1-q & q \\ p & 1-p \end{bmatrix}, \tag{2.5}$$

where $i, j \in \mathbb{W} \triangleq \{0, 1\}$ are the state of the Markov chain. Without loss of generality, let $\gamma(0) = 1$ and the failure rate p and the recovery rate q of the channel satisfy $p, q \in (0, 1)$. It is worth noting that a smaller value of p and a larger value of q indicate a more reliable channel.

Remark 2.1. When the above transition probability matrix is $\begin{bmatrix} p & 1-p \\ p & 1-p \end{bmatrix}$ with $0 \leq p \leq 1$, the above two-state Markov process is reduced to a Bernoulli process [37].

Consider the control law for the LDS (2.1) in the form

$$u(k) = \gamma(k)Lx(k), \tag{2.6}$$

where L is to be designed the control gain matrix. $\{\gamma(k), k \geq 0\}$ is a Markov packet dropout process satisfying (2.5). Then, the resulting closed-loop LDS follows that

$$x(k+1) = \left[\overline{A} + \gamma(k)\overline{B}L\right]x(k) + Gw(k),$$
$$z(k) = \left[C + \gamma(k)D_1L\right]x(k) + D_2w(k), \tag{2.7}$$

where $\overline{A} = A + \Delta A(k)$ and $\overline{B} = B + \Delta B(k)$.

Now, we define two models according to the value of $\gamma(k)$. If $\gamma(k) = 1$, we define the Model 1 at time $k+1$ as follows:

$$x(k+1) = \left(\overline{A} + \overline{B}L_{\zeta}\right)x(k) + Gw(k),$$
$$z(k) = \left(C + D_1L_{\zeta}\right)x(k) + D_2w(k). \tag{2.8}$$

If $\gamma(k) = 0$, we define the Model 2 at time $k+1$ as follows:

$$x(k+1) = \overline{A}x(k) + Gw(k),$$
$$z(k) = Cx(k) + D_2w(k), \tag{2.9}$$

where the selection of L_{ζ} in (2.7) is according to the model of $x(k)$ for all $\zeta \in \{1,0\}$, that is to say, if $x(k)$ is at Model 1, which is $\gamma(k-1) = 1,\ L_{\zeta} = L_1$, otherwise, if $x(k)$ is at Model 2, which is $\gamma(k-1) = 0, L_{\zeta} = L_0$.

Then, (2.7) can be regarded as a closed-loop LDS with Markovian jumps described by

$$x(k+1) = \hbar(1)\left[\left(\overline{A} + \overline{B}L_{\zeta}\right)x(k) + Gw(k)\right] + \hbar(2)\left[\overline{A}x(k) + Gw(k)\right],$$
$$z(k) = \hbar(1)\left[\left(C + D_1L_{\zeta}\right)x(k) + D_2w(k)\right] + \hbar(2)\left[Cx(k) + D_2w(k)\right], \tag{2.10}$$

where $\hbar(a)$, $a \in \{1,2\}$ denotes the mode indicator function. $\hbar(1)$ corresponds to a mode with feedback, and $\hbar(2)$ corresponds to a mode without feedback. It is noted that it yields $\hbar(a) = 1$ when at time $k+1$ be $a \in \{1,2\}$ and $\hbar(b) = 0$ for $b \neq a$. The mode transition probabilities of Markovian jump LDS (2.10) is given by

$$\wp\left\{\eta_v(k+1) = v \mid \eta_u(k) = u\right\} = \pi_{uv}, \tag{2.11}$$

where $\pi_{uv} \geq 0$ for all $u, v \in \{1,2\}$ and $\sum_{v=1}^{2} \pi_{uv} = 1$. $\eta_v(k) = 1$ implies $\hbar(1) = 1,\ \hbar(2) = 0$, which the communication transmission succeeds, and $\eta_v(k) = 2$ implies $\hbar(1) = 0,\ \hbar(2) = 1$, which the communication dropout occurs. Thus, compared to (2.5), it follows that $\pi_{11} = 1 - p,\ \pi_{12} = p,\ \pi_{21} = q,\ \pi_{22} = 1 - q$.

Definition 2.2 (stochastic finite-time stability (SFTS)). The closed-loop Markovian jump LDS with $w(k) = 0$ (2.10) is said to be SFTS with respect to $(\delta_x, \epsilon, R, N)$, where $0 < \delta_x < \epsilon$, R is a symmetric positive-definite matrix and $N \in \mathbb{Z}_{k\geq 0}$, if

$$\mathbf{E}\left\{x^T(0)Rx(0)\right\} \leq \delta_x^2 \Longrightarrow \mathbf{E}\left\{x^T(k)Rx(k)\right\} < \epsilon^2, \quad \forall k \in \{1, 2, \ldots, N\}. \tag{2.12}$$

Definition 2.3 (stochastic finite-time boundedness (SFTB)). The closed-loop LDS with Markovian jumps (2.10) is said to be SFTB with respect to $(\delta_x, \epsilon, R, N, d)$, where $0 < \delta_x < \epsilon$, R is a symmetric positive-definite matrix and $N \in \mathbb{Z}_{k\geq 0}$, if the relation condition (2.12) holds.

Definition 2.4 (stochastic $\mathscr{H}_\infty$ finite-time boundedness (S$\mathscr{H}_\infty$FTB)). The closed-loop LDS with Markovian jumps (2.10) is said to be S$\mathscr{H}_\infty$FTB with respect to $(\delta_x, \epsilon, \gamma, R, N, d)$, where $0 < \delta_x < \epsilon$, R is a symmetric positive-definite matrix and $N \in \mathbb{Z}_{k\geq 0}$, if the closed-loop LDS with Markovian jumps (2.10) is SFTB with respect to $(\delta_x, \epsilon, R, N, d)$ and under the zero-initial condition the output $z(k)$ satisfies

$$\mathbf{E}\left\{\sum_{j=0}^{N} z^T(j)z(j)\right\} < \gamma^2 \sum_{j=0}^{N} w^T(j)w(j) \tag{2.13}$$

for any nonzero $w(k)$ which satisfies (2.2), where γ is a prescribed positive scalar.

Lemma 2.5 (see [38]). *The linear matrix inequality* $\begin{bmatrix} X_{11} & * \\ X_{21} & X_{22} \end{bmatrix} < 0$ *is equivalent to* $X_{22} < 0$ *and* $X_{11} - X_{21}^T X_{22}^{-1} X_{21} < 0$, *where* $X_{11} = X_{11}^T$ *and* $X_{22} = X_{22}^T$.

Lemma 2.6 (see [38]). *For matrices* X, Y, *and* Z *of appropriate dimensions, where* X *is a symmetric matrix, then* $X + YF(t)Z + [YF(t)Z]^T < 0$ *holds for all matrix* $F(t)$ *satisfying* $F^T(t)F(t) \leq I$ *for all* $t \in \mathbb{R}$, *if and only if there exists a positive constant* ϱ, *such that the inequality* $X + \varrho YY^T + \varrho^{-1}Z^T Z < 0$ *holds.*

In this paper, the feedback gain matrices L_1 and L_0 with Markov packet dropout of failure rate p and recovery rate q will be designed to guarantee the states of the closed-loop Markovian jump LDS (2.10) S$\mathscr{H}_\infty$FTB.

3. Main Results

In this section, for the given failure rate p and recovery rate q with $p, q \in (0, 1)$, we will design a state feedback controller that assures S$\mathscr{H}_\infty$FTB of the Markovian jump LDS (2.10).

Theorem 3.1. *For the given failure rate* p *and recovery rate* q *with* $p, q \in (0, 1)$, *the closed-loop Markovian jump LDS* (2.10) *is SFTB with respect to* $(\delta_x, \epsilon, R, N, d)$, *if there exist scalars* $\mu \geq 1$,

$\gamma > 0$, two symmetric positive-definite matrices P_1, P_2, and a set of feedback control matrices $\{L_\zeta,\ \zeta \in \{1,0\}\}$, such that the following inequalities hold:

$$\begin{bmatrix} -\mu P_1 & * & * & * \\ 0 & -\gamma^2\mu^{-N} I & * & * \\ \overline{A} + \overline{B}L_1 & G & -\dfrac{1}{1-p}P_1^{-1} & * \\ \overline{A} & G & 0 & -\dfrac{1}{p}P_2^{-1} \end{bmatrix} < 0, \quad (3.1)$$

$$\begin{bmatrix} -\mu P_2 & * & * & * \\ 0 & -\gamma^2\mu^{-N} I & * & * \\ \overline{A} + \overline{B}L_0 & G & -\dfrac{1}{q}P_1^{-1} & * \\ \overline{A} & G & 0 & -\dfrac{1}{1-q}P_2^{-1} \end{bmatrix} < 0, \quad (3.2)$$

$$\sup_{a\in\{1,2\}} \left\{\lambda_{\max}\left(\widetilde{P}_a\right)\right\}\mu^N \delta_x^2 + \gamma^2 d^2 < \inf_{a\in\{1,2\}} \left\{\lambda_{\min}\left(\widetilde{P}_a\right)\right\}\epsilon^2, \quad (3.3)$$

where $\widetilde{P}_a = R^{-1/2}P_a R^{-1/2}$ for all $a \in \{1,2\}$.

Proof. Assume the mode at time k be $a \in \{1,2\}$. Taking into account that if $a = 1$, then we have $\gamma(k-1) = 1$ and $L_\zeta = L_1$, otherwise if $a = 2$, then $\gamma(k-1) = 0$ and $L_\zeta = L_0$. Consider the following Lyapunov-Krasovskii functional candidate for the Markov jump LDS (2.10):

$$V\left(x(k), \eta_v(k) = a\right) = x^T(k) P_a x(k). \quad (3.4)$$

Then, we have

$$\begin{aligned} \mathbf{E}\{V(k+1)\} &= \mathbf{E}\left\{\sum_{v=1}^{2} \wp\{\eta_v(k+1) = v \mid \eta_v(k) = a\} \times x^T(k+1)P_v x(k+1)\right\} \\ &= \pi_{a1}\left[\left(\overline{A} + \overline{B}L_\zeta\right)x(k) + Gw(k)\right]^T P_1 \left[\left(\overline{A} + \overline{B}L_\zeta\right)x(k) + Gw(k)\right] \\ &\quad + \pi_{a2}\left[\overline{A}x(k) + Gw(k)\right]^T P_2 \left[\overline{A}x(k) + Gw(k)\right]. \end{aligned} \quad (3.5)$$

Denote

$$\Theta(x(k), w(k), a) \triangleq \mathbf{E}\{V(k+1)\} - \mu V(k) - \gamma^2\mu^{-N} w^T(k) w(k). \quad (3.6)$$

Taking into account that if $a = 1$, then $L_\zeta = L_1$, otherwise $a = 2$, then $L_\zeta = L_0$. Noting that $\pi_{11} = 1 - p,\ \pi_{12} = p,\ \pi_{21} = q,\ \pi_{22} = 1 - q$. Thus, when $a = 1$, it follows that

$$\begin{aligned}\Theta(x(k), w(k), 1) &= (1-p)\left[\left(\overline{A} + \overline{B}L_1\right)x(k) + Gw(k)\right]^T P_1\left[\left(\overline{A} + \overline{B}L_1\right)x(k) + Gw(k)\right] \\ &\quad + p\left[\overline{A}x(k) + Gw(k)\right]^T P_2\left[\overline{A}x(k) + Gw(k)\right] \\ &\quad - \mu x^T(k)P_1x(k) - \gamma^2\mu^{-N}w^T(k)w(k) \\ &= \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}^T \left\{ \Phi_1^T \begin{bmatrix} (1-p)P_1 & * \\ 0 & pP_2 \end{bmatrix} \Phi_1 - \begin{bmatrix} \mu P_1 & * \\ 0 & \gamma^2\mu^{-N}I \end{bmatrix} \right\} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix},\end{aligned} \tag{3.7}$$

where

$$\Phi_1 = \begin{bmatrix} \overline{A} + \overline{B}L_1 & G \\ \overline{A} & G \end{bmatrix}. \tag{3.8}$$

By Lemma 2.5, it follows from (3.1) and (3.7) that

$$\Theta(x(k), w(k), 1) < 0. \tag{3.9}$$

When $a = 2$, taking into account condition (3.2), the similar to (3.9), we can derive the following inequality:

$$\Theta(x(k), w(k), 2) < 0. \tag{3.10}$$

Thus, for all $a \in \{1, 2\}$, we have

$$\Theta(x(k), w(k), a) < 0. \tag{3.11}$$

That is to say, for all $a \in \{1, 2\}$, it follows that

$$\mathbf{E}\{V(k+1)\} < \mu V(k) + \gamma^2\mu^{-N}w^T(k)w(k). \tag{3.12}$$

By (3.12), it is obvious that

$$\mathbf{E}\{V(k+1)\} < \mu\mathbf{E}\{V(k)\} + \gamma^2\mu^{-N}w^T(k)w(k). \tag{3.13}$$

From (2.2) and (3.13) and noting that $\mu \geq 1$, we have

$$\begin{aligned}\mathbf{E}\{V(k)\} &< \mu^k\mathbf{E}\{V(0)\} + \gamma^2\mu^{-N}\sum_{j=0}^{k-1}\mu^{k-j-1}w^T(j)w(j) \\ &\leq \mu^k\mathbf{E}\{V(0)\} + \gamma^2\mu^{-N}\mu^k d^2.\end{aligned} \tag{3.14}$$

Let $\widetilde{P}_a = R^{-1/2} P_a R^{-1/2}$ and noting that $\mathbf{E}\{x^T(0)Rx(0)\} \le \delta_x^2$, we have

$$\begin{aligned}
\mathbf{E}\{V(0)\} &= \mathbf{E}\left\{x^T(0)P_a x(0)\right\} \\
&= \mathbf{E}\left\{x^T(0)R^{1/2}\widetilde{P}_a R^{1/2} x(0)\right\} \\
&\le \sup_{a\in\{1,2\}} \left\{\lambda_{\max}\left(\widetilde{P}_a\right)\right\} \mathbf{E}\left\{x^T(0)Rx(0)\right\} \\
&\le \sup_{a\in\{1,2\}} \left\{\lambda_{\max}\left(\widetilde{P}_a\right)\right\} \delta_x^2.
\end{aligned} \tag{3.15}$$

On the other hand, for all $a \in \{1,2\}$, we have

$$\begin{aligned}
\mathbf{E}\{V(k)\} &= \mathbf{E}\left\{x^T(k)P_a x(k)\right\} \\
&= \mathbf{E}\left\{x^T(k)R^{1/2}\widetilde{P}_a R^{1/2} x(k)\right\} \\
&\ge \inf_{a\in\{1,2\}} \left\{\lambda_{\min}\left(\widetilde{P}_a\right)\right\} \mathbf{E}\left\{x^T(k)Rx(k)\right\}.
\end{aligned} \tag{3.16}$$

Combing with (3.14)–(3.16), we can derive

$$\mathbf{E}\left\{x^T(k)Rx(k)\right\} < \frac{\sup_{a\in\{1,2\}}\left\{\lambda_{\max}\left(\widetilde{P}_a\right)\right\}\mu^k\delta_x^2 + \gamma^2\mu^{-N}\mu^k d^2}{\inf_{a\in\{1,2\}}\left\{\lambda_{\min}\left(\widetilde{P}_a\right)\right\}}. \tag{3.17}$$

Noting condition (3.3), it is obvious that $\mathbf{E}\{x^T(k)Rx(k)\} < \epsilon^2$ for all $k \in \{1,2,\dots,N\}$. This completes the proof of this theorem. □

Theorem 3.2. *For the given failure rate p and recovery rate q with $p, q \in (0,1)$, the closed-loop Markovian jump LDS* (2.10) *is $S\mathscr{H}_\infty$FTB with respect to $(\delta_x, \epsilon, \gamma, R, N, d)$, if there exist scalars $\mu \ge 1$, $\gamma > 0$, two symmetric positive-definite matrices P_1, P_2, and a set of feedback control matrices $\{L_\zeta,\ \zeta \in \{1,0\}\}$, such that* (3.3) *and the following inequalities hold:*

$$\begin{bmatrix}
-\mu P_1 & * & * & * & * \\
0 & -\gamma^2\mu^{-N} I & * & * & * \\
\overline{A} + \overline{B}L_1 & G & -\dfrac{1}{1-p}P_1^{-1} & * & * \\
\overline{A} & G & 0 & -\dfrac{1}{p}P_2^{-1} & * \\
C + D_1 L_1 & D_2 & 0 & 0 & -I
\end{bmatrix} < 0, \tag{3.18}$$

$$\begin{bmatrix}
-\mu P_2 & * & * & * & * \\
0 & -\gamma^2\mu^{-N} I & * & * & * \\
\overline{A} + \overline{B}L_0 & G & -\dfrac{1}{q}P_1^{-1} & * & * \\
\overline{A} & G & 0 & -\dfrac{1}{1-q}P_2^{-1} & * \\
C & D_2 & 0 & 0 & 0 \;\; -I
\end{bmatrix} < 0, \tag{3.19}$$

where $\widetilde{P}_a = R^{-1/2} P_a R^{-1/2}$ for all $a \in \{1,2\}$.

Proof. Noting that

$$\Upsilon_1 \triangleq \begin{bmatrix} (C+D_1L_1)^T(C+D_1L_1) & * \\ D_2^T(C+D_1L_1) & D_2^TD_2 \end{bmatrix} = \begin{bmatrix} (C+D_1L_1)^T \\ D_2^T \end{bmatrix} \begin{bmatrix} C+D_1L_1 & D_2 \end{bmatrix} \geq 0, \tag{3.20}$$

$$\Upsilon_2 \triangleq \begin{bmatrix} C^TC & * \\ D_2^TC & D_2^TD_2 \end{bmatrix} = \begin{bmatrix} C^T \\ D_2^T \end{bmatrix} \begin{bmatrix} C & D_2 \end{bmatrix} \geq 0. \tag{3.21}$$

Applying Lemma 2.5, it follows from (3.18) and (3.19) that conditions (3.1) and (3.2) hold. Therefore, the Morkovian jump LDS (2.10) is stochastic finite-time boundedness according to Theorem 3.1.

Then, we only need to prove (2.13) satisfied under zero-value initial condition. Let us assume the mode at time k be $a \in \{1,2\}$. Taking into account that if $a = 1$, then we have $\gamma(k-1) = 1$ and $L_\zeta = L_1$, otherwise if $a = 2$, then $\gamma(k-1) = 0$ and $L_\zeta = L_0$. Let us choose $V(x(k), \eta_v(k) = a) = x^T(k)P_a x(k)$ for the Markovian jump LDS (2.10). We denote

$$\Lambda(x(k), w(k), a) \triangleq \mathbf{E}\{V(k+1)\} - \mu V(k) + z^T(k)z(k) - \gamma^2\mu^{-N}w^T(k)w(k). \tag{3.22}$$

Thus, when $a = 1$, we have

$$\begin{aligned} &\Lambda(x(k), w(k), 1) \\ &\quad = (1-p)\left[\left(\overline{A}+\overline{B}L_1\right)x(k) + Gw(k)\right]^T P_1 \left[\left(\overline{A}+\overline{B}L_1\right)x(k) + Gw(k)\right] \\ &\qquad + p\left[\overline{A}x(k) + Gw(k)\right]^T P_2 \left[\overline{A}x(k) + Gw(k)\right] \\ &\qquad + \left[(C+D_1L_1)x(k) + D_2w(k)\right]^T \left[(C+D_1L_1)x(k) + D_2w(k)\right] \\ &\qquad - \mu x^T(k)P_1x(k) - \gamma^2\mu^{-N}w^T(k)w(k) \\ &\quad = \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}^T \left\{ \Phi_1^T \begin{bmatrix} (1-p)P_1 & * \\ 0 & pP_2 \end{bmatrix} \Phi_1 - \begin{bmatrix} \mu P_1 & * \\ 0 & \gamma^2\mu^{-N}I \end{bmatrix} + \Upsilon_1 \right\} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}, \end{aligned} \tag{3.23}$$

where Φ_1, Υ_1 are the same as the above. Thus, according to Lemma 2.5, we can obtain from (3.18) and (3.23)

$$\Lambda(x(k), w(k), 1) < 0. \tag{3.24}$$

When $a = 2$, taking into account condition (3.19) and (3.21), the similar to the above deduction, we can derive that the following inequality holds:

$$\Lambda(x(k), w(k), 2) < 0. \tag{3.25}$$

Thus, for all $a \in \{1,2\}$, we can obtain

$$\Lambda(x(k), w(k), a) = \mathbf{E}\{V(k+1)\} - \mu V(k) + z^T(k)z(k) - \gamma^2\mu^{-N}w^T(k)w(k) < 0. \tag{3.26}$$

According to (3.26), it is obvious that

$$\mathbf{E}\{V(k+1)\} < \mu\mathbf{E}\{V(k)\} - \mathbf{E}\left\{z^T(k)z(k)\right\} + \gamma^2\mu^{-N}w^T(k)w(k). \tag{3.27}$$

From (3.27), we have

$$\mathbf{E}\{V(k)\} < \mu^k\mathbf{E}\{V(0)\} - \sum_{j=0}^{k-1}\mu^{k-j-1}\mathbf{E}\left\{z^T(j)z(j)\right\} + \gamma^2\mu^{-N}\sum_{j=0}^{k-1}\mu^{k-j-1}w^T(j)w(j). \tag{3.28}$$

Under the zero-value initial condition and noting that $V(k) \geq 0$ for all $k \in \mathbb{Z}_{k\geq 0}$, we have

$$\sum_{j=0}^{k-1}\mu^{k-j-1}\mathbf{E}\left\{z^T(j)z(j)\right\} < \gamma^2\mu^{-N}\sum_{j=0}^{k-1}\mu^{k-j-1}w^T(j)w(j). \tag{3.29}$$

From (3.29) and noting that $\mu \geq 1$, we have

$$\begin{aligned}\mathbf{E}\left\{\sum_{j=0}^{N}z^T(j)z(j)\right\} &= \sum_{j=0}^{N}\mathbf{E}\left\{z^T(j)z(j)\right\} \leq \sum_{j=0}^{N}\mathbf{E}\left\{\mu^{N-j}z^T(j)z(j)\right\} \\ &< \gamma^2\mu^{-N}\sum_{j=0}^{N}\mu^{N-j}w^T(j)w(j) \leq \gamma^2\sum_{j=0}^{N}w^T(j)w(j).\end{aligned} \tag{3.30}$$

Thus, this completes the proof of the theorem. □

Denoting $X_1 = P_1^{-1},\ X_2 = P_2^{-1},\ L_1 = Y_1X_1^{-1},\ L_0 = Y_0X_2^{-1}$ and applying Lemmas 2.5 and 2.6, one can obtain from Theorem 3.2 the following results on the stochastic $\mathscr{H}_\infty$ finite-time stabilization.

Theorem 3.3. *For the given failure rate p and recovery rate q with $p, q \in (0,1)$, there exists a state feedback controller $u(t) = L_\zeta x(t),\ \zeta \in \{1,0\}$ with $L_1 = Y_1X_1^{-1}$ and $L_0 = Y_0X_2^{-1}$ such that the closed-loop Markovian jump LDS* (2.10) *is $S\mathscr{H}_\infty$FTB with respect to $(\delta_x, \epsilon, \gamma, R, N, d)$, if there exist scalars*

$\mu \geq 1$, $\gamma > 0$, $\epsilon_1 > 0$, $\epsilon_2 > 0$, two symmetric positive-definite matrices X_1, X_2, and a set of feedback control matrices $\{L_\zeta,\ \zeta \in \{1,0\}\}$, such that the following inequalities hold:

$$\begin{bmatrix} -\mu X_1 & * & * & * & * & * & * \\ 0 & -\gamma^2\mu^{-N} I & * & * & * & * & * \\ AX_1 + BY_1 & G & -\dfrac{1}{1-p}X_1 + \epsilon_1 FF^T & * & * & * & * \\ AX_1 & G & 0 & -\dfrac{1}{p}X_2 + \epsilon_1 FF^T & * & * & * \\ CX_1 + D_1Y_1 & D_2 & 0 & 0 & -I & * & * \\ E_1X_1 + E_2Y_1 & 0 & 0 & 0 & 0 & -\epsilon_1 I & * \\ E_1X_1 & 0 & 0 & 0 & 0 & 0 & -\epsilon_1 I \end{bmatrix} < 0, \quad (3.31)$$

$$\begin{bmatrix} -\mu X_2 & * & * & * & * & * & * \\ 0 & -\gamma^2\mu^{-N} I & * & * & * & * & * \\ AX_2 + BY_0 & G & -\dfrac{1}{q}X_1 + \epsilon_2 FF^T & * & * & * & * \\ AX_2 & G & 0 & -\dfrac{1}{1-q}X_2 + \epsilon_2 FF^T & * & * & * \\ CX_2 & D_2 & 0 & 0 & -I & * & * \\ E_1X_2 + E_2Y_0 & 0 & 0 & 0 & 0 & -\epsilon_2 I & * \\ E_1X_2 & 0 & 0 & 0 & 0 & 0 & -\epsilon_2 I \end{bmatrix} < 0, \quad (3.32)$$

$$\sup_{a\in\{1,2\}} \left\{\lambda_{\max}\left(\widetilde{X}_a\right)\right\}\mu^N \delta_x^2 + \gamma^2 d^2 < \inf_{a\in\{1,2\}} \left\{\lambda_{\min}\left(\widetilde{X}_a\right)\right\}\epsilon^2, \quad (3.33)$$

where $\widetilde{X}_a = R^{-1/2}X_a^{-1}R^{-1/2}$ for all $a \in \{1,2\}$.

Remark 3.4. It is easy to check that condition (3.33) is guaranteed by imposing the conditions for all $a \in \{1,2\}$:

$$\lambda R^{-1} < X_a < R^{-1}, \quad \begin{bmatrix} \mu^{-N}\left(\gamma^2 d^2 - \epsilon^2\right) & * \\ \delta_x & -\lambda \end{bmatrix} < 0. \quad (3.34)$$

It follows that conditions (3.31), (3.32), and (3.34) are not strict LMIs; however, once we fix the parameter μ, the conditions can be turned into LMI-based feasibility problem:

Remark 3.5. From the above discussion, we can obtain that the feasibility of conditions stated in Theorem 3.3 can be turned into the following LMIs based feasibility problem

$$\begin{aligned} \min \quad & \left(\gamma^2 + \epsilon^2\right) \\ & X_1, X_2, Y_1, Y_0, \epsilon_1, \epsilon_2, \lambda \\ \text{s.t.} \quad & \text{LMIs (3.31), (3.32), and (3.34)} \end{aligned} \quad (3.35)$$

with a fixed parameter μ. Furthermore, we can also find the parameter μ by an unconstrained nonlinear optimization approach, which a locally convergent solution can be obtained by using the program *fminsearch* in the optimization toolbox of Matlab.

Remark 3.6. If we can find feasible solution with the parameter $\mu = 1$, by the above discussion, we can obtain that the designed controller can ensure both stochastic finite-time boundedness and robust stochastic stabilization of the family of network control systems.

4. Numerical Examples

In this section, we present two examples to illustrate the proposed methods.

Example 4.1. Consider a Morkovian jump LDS (2.10) with parameters as

$$A = \begin{bmatrix} 1.5 & 0 \\ 0.2 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 1 \\ 0.5 & 1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 0.5 \\ 0 & 3 \end{bmatrix}, \tag{4.1}$$
$$F = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.4 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.4 & 0 \\ 0.5 & 0.2 \end{bmatrix},$$

and $d = 1$, $\Delta(k) = \operatorname{diag}\{\Delta_1(k), \Delta_2(k)\}$, where $\Delta_i(k)$ satisfies $|\Delta_i(k)| \leq 1$ for all $i \in \{1,2\}$ and $k \in \mathbb{Z}_{k\geq 0}$. Moreover, we assume the failure rate $p = 0.3$ and the recovery rate $q = 0.6$.

Then, we chose $R = I_3$, $\delta_x = 1$, $N = 5$, and $\mu = 1.8$, Theorem 3.3 yields to $\gamma = 27.1939$, $\epsilon = 28.7912$, and

$$X_1 = \begin{bmatrix} 0.3069 & 0.2090 \\ 0.2090 & 0.8775 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0.2612 & -0.1408 \\ -0.1408 & 0.6357 \end{bmatrix},$$
$$Y_1 = \begin{bmatrix} -0.5248 & -0.4726 \\ -0.0747 & -0.6270 \end{bmatrix}, \quad Y_0 = \begin{bmatrix} -0.4255 & 0.0877 \\ 0.0762 & -0.4695 \end{bmatrix}, \tag{4.2}$$
$$\epsilon_1 = 1.6732, \quad \epsilon_2 = 0.6003, \quad \lambda = 0.2127.$$

Thus, we can obtain the following state feedback controller gains

$$L_1 = \begin{bmatrix} -1.6031 & -0.1567 \\ 0.2903 & -0.7837 \end{bmatrix}, \quad L_0 = \begin{bmatrix} -1.7652 & -0.2529 \\ -0.1206 & -0.7653 \end{bmatrix}. \tag{4.3}$$

Furthermore, let $R = I_3$, $\delta_x = 1$, and $N = 5$, by Theorem 3.3, the optimal bound with minimum value of $\gamma^2 + \epsilon^2$ relies on the parameter μ. We can find feasible solution when $1.12 \leq \mu \leq 34.13$. Figures 1 and 2 show the optimal value with different value of μ. Then, by using the program *fminsearch* in the optimization toolbox of Matlab starting at $\mu = 1.8$, the locally convergent solution can be derived as

$$L_1 = \begin{bmatrix} -1.5669 & -0.1166 \\ 0.3827 & -0.8124 \end{bmatrix}, \quad L_0 = \begin{bmatrix} -1.6611 & -0.1668 \\ 0.0456 & -0.7598 \end{bmatrix}, \tag{4.4}$$

with $\mu = 1.5694$ and the optimal value $\gamma = 26.2353$ and $\epsilon = 27.4932$.

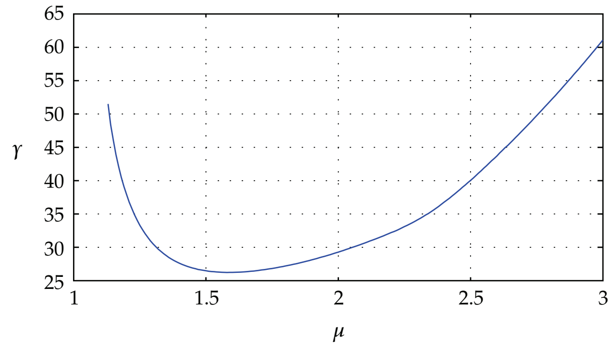

Figure 1: The local optimal bound of γ.

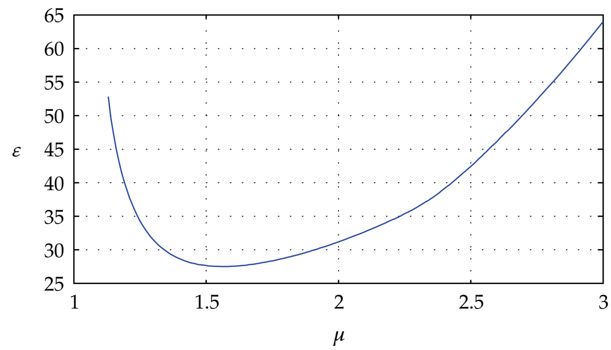

Figure 2: The local optimal bound of ϵ.

Example 4.2. Consider a Morkovian jump LDS (2.10) with

$$A = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.8 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \tag{4.5}$$

and the failure rate $p = 0.3$ and the recovery rate $q = 0.9$. In addition, the other matrices parameters are the same as Example 4.1.

Then, let $R = I_3$ and $\delta_x = 1$, by Theorem 3.3, we can find feasible solution when $\mu = 1$. Furthermore, when $\mu = 1$, it yields the optimal value $\gamma = 4.7057$ and $\epsilon = 5.0791$ and the following optimized state feedback controller gains:

$$L_1 = \begin{bmatrix} -0.3412 & -0.0380 \\ -0.4900 & -1.1045 \end{bmatrix}, \qquad L_0 = \begin{bmatrix} -0.5649 & -0.0685 \\ 0.6100 & -0.7019 \end{bmatrix}. \tag{4.6}$$

Thus, the above LDS with Morkovian jumps is stochastically stable and the calculated minimum $\mathscr{H}_\infty$ performance γ satisfies $\|T_{wz}\| < 4.7057$.

5. Conclusions

This paper addresses the $S\mathscr{H}_\infty$FTB control problems for one family of linear discrete-time systems over networks with packet dropout. Under assuming packet loss being a time homogenous Markov process, the class of linear discrete-time systems can be regarded as Markovian jump systems. Sufficient conditions are given for the resulting closed-loop linear discrete-time Markovian jump system to be $S\mathscr{H}_\infty$FTB, and state feedback controllers are designed to guarantee $S\mathscr{H}_\infty$FTB of the class of linear systems with Markov jumps. The $S\mathscr{H}_\infty$FTB criteria can be tackled in the form of linear matrix inequalities with a fixed parameter. As an auxiliary result, we also give sufficient conditions on the robust stochastic stabilization of the class of linear discrete-time systems with data packet dropout. Finally, simulation results are also given to show the validity of the proposed approaches.

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References

[1] W. Zhang, M. S. Branicky, and S. M. Phillips, "Stability of networked control systems," *IEEE Control Systems Magazine*, vol. 21, no. 1, pp. 84–99, 2001.
[2] X. He, Z. Wang, Y. D. Ji, and D. Zhou, "Fault detection for discrete-time systems in a networked environment," *International Journal of Systems Science*, vol. 41, no. 8, pp. 937–945, 2010.
[3] N. E. Nahi, "Optimal recursive estimation with uncertain observation," *IEEE Transactions on Information Theory*, vol. 15, no. 4, pp. 457–462, 1969.
[4] R. W. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Transactions on Automatic Control*, vol. 45, no. 7, pp. 1279–1289, 2000.
[5] D. Liberzon, "Hybrid feedback stabilization of systems with quantized signals," *Automatica*, vol. 39, no. 9, pp. 1543–1554, 2003.
[6] M. Yu, L. Wang, and T. Chu, "Stabilization of networked control systems with data packet dropout and transmission delays: continuous-time case," *European Journal of Control*, vol. 11, no. 1, pp. 40–55, 2005.
[7] X. Zhang, G. Lu, and Y. Zheng, "Stabilization of networked stochastic time-delay fuzzy systems with data dropout," *IEEE Transactions on Fuzzy Systems*, vol. 16, no. 3, pp. 798–807, 2008.
[8] K. Tsumura, H. Ishii, and H. Hoshina, "Tradeoffs between quantization and packet loss in networked control of linear systems," *Automatica*, vol. 45, no. 12, pp. 2963–2970, 2009.
[9] Y. Niu, T. Jia, X. Wang, and F. Yang, "Output-feedback control design for NCSs subject to quantization and dropout," *Information Sciences*, vol. 179, no. 21, pp. 3804–3813, 2009.
[10] Y. Ishido, K. Takaba, and D. E. Quevedo, "Stability analysis of networked control systems subject to packet-dropouts and finite-level quantization," *Systems & Control Letters*, vol. 60, no. 5, pp. 325–332, 2011.
[11] O. C. Imer, S. Yüksel, and T. Başar, "Optimal control of LTI systems over unreliable communication links," *Automatica*, vol. 42, no. 9, pp. 1429–1439, 2006.
[12] Z. Wang, D. W. C. Ho, and X. Liu, "Variance-constrained filtering for uncertain stochastic systems with missing measurements," *IEEE Transactions on Automatic Control*, vol. 48, no. 7, pp. 1254–1258, 2003.

[13] Z. Wang, D. W. C. Ho, and X. Liu, "Variance-constrained control for uncertain stochastic systems with missing measurements," *IEEE Transactions on Systems, Man, and Cybernetics Part A*, vol. 35, no. 5, pp. 746–753, 2005.
[14] H. Dong, Z. Wang, and H. Gao, "$\mathscr{H}_\infty$ fuzzy control for systems with repeated scalar nonlinearities and random packet losses," *IEEE Transactions on Fuzzy Systems*, vol. 17, no. 2, pp. 440–450, 2009.
[15] Y. Zhao, H. Gao, and T. Chen, "Fuzzy constrained predictive control of non-linear systems with packet dropouts," *IET Control Theory and Applications*, vol. 4, no. 9, pp. 1665–1677, 2010.
[16] N. M. Krasovskii and E. A. Lidskii, "Analytical design of controllers in systems with random attributes," *Automation and Remote Control*, vol. 22, no. 9–11, pp. 1021–1025, 1141–1146, 1289–1294, 1961.
[17] X. Mao, "Stability of stochastic differential equations with Markovian switching," *Stochastic Processes and their Applications*, vol. 79, no. 1, pp. 45–67, 1999.
[18] C. E. de Souza, "Robust stability and stabilization of uncertain discrete-time Markovian jump linear systems," *IEEE Transactions on Automatic Control*, vol. 51, no. 5, pp. 836–841, 2006.
[19] P. Shi, Y. Xia, G. P. Liu, and D. Rees, "On designing of sliding-mode control for stochastic jump systems," *IEEE Transactions on Automatic Control*, vol. 51, no. 1, pp. 97–103, 2006.
[20] S. K. Nguang, W. Assawinchaichote, and P. Shi, "Robust $\mathscr{H}_\infty$ control design for fuzzy singularly perturbed systems with Markovian jumps: an LMI approach," *IET Control Theory and Applications*, vol. 1, no. 4, pp. 893–908, 2007.
[21] L. Wu, P. Shi, and H. Gao, "State estimation and sliding-mode control of markovian jump singular systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 5, Article ID 5406129, pp. 1213–1219, 2010.
[22] Z. Shu, J. Lam, and J. Xiong, "Static output-feedback stabilization of discrete-time Markovian jump linear systems: a system augmentation approach," *Automatica*, vol. 46, no. 4, pp. 687–694, 2010.
[23] L. Weiss and E. F. Infante, "Finite time stability under perturbing forces and on product spaces," *IEEE Transactions on Automatic Control*, vol. 12, pp. 54–59, 1967.
[24] F. Amato, M. Ariola, and P. Dorato, "Finite-time control of linear systems subject to parametric uncertainties and disturbances," *Automatica*, vol. 37, no. 9, pp. 1459–1463, 2001.
[25] W. Zhang and X. An, "Finite-time control of linear stochastic systems," *International Journal of Innovative Computing, Information and Control*, vol. 4, no. 3, pp. 689–696, 2008.
[26] R. Ambrosino, F. Calabrese, C. Cosentino, and G. de Tommasi, "Sufficient conditions for finite-time stability of impulsive dynamical systems," *IEEE Transactions on Automatic Control*, vol. 54, no. 4, pp. 861–865, 2009.
[27] G. Garcia, S. Tarbouriech, and J. Bernussou, "Finite-time stabilization of linear time-varying continuous systems," *IEEE Transactions on Automatic Control*, vol. 54, no. 2, pp. 364–369, 2009.
[28] F. Amato, R. Ambrosino, M. Ariola, and C. Cosentino, "Finite-time stability of linear time-varying systems with jumps," *Automatica*, vol. 45, no. 5, pp. 1354–1358, 2009.
[29] D. Yang and K. Y. Cai, "Finite-time quantized guaranteed cost fuzzy control for continuous-time nonlinear systems," *Expert Systems with Applications*, vol. 37, no. 10, pp. 6963–6967, 2010.
[30] F. Amato, M. Ariola, and C. Cosentino, "Finite-time control of discrete-time linear systems: analysis and design conditions," *Automatica*, vol. 46, no. 5, pp. 919–924, 2010.
[31] S. He and F. Liu, "Stochastic finite-time boundedness of Markovian jumping neural network with uncertain transition probabilities," *Applied Mathematical Modelling*, vol. 35, no. 6, pp. 2631–2638, 2011.
[32] Y. Zhang, C. Liu, and X. Mu, "Robust finite-time $\mathscr{H}_\infty$ control of singular stochastic systems via static output feedback," *Applied Mathematics and Computation*, vol. 218, no. 9, pp. 5629–5640, 2012.
[33] J. Xiong and J. Lam, "Stabilization of linear systems over networks with bounded packet loss," *Automatica*, vol. 43, no. 1, pp. 80–87, 2007.
[34] X. Zhang, Y. Zheng, and G. Lu, "Stochastic stability of networked control systems with network-induced delay and data dropout," *Journal of Control Theory and Applications*, vol. 6, no. 4, pp. 405–409, 2008.
[35] J. Yu, L. Wang, G. Zhang, and M. Yu, "Output feedback stabilisation of networked control systems via switched system approach," *International Journal of Control*, vol. 82, no. 9, pp. 1665–1677, 2009.
[36] Y. B. Zhao, Y. Kang, G. P. Liu, and D. Rees, "Stochastic stabilization of packet-based networked control systems," *International Journal of Innovative Computing, Information and Control*, vol. 7, no. 5 A, pp. 2441–2455, 2011.
[37] P. Seiler and R. Sengupta, "An $\mathscr{H}_\infty$ approach to networked control," *IEEE Transactions on Automatic Control*, vol. 50, no. 3, pp. 356–364, 2005.
[38] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequality in Systems and Control Theory*, SIAM Studies in Applied Mathematics, SIAM, Philadelphia, Pa, USA, 1994.


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