Mathematical Problems in Engineering
Volume 7, pp. 177–203
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Sufficient Conditions for Lagrange, Mayer, and Bolza Optimization Problems

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(Received 7 February 2000)

The Maximum Principle [2, 13] is a well known necessary condition for optimality. This condition, generally, is not sufficient. In [3], the author proved that if there exists *regular synthesis* of trajectories, the Maximum Principle also is a *sufficient* condition for time-optimality. In this article, we generalize this result for Lagrange, Mayer, and Bolza optimization problems.[1]

Keywords: Optimization; Maximum principle; Dynamic programming; Bellman inequality; Variational calculus

AMS Math. Classification Index: Primary: 05C38, 15A15; Secondary: 05A15, 15A18

1. BELLMAN'S DYNAMIC PROGRAMMING FOR OPTIMIZATION PROBLEM IN LAGRANGE FORM

Consider controlled object

$$\dot{x} = f(x, u), \tag{1}$$

where $x = (x^1, \dots, x^n)^T \in R^n$ is the *state* of the object and $u = (u^1, \dots, u^r)^T$ is the *control* that can run over a given set $U \subset R^r$:

$$u \in U. \tag{2}$$

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[1] The research supported by Deutsche Forschunsgemeinschaft.

The right-hand side $f(x,u)=(f^1(x,u),\dots,f^n(x,u))^T$ is assumed to be smooth with respect to x and continuous with respect to x, u. The upper index T is used, since we consider x, u and $f(x,u)$ as *contravariant* vectors (*i.e.*, as column vectors).

A process $x(t)$, $u(t)$, $t_0 \le t \le t_1$, is said to be *admissible* if the function $u(t)$ with values in U is piecewise continuous, $x(t)$ is a continuous, piecewise differentiable function, and $\dot{x}(t)=f(x(t),u(t))$, $t_0 \le t \le t_1$, except for a finite number of discontinuity points of $u(t)$. The moments t_0, t_1 are not fixed, *i.e.*, they can vary for different admissible processes.

We assume that a terminal set $M_1 \subset R^n$ is given. Denote by G the *controllability region*, *i.e.*, the set of all initial points $x_0 \in R^n$ which can be transfered to M_1. In other words, $x_0 \in G$ if either $x_0 \in M_1$ or there exists an admissible process $x(t)$, $u(t)$, $t_0 \le t \le t_1$, such that $x(t_0)=x_0$ and $x(t_1) \in M_1$. In application, M_1 usually is a smooth manifold. But in this Section, we only assume that M_1 is *closed in* G, *i.e.*, for every compact set $P \subset G$, the intersection $M_1 \cap P$ is compact, too.

Moreover, a function $f^0(x,u)$ is given that also is smooth with respect to x and continuous with respect to $x, u\,(x \in G, u \in U)$. For every admissible process $x(t), u(t), t_0 \le t \le t_1$, define *Lagrange performance index*

$$J^L = \int_{t_0}^{t_1} f^0(x(t),u(t))dt. \tag{3}$$

Let $x(t)$, $u(t)$, $t_0 \le t \le t_1$, be an admissible process with $x(t_1) \in M_1$. We say that the process is realized *within* $G \backslash M_1$ if $x(t) \in G \backslash M_1$ for $t_0 \le t < t_1$.

Lagrange Optimization Problem Find admissible control transferring a given initial point x_0 to M_1 within $G \backslash M_1$ such that functional (3) attains its minimal value. The process that solves the problem is said to be *L-optimal.*

In Example 1 (Section 5), we illustrate the requirement that the control transfers x_0 to M_1 *within* $G \backslash M_1$.

Now we explain Bellman's idea for solution. Let $x(t)$, $u(t)$, $t_0 \le t \le t_1$, be the optimal process transferring a point x_0 to M_1 within $G \backslash M_1$. We put

$$\omega(x_0) = -\int_{t_0}^{t_1} f^0(x(t),u(t))dt. \tag{4}$$

Assume that $\omega(x)$ is smooth on $G\backslash M_1$ and continuous on G. Starting from x_0 and moving under action of a constant control $u_0 \in U$ during a time $dt > 0$, we arrive to the point $x_0 + dx$ where $dx = f(x_0, u_0)dt$. Assume, furthermore, that from $x_0 + dx$ we move optimally to M_1. Then, during the whole moving $x_0 \to (x_0 + dx) \to M_1$, we realize the value $f^0(x_0, u_0)dt - \omega(x_0 + dx)$ of the functional (3). Evidently, this value is not less than the *optimal* value $-\omega(x_0)$:

$$-\omega(x_0) \leq f^0(x_0, u_0)dt - \omega(x_0 + dx).$$

Since

$$\omega(x_0 + dx) - \omega(x_0) = \sum_{i=1}^{n} \frac{\partial \omega(x_0)}{\partial x^i} dx^i = \sum_{i=1}^{n} \frac{\partial \omega(x_0)}{\partial x^i} f^i(x_0, u_0)dt,$$

we obtain, replacing x_0, u_0 by x, u, respectively, *Bellman inequality*

$$\sum_{i=1}^{n} \frac{\partial \omega(x)}{\partial x^i} f^i(x, u) \leq f^0(x, u), \quad x \in G \setminus M_1, \ u \in U. \tag{5}$$

Along the *optimal* process $x(t)$, $u(t)$, $t_0 \leq t \leq t_1$, transferring the point x_0 to M_1 within $G\backslash M_1$, Bellman inequality turns into the equality:

$$\sum_{i=1}^{n} \frac{\partial \omega(x(t))}{\partial x^i} f^i(x(t), u(t)) = f^0(x(t), u(t)), \quad t_0 \leq t \leq t_1.$$

The aforesaid gives the main idea of Bellman Dynamic Programming (we don't consider here the Dynamic Programming Method for *discrete* processes and for other extremal problems [1]).

Remark that in the classical Variational Calculus, the set $U \subset R^n$ is *open* and (assuming that for every $x_0 \in G\backslash M_1$ the optimal control exists) the function $\omega(x)$ is *smooth*. But for non-classical case (say if U is a *closed* subset of R^r), in general, Bellman function $\omega(x)$ is non-smooth (*cf.* for example, [10, 11]), *i.e.*, it is impossible to write and to apply Bellman inequality (5).

2. IMPROVED FORM OF DYNAMIC PROGRAMMING

First we formulate a more general problem:

Auxiliary Optimization Problem Let $G \subset R^n$ be the controllability region for controlled object (1), (2). Let, furthermore, $g(x)$ be a smooth function defined on G. Instead of (3), consider performance index

$$J^A = g(x_0) + \int_{t_0}^{t_1} f^0(x(t), u(t))dt. \tag{6}$$

The problem is: Find an admissible process transferring a given initial point x_0 to M_1 within $G \backslash M_1$ such that functional (6) attains its minimal value. The process solving the problem is said to be A-optimal.

Certainly, Auxiliary optimization problem and Lagrange one are equivalent. Indeed, for every fixed initial point x_0, functional (6) differs from (3) only by the constant summand $g(x_0)$, and consequently both the functionals attain their minimal values for the same process $u(t)$, $x(t)$. Nevertheless, the additional summand $g(x_0)$ in (6) is convenient for Mayer and Bolza optimization problems considered below.

DEFINITION 1 Let K be a bounded, closed, s-dimensional convex polyhedron and φ be a smooth, nondegenerate, one-to-one mapping of K onto its image $\varphi(K) \subset R^n$. Then $\varphi(K)$ is said to be a *curved s-dimensional polyhedron* in R^n. Let, furthermore, $G \subset R^n$ be an open set. A set $Q \subset G$ is said to be *piecewise smooth* if Q is representable as the union of a family of curved polyhedra such that every compact set $M \subset G$ has points in common only with a finite number of them. If each polyhedron in this representation has dimension $\leq q$, then Q is a *piecewise smooth set of dimension* $\leq q$. It is clear that every piecewise smooth set $Q \subset G$ is closed in G.

MAIN LEMMA *Let $G \subset R^n$ be an open set and $Q \subset G$ be a piecewise smooth set of dimension $\leq n-1$. Let, furthermore, $\omega(x)$ be a continuous scalar function defined on G such that on $G \backslash (M_1 \cup Q)$ the function $\omega(x)$ is smooth and satisfies Bellman inequality*

$$\sum_{i=1}^{n} \left(\frac{\partial \omega(x)}{\partial x^i} + \frac{\partial g(x)}{\partial x^i} \right) f^i(x, u) \leq f^0(x, u). \tag{7}$$

Then for each admissible process $x(t)$, $u(t)$, $t_0 \leq t \leq t_1$, where $x(t)$ is situated in $G \backslash M_1$ for $t_0 \leq t < t_1$, the following estimate holds:

$$g(x(t_0)) + \int_{t_0}^{t_1} f^0(x(t), u(t))dt \geq \omega(x(t_1)) - \omega(x(t_0)) + g(x(t_1)).$$

Proof Let ε be a positive number. The trajectory $x(t)$, $t_0 \leq t \leq t_1 - \varepsilon$, has no points in common with M_1. Since M_1 is closed in the open set G, there exists a neighborhood W of the point $x_0 = x(t_0)$ such that every solution $\bar{x}(t)$, $t_0 \leq t \leq t_1 - \varepsilon$, of the equation $\dot{x} = f(x, u(t))$ with any initial condition $\bar{x}(t_0) \in W$ has no common points with M_1. Furthermore, by Lemma 3.18 in monograph [4] (*cf.* also [3]), there is a point $\bar{x}_0 \in W$ such that the solution $\bar{x}(t)$, $t_0 \leq t \leq t_1 - \varepsilon$, of the equation $\dot{x} = f(x, u(t))$ with $\bar{x}(t_0) = \bar{x}_0$ has only a finite number of points in common with Q. In other words, there are moments $\theta_1 < \cdots < \theta_k < t_1 - \varepsilon$ such that $x(t) \in G \backslash (M_1 \cup Q)$ for $t \in [t_0, t_1 - \varepsilon] \backslash \{\theta_1, \ldots, \theta_k\}$. Denote t_0, $t_1 - \varepsilon$ by θ_0, θ_{k+1}, respectively. Then

$$x(t) \in G \backslash (M_1 \cup Q) \quad \text{for } \theta_i < t < \theta_{i+1}, \ i = 0, 1, \ldots, k.$$

For any moments τ_i, τ_{i+1} with $\theta_i < \tau_i < \tau_{i+1} < \theta_{i+1}$ we have

$$\begin{aligned} \frac{d}{dt}(\omega(\bar{x}(t)) + g(\bar{x}(t))) &= \sum_{i=1}^{n} \left(\frac{\partial \omega(\bar{x}(t))}{\partial x^i} + \frac{\partial g(\bar{x}(t))}{\partial x^i} \right) \frac{d\bar{x}^i(t)}{dt} \\ &= \sum_{i=1}^{n} \left(\frac{\partial \omega(\bar{x}(t))}{\partial x^i} + \frac{\partial g(\bar{x}(t))}{\partial x^i} \right) f^i(\bar{x}(t), u(t)) \\ &\leq f^0(\bar{x}(t), u(t)) \end{aligned}$$

on the segment $[\tau_i, \tau_{i+1}]$ (*cf.* (7)). Integrating, we obtain

$$\begin{aligned} &(\omega(\bar{x}(\tau_{i+1})) + g(\bar{x}(\tau_{i+1}))) - (\omega(\bar{x}(\tau_i)) + g(\bar{x}(\tau_i))) \\ &\quad \leq \int_{\tau_i}^{\tau_{i+1}} f^0(\bar{x}(t), u(t)) dt. \end{aligned}$$

As $\tau_i \to \theta_i$, $\tau_{i+1} \to \theta_{i+1}$, we conclude (by continuity of the function ω)

$$\begin{aligned} &(\omega(\bar{x}(\theta_{i+1})) + g(\bar{x}(\theta_{i+1}))) - (\omega(\bar{x}(\theta_i)) + g(\bar{x}(\theta_i))) \\ &\quad \leq \int_{\theta_i}^{\theta_{i+1}} f^0(\bar{x}(t), u(t)) dt, \quad i = 0, 1, \ldots, k. \end{aligned}$$

Summing up these inequalities over $i = 0, 1, \ldots, k$, we find

$$\begin{aligned} \int_{t_0}^{t_1 - \varepsilon} f^0(\bar{x}(t), u(t)) dt \geq\; &(\omega(\bar{x}(t_1 - \varepsilon)) + g(\bar{x}(t_1 - \varepsilon))) \\ &- (\omega(\bar{x}(t_0)) + g(x(t_0))). \end{aligned}$$

Finally, as $\bar{x}_0 \to x_0$ (*i.e.*, $\varepsilon \to 0$ and the neighborhood W becomes less), we obtain the inequality indicated in the Main Lemma. ■

Now we are able to establish a sufficient condition for A-optimality in improved form of Dynamic Programming.

THEOREM 1 *Assume that the controllability region for the controlled object* (1), (2) *with the terminal set* M_1 *is an open set* $G \subset R^n$ *and* M_1 *is closed in G. Assume, furthermore, that a continuous scalar function* $\omega(x)$ *is defined on G such that*

(i) $\omega(x) = -g(x)$ *on* M_1;
(ii) *for every point* $x_0 \in G \backslash M_1$, *there exists an admissible control* $u(t; x_0)$ *transferring* x_0 *to* M_1 *such that the corresponding trajectory* $x(t, x_0)$ *is situated in* $G \backslash M_1$ *except for the terminal point and the corresponding value of functional* (6) *is equal to* $-\omega(x_0)$;
(iii) *there exists a piecewise smooth set* $Q \subset G$ *with dim* $Q \leq n-1$ *such that on* $G \backslash (M_1 \cup Q)$ *the function* $\omega(x)$ *is smooth and for all* $x \in G \backslash (M_1 \cup Q)$, $u \in U$ *Bellman inequality* (7) *holds.*

Then all controls $u(t; x_0)$ *are* A*-optimal.*

Proof By Main Lemma, for every admissible control $u(t)$, $t_0 \leq t \leq t_1$, transferring x_0 to M_1 within $G \backslash M_1$, the estimate

$$g(x_0) + \int_{t_0}^{t_1} f^0(x(t), u(t))dt \geq \omega(x(t_1)) - \omega(x_0) + g(x(t_1)) = -\omega(x_0)$$

holds, *i.e.*, it is impossible to arrive to M_1 (within $G \backslash M_1$) with lesser value of functional (6) than $-\omega(x_0)$. Consequently all controls $u(t; x_0)$ are A-optimal. ■

For $g(x) \equiv 0$, the Auxiliary optimization problem turns into Lagrange one. Thus, putting $g(x) \equiv 0$ in Theorem 1, we obtain the following sufficient condition of optimality for Lagrange problem:

THEOREM 2 *Assume that the controllability region for the controlled object* (1), (2) *with the terminal set* M_1 *is an open set* $G \subset R^n$ *and* M_1 *is closed in G. Assume, furthermore, that a continuous scalar function* $\omega(x)$ *is defined on G such that*

(i) $\omega(x) = 0$ *on* M_1;
(ii) *for every point* $x_0 \in G\backslash M_1$, *there exists an admissible control* $u(t; x_0)$ *transferring* x_0 *to* M_1 *such that the corresponding trajectory* $x(t, x_0)$ *is situated in* $G\backslash M_1$ *except for the terminal point and the corresponding value of functional* (3) *is equal to* $-\omega(x_0)$;
(iii) *there exists a piecewise smooth set* $Q \subset G$ *with dim* $Q \leq n-1$ *such that on* $G\backslash(M_1 \cup Q)$ *the function* $\omega(x)$ *is smooth and for all* $x \in G\backslash(M_1 \cup Q)$, $u \in U$ *Bellman inequality* (5) *holds.*

Then all controls $u(t; x_0)$ *are L-optimal.* ■

In particular, if $f^0(x, u) \equiv 1$, then (3) has the form $J^L = \int_{t_0}^{t_1} dt = t_1 - t_0$, *i.e.*, Lagrange optimization problem is reduced to time-optimality. Thus from Theorem 2 we obtain the following result (proved earlier [3, 4]):

THEOREM 3 *Assume that the controllability region for the controlled object* (1), (2) *with the terminal set* M_1 *is an open set* $G \subset R^n$ *and* M_1 *is closed in G. Assume, furthermore, that a continuous scalar function* $\omega(x)$ *is defined on G such that*

(i) $\omega(x) = 0$ *on* M_1;
(ii) *for every point* $x_0 \in G\backslash M_1$ *there exists an admissible control* $u(t; x_0)$ *transferring* x_0 *to* M_1 *in the time* $-\omega(x_0)$;
(iii) *there exists a piecewise smooth set* $Q \subset G$ *with dim* $Q \leq n-1$ *such that on* $G\backslash(M_1 \cup Q)$ *the function* $\omega(x)$ *is smooth and for all* $x \in G\backslash(M_1 \cup Q)$, $u \in U$ *satisfies Bellman inequality*

$$\sum_{i=1}^{n} \frac{\partial \omega(x)}{\partial x^i} f^i(x, u) \leq 1.$$

Then all controls $u(t; x_0)$ *are time-optimal.* ■

We now consider one problem more.

Mayer Optimization Problem Let $g(x)$ be a smooth function defined on G. Instead of (3), consider *Mayer performance index*

$$J^M = g(x(t_1)). \tag{8}$$

The problem is: Find an admissible process transferring a given initial point x_0 to M_1 within $G\backslash M_1$ such that functional (8) attains its minimal value. The process solving the problem is said to be *M-optimal.*

To solve this problem, we replace the function $f^0(x,u)$ by

$$\frac{d}{dt}g(x) = \sum_{i=1}^{n} \frac{\partial g(x)}{\partial x^i} f^i(x,u). \tag{9}$$

Then the functional (6) takes the form

$$\begin{aligned} J^A &= g(x_0) + \int_{t_0}^{t_1} \frac{d}{dt} g(x) dt \\ &= g(x_0) + (g(x(t_1)) - g(x_0)) = g(x(t_1)) = J^M. \end{aligned}$$

Thus, replacing $f^0(x,u)$ by (9) in Theorem 1, we obtain the following result:

THEOREM 4 *Assume that the controllability region for the controlled object* (1), (2) *with the terminal set* M_1 *is an open set* $G \subset R^n$ *and* M_1 *is closed in* G. *Assume, furthermore, that a continuous scalar function* $\omega(x)$ *is defined on* G *such that*

(i) $\omega(x) = -g(x)$ *on* M_1;
(ii) *for every point* $x_0 \in G \backslash M_1$ *there exists an admissible control* $u(t; x_0)$ *transferring* x_0 *to* M_1 *such that the corresponding trajectory* $x(t, x_0)$ *is situated in* $G \backslash M_1$ *except for the terminal point and the corresponding value of functional* (8) *is equal to* $-\omega(x_0)$;
(iii) *there exists a piecewise smooth set* $Q \subset G$ *with* $\dim Q \le n-1$ *such that on* $G \backslash (M_1 \cup Q)$ *the function* $\omega(x)$ *is smooth and for all* $x \in G \backslash (M_1 \cup Q)$, $u \in U$ *satisfies Bellman inequality*

$$\sum_{i=1}^{n} \frac{\partial \omega(x)}{\partial x^i} f^i(x,u) \le 0. \tag{10}$$

Then all controls $u(t; x_0)$ *are M-optimal.*

Finally, consider Bolza optimization problem that is a combination of Lagrange and Mayer ones.

Bolza Optimization Problem Let $g(x)$ be a smooth function defined on G. Consider *Bolza performance index*:

$$J^B = g(x(t_1)) + \int_{t_0}^{t_1} f^0(x(t), u(t)) dt. \tag{11}$$

The problem is: Find an admissible process transferring a given initial point x_0 to M_1 within $G\backslash M_1$ such that functional (11) attains its minimal value. The process solving the problem is said to be *B-optimal.*

To solve this problem, we replace the function $f^0(x,u)$ by

$$f^0(x,u)+\frac{d}{dt}g(x)=f^0(x,u)+\sum_{i=1}^{n}\frac{\partial g(x)}{\partial x^i}f^i(x,u). \tag{12}$$

Then functional (6) takes the form

$$\begin{aligned} J^A &= g(x_0)+\int_{t_0}^{t_1}\left(f^0(x,u)+\frac{d}{dt}g(x)\right)dt \\ &= g(x(t_1))+\int_{t_0}^{t_1} f^0(x,u)dt = J^B. \end{aligned}$$

Thus, replacing $f^0(x,u)$ by (12) in Theorem 1, we obtain the following result:

THEOREM 5 *Assume that the controllability region for the controlled object* (1), (2) *with the terminal set* M_1 *is an open set* $G\subset R^n$ *and* M_1 *is closed in* G. *Assume, furthermore, that a continuous scalar function* $\omega(x)$ *is defined on* G *such that*

(i) $\omega(x)=-g(x)$ *on* M_1;
(ii) *for every point* $x_0\in G\backslash M_1$ *there exists an admissible control* $u(t;x_0)$ *transferring* x_0 *to* M_1 *such that the corresponding trajectory* $x(t,x_0)$ *is situated in* $G\backslash M_1$ *except for the terminal point and the corresponding value of functional* (11) *is equal to* $-\omega(x_0)$;
(iii) *there exists a piecewise smooth set* $Q\subset G$ *with dim* $Q\leq n-1$ *such that on* $G\backslash(M_1\cup Q)$ *the function* $\omega(x)$ *is smooth and for all* $x\in G\backslash(M_1\cup Q)$, $u\in U$ *satisfies Bellman inequality*

$$\sum_{i=1}^{n}\frac{\partial\omega(x)}{\partial x^i}f^i(x,u)\leq f^0(x,u). \tag{13}$$

Then all controls $u(t;x_0)$ *are B-optimal.*

3. REGULAR SYNTHESIS

In general, it is difficult to find $\omega(x)$ by the Dynamic Programming Method. In the sequel, we show that the Maximum Principle allows to obtain a sufficient condition for optimality *without* knowledge the function $\omega(x)$. Moreover, this sufficient condition helps *to construct* $\omega(x)$.

Recall that in article [5], there is Pontryagin's conjecture that the Maximum Principle is a sufficient condition of optimality (in local sense). This conjecture is completely wrong: in pure form, the Maximum Principle *is not* a sufficient condition for optimality (in general, nonlinear case). We give below a sufficient condition of optimality, combining the statement of the Maximum Principle (that was proved in [2] as a correct *necessary* condition for optimality in global sense, *cf.* also [6, 13]) with Feldbaum's idea of optimal synthesis [12], and Bellman's Dynamic Programming Method [1]. We note that for the case of classical Variational Calculus (when the control region U is *open*), the sufficient condition given below is analogous to well known Weierstrass sufficient condition (using a field of extremal).

Moreover, remark that performance indices (3), (6), (8), (11) have integral form, *i.e.*, they are connected with the admissible process $u(t)$, $x(t)$, $t_0 \leq t \leq t_1$, in the large. At the same time, the Maximum Principle (*cf.* condition (E^L) below) has a local form, *i.e.*, it is formulated separately for every $t \in [t_0, t_1]$. By this reason, we don't consider below the Auxiliary optimization problem, since *locally* it does not differ from Lagrange one.

DEFINITION 2 We say that the for the controlled object (1), (2) with the performance index (3) and the terminal set M_1, the *regular synthesis* is realized if (i) the controllability region G is an open set in R^n; (ii) M_1 is a piecewise smooth set in G; (iii) some piecewise smooth sets $N \subset G$ and $P^0 \subset P^1 \subset \cdots \subset P^{n-1} \subset G$ are given such that the following conditions are satisfied:

(A) Each connected component of the set $P^i \backslash P^{i-1}$ is an i-dimensional smooth manifold, $i = 1, \ldots, n-1$. These connected components are said to be *i-dimensional cells*. The points of the set P^0 are 0-dimensional cells. The set N has dimension $\leq (n-1)$. The set $G \backslash (N \cup P^{n-1} \cup M_1)$ is open, and its connected components are *n-dimensional cells*. Each cell has no points in common with $N \cup M_1$.

(B) All cells are distributed on cells of the first type or the second type. All n-dimensional cells are cells of the first type. All 0-dimensional cells are cells of the second type. On every cell of the first type a function $v(x)$ is defined that is smooth with respect to x, u.

(C) If σ is an i-dimensional cell of the first type, then through every point of σ, a unique trajectory of the equation

$$\dot{x} = f(x, v(x)) \tag{14}$$

passes along σ. There is an $(i-1)$-dimensional cell $\Pi(\sigma)$ (or an $(i-1)$-dimensional curved polyhedron $\Pi(\sigma) \subset M_1$) such that every trajectory of (14), going along σ, leaves σ in a finite time and comes to a point of $\Pi(\sigma)$ in a nonzero angle.

Even if σ is an i-dimensional cell of the second type, then there exists an $(i+1)$-dimensional cell $\Sigma(\sigma)$ of the first type such that for every point $x \in \sigma$ there is a unique trajectory of Eq. (14) which emanates from x and goes along $\Sigma(\sigma)$; in this case the function $v(x)$ is smooth on $\sigma \cup \Sigma(\sigma)$.

(D) The above conditions allow to prolong the trajectories of Eq. (14) from cell to cell. Each trajectory prolonged in such a way goes along a finite number of cells and arrives to M_1. These trajectories are *indicated* ones. Thus a unique indicated trajectory emanates from every point of each cell and arrives to M_1. Also from every point of the set N a trajectory (maybe, non-unique) of the Eq. (14) emanates which arrives to the terminal set M_1 and is named *indicated*, too.

(E^L) Introduce the Hamiltonian function

$$H^L(\psi_0^L, \psi^L, x, u) = \sum_{\alpha=0}^{n} \psi_\alpha^L f^\alpha(x, u),$$

where ψ_0 is a real number and $\psi = (\psi_1, \ldots, \psi_n)$ is an auxiliary covariant vector (*i.e.*, a row vector). Furthermore, write the following *conjugate system*:

$$\dot{\psi}_i^L = -\frac{\partial H^L(\psi_0^L, \psi^L, x(t), v(x(t)))}{\partial x^i}$$
$$= -\sum_{\alpha=0}^{n} \psi_\alpha^L \frac{\partial f^\alpha(x(t), v(x(t)))}{\partial x^i}, \quad i = 1, \ldots, n. \tag{15}$$

Every indicated trajectory $x(t)$, $t_0 \le t \le t_1$, satisfies the Maximum Principle, *i.e.*, there exists a number $\psi_0^L \le 0$ and a solution $\psi^L(t) = (\psi_1^L(t), \dots, \psi_n^L(t))$ of (15), $\psi^L(t)$ being nontrivial if $\psi_0^L = 0$, such that for every $t \in [t_0, t_1]$ the maximum condition

$$\max_{u \in U} H^L(\psi_0^L, \psi^L(t), x(t), u) = H^L(\psi_0^L, \psi^L(t), x(t), v(x(t))) \equiv 0 \qquad (16)$$

holds. Moreover, if the indicated trajectory going along a cell σ of the first type arrives at a moment t to the point $x(t) \in \Pi(\sigma)$ and $\Pi(\sigma)$ has a positive dimension, then the vector $\psi^L(t_1)$ is orthogonal to $\Pi(\sigma)$ at the point $x(t)$ (the *transversality condition*).

(F) The integral (3) taken along indicated trajectories is a continuous function of the initial point x_0. In particular, if for a point $x_0 \in N$, there are several indicated trajectories emanating from x_0, then for all these trajectories, integral (3) takes the same value.

We remark that in condition (E^L) there is the upper index (L) which is related with the specific character of the Lagrange problem. The statements of the condition (E) for Mayer and Bolza problems are given below.

For $f^0 \equiv 1$ and $M_1 = \{x_1\}$, the above definition of the regular synthesis is contained (in a little varied form) in [3, 4]. As Brunowski proved [9], for every linear controlled object $\dot{x} = Ax + Bu$ with polyhedral control region U containing the origin in its interior, the regular synthesis for time-optimization problem exists (under some "general position condition").

4. SUFFICIENT CONDITIONS OF OPTIMALITY IN THE FORM OF REGULAR SYNTHESIS

THEOREM 6 *For the controlled object* (1), (2) *with the performance index* (3) *and a terminal set* M_1, *if the regular synthesis is realized, then all indicated trajectories of this synthesis are L-optimal.*

Proof In the proof, we call L-optimal processes simply *optimal*, since no other sense of optimality is considered. Moreover, we will not write the superscripts L for H^L and ψ^L because only this sense of H and ψ is considered.

If $x_0 \in G \backslash M_1$, denote by $-\omega(x_0)$ integral (3) along the indicated trajectory going from x_0 to M_1. Even if $x_0 \in M_1$, we put $\omega(x_0)=0$. Thus regular synthesis allows *to define* the function $\omega(x)$.

The following assertion is the key for the proof:

Let σ be an arbitrary n-dimensional cell and $x_0 \in \sigma$. Denote by $x(t)$, $t_0 \leq t \leq t_1$, the indicated trajectory that transfers x_0 to the terminal set M_1 and by $u(t)=v(x(t))$, $t_0 \leq t \leq t_1$, the corresponding control. Let $\psi_0 \leq 0$ and $\psi(t)$ satisfy the Maximum Principle, cf. condition (E^L) *in definition of regular synthesis. Then*

$$\psi(t_0) = -\psi_0 \text{ grad } \omega(x(t_0)). \tag{17}$$

We prove this assertion. Choose arbitrary real number t_0, and let $t_0+\theta_1(x_0)$ be the moment at which the solution $x(t)$ of (14), emanating at the moment t_0 from x_0, arrives to the cell $\Pi(\sigma)$, *i.e.*, $\theta_1(x_0)$ is the transferring time from x_0 to a point $\xi_1(x_0) \in \Pi(\sigma)$. By Theorem on dependence of solutions of differential equations on initial points, $\xi_1(x_0)$ and $\theta_1(x_0)$ are continuously differentiable functions of $x_0 \in \sigma$ (since the trajectories of (14) arrive to $\Pi(\sigma)$ in a nonzero angle). Furthermore, from the point $\xi_1(x_0)$ the solution $x(t)$ passes along the cell $\Pi(\sigma)$ (if $\Pi(\sigma)$ is a cell of the first type) or along the cell $\Sigma(\Pi(\sigma))$ (if $\Pi(\sigma)$ is a cell of the second type). As above, the point $\xi_2(x_0)$ at which the trajectory leaves the cell $\Pi(\sigma)$ (or $\Sigma(\Pi(\sigma))$) and the time $\theta_2(x_0)$ during which the trajectory passes along this cell are continuously differentiable functions of $x_0 \in \sigma$. Continuing, we obtain that the time $t_1-t_0=\theta_1(x_0)+\theta_2(x_0)+\cdots$ during which the indicated trajectory $x(t)$ passes from x_0 to M_1 is a continuously differentiable function of $x_0 \in \sigma$. Consequently $\omega(x_0)=-J^L$ (*cf.* (3)) is a continuously differentiable function of x_0 on the union of all n-dimensional cells, *i.e.*, on $G\backslash(M_1 \cup Q)$ where $Q=N \cup P^{n-1}$ is a piecewise smooth set of dimension $n-1$.

Let δw be an arbitrary vector whose length we consider as an infinitesimal of the first order. Denote by $\bar{x}(t)$ the indicated trajectory transferring the point $x(t_0)+\delta w$ to M_1. We may suppose (changing t by $t+\text{const}$, if necessary) that $\bar{x}(t)$ arrives to M_1 at the same moment t_1 as the trajectory $x(t)$. Denote by $\tau_0=t_0+\delta t$ the moment at which $\bar{x}(t)$ starts from $x(t_0)+\delta w$.

Furthermore, denote the difference $\bar{x}(t) - x(t)$ by $\delta x(t)$. Then

$$\delta w = \bar{x}(\tau_0) - x(t_0) = \delta x(t_0) + f(x(t_0), u(t_0))\delta t \tag{18}$$

(here and in the sequel we write the equalities up to infinitesimal of higher order). The trajectory $\bar{x}(t)$ corresponds to the control

$$\bar{u}(t) = v(\bar{x}(t)) = v(x(t) + \delta x(t)) = u(t) + \sum_{\beta=1}^{n} \frac{\partial v(x(t))}{\partial x^\beta} \delta x^\beta(t).$$

We have $\omega(x(t_1)) = \omega(\bar{x}(t_1)) = 0$ (since $\omega \equiv 0$ on M_1). Consequently,

$$\begin{aligned}\int_{t_0}^{t_1} f^0(\bar{x}(t), \bar{u}(t))dt &= -\omega(\bar{x}(\tau_0)) + \int_{t_0}^{\tau_0} f^0(\bar{x}(t), \bar{u}(t))dt \\ &= -\omega(x(t_0)) - \langle \text{grad } \omega(x(t_0)), \delta w\rangle \\ &\quad + f^0(\bar{x}(\tau_0), \bar{u}(\tau_0))\delta t \\ &= \int_{t_0}^{t_1} f^0(x(t), u(t))dt + f^0(x(t_0), u(t_0))\delta t \\ &\quad - \langle \text{grad } \omega(x(t_0)), \delta w\rangle.\end{aligned} \tag{19}$$

Let $\psi_0 \leq 0$ and $\psi(t) = (\psi_1(t), \ldots, \psi_n(t))$ be the solution of conjugate system (15) that corresponds to the indicated trajectory $x(t)$, *cf.* condition (E^L) in definition of regular synthesis.

Taking into account the transversality condition $\langle \psi(t_1), \delta x(t_1)\rangle = 0$ and the equality

$$\langle \psi(t_0), f(x(t_0), u(t_0))\rangle = \sum_{i=1}^{n} \psi_i(t_0) f^i(x(t_0), u(t_0)) = -\psi_0 f^0(x(t_0), u(t_0))$$

(*cf.* (16)), we obtain by (18)

$$\begin{aligned}-\langle \psi(t_0), \delta w\rangle &= \langle \psi(t_1), \delta x(t_1)\rangle - \langle \psi(t_0), \delta x(t_0)\rangle \\ &\quad - \langle \psi(t_0), f(x(t_0), u(t_0))\rangle \delta t \\ &= \int_{t_0}^{t_1} \frac{d}{dt}\langle \psi(t), \delta x(t)\rangle dt + \psi_0 f^0(x(t_0), u(t_0))\delta t \\ &= \int_{t_0}^{t_1} \sum_{i=0}^{n}\sum_{\gamma=1}^{r} \left(\psi_i \frac{\partial f^i}{\partial u^\gamma}\right)\delta u^\gamma dt - \int_{t_0}^{t_1} \sum_{\gamma=1}^{r} \left(\psi_0 \frac{\partial f^0}{\partial u^\gamma}\right)\delta u^\gamma dt \\ &\quad + \psi_0 f^0(x(t_0), u(t_0))\delta t \\ &= \int_{t_0}^{t_1} \delta H dt - \int_{t_0}^{t_1} \psi_0 \, \delta f^0 dt + \psi_0 f^0(x(t_0), u(t_0))\delta t.\end{aligned}$$

The first summand in the right-hand side is *nonpositive* (by the maximum condition (16)). Hence, by (19),

$$-\langle\psi(t_0),\delta w\rangle \leq -\int_{t_0}^{t_1}\psi_0\,\delta f^0 dt + \psi_0 f^0(x(t_0),u(t_0))\delta t$$
$$= \psi_0\langle \text{grad } \omega(x(t_0)),\delta w\rangle,$$

i.e.,

$$\langle\psi(t_0) + \psi_0 \text{ grad } \omega(x(t_0)),\delta w\rangle \geq 0. \tag{20}$$

Since inequality (20) holds for every vector δw, this implies (17).

Now we can complete the proof of Theorem 6. In the notation of the key-assertion, we obtain from (17):

$$H(\psi(t_0),x(t_0),u) = \psi_0 f^0(x(t_0),u) + \sum_{i=1}^{n}\psi_i(t_0)f^i(x(t_0),u)$$
$$= |\psi_0|\left(-f^0(x(t_0),u) + \sum_{i=1}^{n}\frac{\partial\omega(x(t_0))}{\partial x^i}f^i(x(t_0),u)\right). \tag{21}$$

Now we consider three possible cases:

(a) $\psi_0 \neq 0$. Then (21) implies (by the maximum condition (16)) that at the point $x_0 = x(t_0)$ and for every it $u \in U$ Bellman inequality (5) holds.

(b) $\psi_0 = 0$, but in every neighborhood of x_0 there are points for which the corresponding ψ_0, $\psi(t)$ satisfy the condition $\psi_0 \neq 0$. In other words, there is a sequence $x^{(1)}, x^{(2)}, \ldots$ convergent to x_0 such that at every point of this sequence, according to the case (a), Bellman inequality (5) holds. For every fixed $u \in U$, we obtain, as $x^{(k)} \to x_0$, that at the point x_0 the Bellman inequality (5) holds for arbitrary taken $u \in U$.

(c) $\psi_0 = 0$ for every point $\bar{x}_0$ in a neighborhood $W \subset \sigma$ of the point x_0 where σ is the n-dimensional cell containing x_0. But we show that this case is impossible. Indeed, in this case, by (16),

$$\sum_{i=1}^{n}\bar{\psi}_i(t)f^i(\bar{x}(t),v(\bar{x}(t))) = 0 \tag{22}$$

where $\bar{x}(t)$ is the indicated trajectory starting from an arbitrary point $\bar{x}_0 \in W$ and $\bar{\psi}(t) = (\bar{\psi}_1(t), \dots, \bar{\psi}_n(t))$ is the corresponding solution of the conjugate system, *cf.* condition (E^L). In other words, $\bar{\psi}(t) \perp f(\bar{x}(t), v(\bar{x}(t)))$. Let $\tau(\bar{x}_0)$ be the moment when the indicated trajectory $\bar{x}(t)$ arrives to $(n-1)$-dimensional cell $\Pi(\sigma)$. Assume at first that $\Pi(\sigma)$ is a cell of the first type. Then at the moment $\tau(\bar{x}_0) + 0$ the vector $\bar{\psi}(t)$ is orthogonal to $f(\bar{x}(t), v(\bar{x}(t)))$ where the trajectory $\bar{x}(t)$ is going along the cell $\Pi(\sigma)$. This is true for every point $\bar{x}_0 \in W$ and hence, at the moment $\tau(\bar{x}_0) + 0$, the vector $\bar{\psi}(t)$ is orthogonal to the cell $\Pi(\sigma)$. But then equality (22) taken for the moment $\tau(\bar{x}_0) - 0$ means that the trajectory $\bar{x}(t)$ *touches* the cell $\Pi(\sigma)$ at the moment of arrival to this cell, what is excluded by condition (C) in the definition of the regular synthesis.

Even if $\Pi(\sigma)$ is a cell of the second type, then we follow the indicated trajectory along the cell $\Sigma(\Pi(\sigma))$ *etc.*, until we arrive to an $(n-1)$-dimensional cell of the *first type*, and we obtain the same contradiction.

Thus Bellman inequality (5) holds inside every n-dimensional cell. Now it follows from Theorem 2 that all indicated trajectories are L-optimal. ■

Remark It is necessary to give an improvement of the above reasoning. Consider the moments $\tau_1, \dots, \tau_k$ at which the indicated trajectory $x(t)$, $t_0 \le t \le t_1$, leaves cells through which it passes:

$$\tau_1 = t_0 + \theta_1(x_0), \tau_2 = t_0 + \theta_1(x_0) + \theta_2(x_0), \dots, \tau_k = t_1.$$

The trajectory $\bar{x}(t)$ leaves the same cells at moments $\bar{\tau}_1, \bar{\tau}_2, \dots, \bar{\tau}_k = \tau_k$ which are *close* to $\tau_1, \tau_2, \dots, \tau_k$, not coinciding with them. Hence the function $\delta x(t) = \bar{x}(t) - x(t)$ is *non-smooth* at the moments $\tau_1, \dots, \tau_{k-1}; \bar{\tau}_1, \dots, \bar{\tau}_{k-1}$, making the above proof of (17) incorrect. Nevertheless, the equality (17) is true. Indeed, denote by ε the length of the vector δw and consider the time-intervals $I_1, I_2, \dots, I_{k-1}$ of the length $2q\varepsilon$ with midpoints at the moments $\tau_1, \dots, \tau_{k-1}$. With suitable number $q > 0$, the moments $\tau_1, \dots, \tau_{k-1}; \bar{\tau}_1, \dots, \bar{\tau}_{k-1}$ are situated in the set $\Delta = I_1 \cup I_2 \cup \dots \cup I_{k-1}$. This means that on $\Theta = [t_0, t_1] \backslash \Delta$ the function $\delta x(t)$ is smooth, and the above calculation is correct. Furthermore, the integrals taken over Δ (in the above proof of (17)) have the value $o(\varepsilon)$, since the measure of Δ is $O(\varepsilon)$ and the integrand is $O(\varepsilon)$. Consequently we conclude that

$$\langle \psi(t_0) + \psi_0 \text{ grad } \omega(x(t_0)), \delta w \rangle + o(\varepsilon) \geq 0$$

(*cf.* (20)), what makes (17) correct, since $\|\delta w\| = \varepsilon$. ■

We now consider Mayer optimization problem (1), (2), (8) with the terminal set M_1 as in Section 2. Assume that $g(x)$ is a twice continuously differentiable function on G. The regular synthesis for this problem is defined as in Section 3, but with some variation in condition (E^L).

DEFINITION 3 We say that for the controlled object (1), (2) with performance index (8) and the terminal set M_1, the *regular synthesis* is realized if the controllability region G is an open set in R^n and, moreover, some piecewise smooth sets N and $P^0 \subset P^1 \subset \cdots \subset P^{n-1}$ are given in G such that conditions (A)–(D), (F) of Definition 2 are satisfied and the condition (E^L) holds in the following varied form:

(E^M) Introduce the Hamiltonian function

$$H^M(\psi^M, x, u) = \sum_{j=1}^{n} \psi_j^M f^j(x, u), \tag{23}$$

where $\psi^M = (\psi_1^M, \ldots, \psi_n^M)$ is an auxiliary vector. Furthermore, write the following *conjugate system*:

$$\begin{aligned} \dot{\psi}_i^M &= -\frac{\partial H^M(\psi^M, x(t), v(x(t)))}{\partial x^i} \\ &= -\sum_{j=1}^{n} \psi_j^M \frac{\partial f^j(x(t), v(x(t)))}{\partial x^i}, \quad i = 1, \ldots, n. \end{aligned} \tag{24}$$

Every indicated trajectory $x(t)$, $t_0 \leq t \leq t_1$, satisfies the Maximum Principle, *i.e.*, there exists a number $\lambda \geq 0$ and a solution $\psi^M(t) = (\psi_1^M(t), \ldots, \psi_n^M(t))$ of (24), $\psi^M(t)$ being nontrivial if $\lambda = 0$, such that for all $t \in [t_0, t_1]$ the maximum condition

$$\max_{u \in U} H^M(\psi^M(t), x(t), u) = H^M(\psi^M(t), x(t), v(x(t))) \equiv 0 \tag{25}$$

holds, and $\psi^M(t_1) + \lambda$ grad $g(x(t_1)) \perp M_1$ at the terminal point $x(t_1)$ (the *transversality condition*).

THEOREM 7 *For the controlled object* (1), (2) *with performance index* (8) *and a smooth terminal manifold* M_1, *if the function* $g(x)$ *is twice continuously differentiable and regular synthesis is realized, then all indicated trajectories of this synthesis are M-optimal.*

Proof As we have seen in Section 2, the considered Mayer's optimization problem is equivalent to Lagrange's problem (1), (2), (3) with

$$f^0(x,u) = \sum_{j=1}^{n} \frac{\partial g(x)}{\partial x^j} f^j(x,u). \tag{26}$$

By Theorem 6, every indicated trajectory $x(t)$, $t_0 \leq t \leq t_1$, satisfies the Maximum Principle, as in condition (E^L), *i.e.*, there exists a number $\psi_0^L \leq 0$ and a solution $\psi^L(t) = (\psi_1^L(t), \dots, \psi_n^L(t))$ of (15) where $\psi^L(t)$ is nontrivial if $\psi_0^L = 0$ such that maximum condition (16) holds and $\psi^L(t_1) \perp M_1$ at the terminal point $x(t_1)$ (the *transversality condition*).

Now we make a change of variables, writing $\psi_j^L + \psi_0^L (\partial g/\partial x^j) = \psi_j^M$. Then the Hamiltonian function indicated in condition (E^L) takes the form

$$\begin{aligned} H^L(\psi_0^L, \psi^L, x, u) &= \psi_0^L \sum_{j=1}^{n} \frac{\partial g(x)}{\partial x^j} + \sum_{j=1}^{n} \psi_j^L f^j(x,u) \\ &= \sum_{j=1}^{n} \left(\psi_j^L + \psi_0^L \frac{\partial g(x)}{\partial x^j} \right) f^j(x,u) = \sum_{j=1}^{n} \psi_j^M f^j(x,u) \\ &= H^M(\psi^M, x, u) \end{aligned}$$

(*cf.* (23)), *i.e.*, maximum condition (16) turns into (25). Furthermore (everywhere the arguments are $x(t)$ and $v(x(t))$)

$$\begin{aligned} \dot{\psi}_i^M &= \dot{\psi}_i^L + \psi_0^L \frac{d}{dt} \frac{\partial g}{\partial x^i} = -\sum_{j=0}^{n} \psi_j^L \frac{\partial f^j}{\partial x^i} + \psi_0^L \sum_{j=1}^{n} \frac{\partial^2 g}{\partial x^i \partial x^j} f^j \\ &= -\psi_0^L \frac{\partial f^0}{\partial x^i} - \sum_{j=1}^{n} \psi_j^L \frac{\partial f^j}{\partial x^i} + \psi_0^L \sum_{j=1}^{n} \frac{\partial^2 g}{\partial x^i \partial x^j} f^j \\ &= -\psi_0^L \left(\sum_{j=1}^{n} \frac{\partial^2 g}{\partial x^j \partial x^i} f^j + \sum_{j=1}^{n} \frac{\partial g}{\partial x^j} \cdot \frac{\partial f^j}{\partial x^i} \right) - \sum_{j=1}^{n} \psi_j^L \frac{\partial f^j}{\partial x^i} \end{aligned}$$

$$+\psi_0^L\sum_{j=1}^{n}\frac{\partial^2 g}{\partial x^i\partial x^j}f^j$$
$$=-\sum_{j=1}^{n}\left(\psi_j^L+\psi_0^L\frac{\partial g}{\partial x^j}\right)\frac{\partial f^j}{\partial x^i}=-\sum_{j=1}^{n}\psi_j^M\frac{\partial f^j}{\partial x^i},$$

i.e., we obtain conjugate system (24). (We remark that $(\partial^2 g/\partial x^j\partial x^i)=(\partial^2 g/\partial x^i\partial x^j)$, since $g(x)$ is twice continuously differentiable.) Finally, the transversality condition $\psi^L(t_1)\perp M_1$ takes the form $\psi^M(t_1)+\lambda\,\mathrm{grad}\,g(x(t_1))\perp M_1$, where $\lambda=-\psi_0^L\geq 0$. Consequently Theorem 6 implies Theorem 7. ■

In conclusion, we consider Bolza's optimization problem (1), (2), (11) with the terminal set M_1 as in Section 2. The following theorem is deduced from Theorem 6 by a reasoning quite analogous to the previous proof, and we give here only the statement.

Definition 4 We say that for the controlled object (1), (2) with performance index (11) and the terminal set M_1, the *regular synthesis* is realized if the controllability region G is an open set in R^n and, moreover, some piecewise smooth sets N and $P^0\subset P^1\subset\cdots\subset P^{n-1}$ are given in G such that conditions (A)–(D), (F) of Definition 2 are satisfied and condition (E^L) holds in the following varied form:

(E^B) Introduce the Hamiltonian function

$$H^B(\psi_0^B,\psi^B,x,u)=\sum_{j=0}^{n}\psi_j^B f^j(x,u),$$

where ψ_0^B is a real number and $\psi^B=(\psi_i^B,\ldots,\psi_n^B)$ is an auxiliary vector. Furthermore, write the following *conjugate system*:

$$\dot\psi_i^B=-\frac{\partial H^B(\psi_0^B,\psi^B,x(t),v(x(t)))}{\partial x^i}$$
$$=-\sum_{j=0}^{n}\psi_j^B\frac{\partial f^j(x(t),v(x(t)))}{\partial x^i},\quad i=1,\ldots,n. \tag{27}$$

Every indicated trajectory $x(t)$, $t_0\leq t\leq t_1$, satisfies the Maximum Principle, *i.e.*, there exists a number $\psi_0^B\leq 0$ and a solution $\psi^B(t)=(\psi_1^B(t),\ldots,\psi_n^B(t))$ of (27), $\psi^B(t)$ being nontrivial if $\psi_0^B=0$, such that for all $t \in [t_0, t_1]$ the maximum condition

$$\max_{u \in U} H^B(\psi_0^B, \psi^B(t), x(t), u) = H^B(\psi_0^B, \psi^B(t), x(t), v(x(t))) \equiv 0$$

holds, and $\psi^B(t_1) + \lambda$ grad $g(x(t_1)) \perp M_1$ at the terminal point $x(t_1)$ (the *transversality condition*).

THEOREM 8 *For the controlled object* (1), (2) *with the performance index* (11) *and a smooth terminal manifold* M_1, *if the function* $g(x)$ *is twice continuously differentiable and regular synthesis is realized, then all indicated trajectories of this synthesis are B-optimal.*

5. EXAMPLES

Example 1 In R^n, consider the controlled object

$$\dot{x}^i = u^i, \quad i = 1, \ldots, n; \quad \|u\| \leq 1,$$

with terminal set $M_1 = \{x : \|x\| \leq 1\}$ and functional (3) of the form

$$f^0(x, u) = \begin{cases} 1 & \text{for } x \notin M_1, \\ 1 - 2(1 - \|x\|^2)^2 & \text{for } x \in M_1, \end{cases}$$

where $\|u\|^2 = (u^1)^2 + \cdots + (u^n)^2$, $\|x\|^2 = (x^1)^2 + \cdots + (x^n)^2$.

Let $u(t)$, $x(t)$, $t_0 \leq t \leq t_1$, be an admissible process transferring an initial point $x_0 \in R^2 \backslash M_1$ to the terminal set M_1, *i.e.*, $x(t_0) = x_0$, $x(t_1) \in M_1$. If the process transfers x_0 to M_1 *within* $R^2 \backslash M_1$ *i.e.*, $x(t) \notin M_1$ for $t_0 \leq t < t_1$, then $f^0(x, u) \equiv 1$ during the process, and L-optimality coincides with time-optimality. In this case, it is easily shown that the optimal process has the form

$$u(t) = -\frac{x_0}{\|x_0\|} = \text{const},$$

and the minimal value of functional (3) is equal to $\|x_0\| - 1$.

On the other hand, consider a process that is defined in a larger segment, coinciding with the above time-optimal process during the time $\|x_0\| - 1$ and situated in M_1 after arriving to the boundary bd M_1 of the set M_1. In the set $\{x : \|x\| < (1/2)\}$, the function $f^0(x, u)$ is *negative*, and it is possible to move in M_1 any time, making the functional J^L negative with any great value $|J^L|$. This means that if we consider admissible processes in whole plane R^2 (including the points in the interior int M_1 of the set M_1), then minimum of J^L *does not exist*. This shows that in Lagrange optimization problem, it is essential to limit ourselves by the processes running only *within* $G\backslash M_1$ (except for the terminal moment). It is possible to construct an analogous example with a one-dimensional terminal set M_1.

Example 2 In Cartesian coordinate system $x^1=x$, $x^2=y$ in R^2, consider (as in [10, 11]) the controlled object

$$\dot{x}=y, \quad \dot{y}=u, \quad |u|\leq 1,$$

with $M_1=\{(0,0)\}, f^0\equiv 1$. Thus the optimization problem is: Starting from a point $x_0=(a,b)\in R^2$, arrive to the origin $x_1=(0,0)$ in the shortest time. The Maximum Principle [2, 6, 13] gives the simplest way to obtain the solution. We explain the result.

Consider the curve $\Gamma=\Gamma^-\cup\Gamma^+$, where each curve Γ^-, Γ^+ is a half-parabola:

$$\Gamma^-: x=-\frac{1}{2}y^2, \quad y\geq 0; \quad \Gamma^+: x=\frac{1}{2}y^2, \quad y\leq 0.$$

If the initial point $x_0=(a,b)$ is situated above Γ (or in the arc Γ^-), the optimal process passes at first along the parabola

$$x=-\frac{1}{2}y^2+\left(a+\frac{1}{2}b^2\right)$$

(with $\dot{y}=u\equiv -1$) from x_0 till the intersection with Γ^+ at a point (c,d), and after that along Γ^+ (with $\dot{y}=u\equiv 1$) till the arrival to the origin. Hence the transferring time is equal to $(b-d)+|d|$, *i.e.*, as it is easily shown,

$$-\omega(x_0)=b+2\sqrt{a+\frac{1}{2}b^2}.$$

Even if x_0 is situated below Γ (or in the arc Γ^+), then

$$-\omega(x_0)=-b+2\sqrt{-a+\frac{1}{2}b^2}.$$

This is obtained with the help of the Maximum Principle. But, as soon as we already *know* $\omega(x_0)$, it is possible to justify the optimality. Indeed, the two above expressions for $\omega(x_0)$ *coincide* along Γ, *i.e.*, the function $\omega(x_0)$ is continuous in the whole plane R^2. Furthermore, $\omega(x_0)$ is smooth in $R^2\backslash\Gamma$. It is easily shown with the help of the above expression for $\omega(x_0)$ that the function $\omega(x_0)$ satisfies Bellman inequality (5) in $R^2\backslash\Gamma$. Finally, for every point $x_0 \in R^2$ there is a trajectory transferring x_0 to the origin in the time $-\omega(x_0)$. Hence, by Theorem 3, these trajectories are time-optimal.

Example 3 In [7], we considered the controlled object

$$\dot{x}^1 = f^1(x^1, x^2, u) = x^2, \quad \dot{x}^2 = f^2(x^1, x^2, u);$$
$$-1 \leq u \leq 1; \quad M_1 = \{(0,0)\}, \ f^0 \equiv 1,$$

that satisfies the following four conditions:

(α) $f^2(0,0,1) > 0, f^2(0,0,-1) < 0$;
(β) $f^2(x^1, x^2, u)$ increases with respect to $u \in [-1,1]$ for any fixed x^1, x^2;
(A) no trajectory comes to infinity in a finite time;
(B) there exists a continuous function $\varphi(x^1, x^2, u)$ with continuous derivatives $(\partial\varphi/\partial x^1)$, $(\partial\varphi/\partial x^2)$ such that

$$x^2 \frac{\partial \varphi}{\partial x^1} + f^2 \frac{\partial \varphi}{\partial x^2} + (\varphi)^2 - \varphi \frac{\partial f^2}{\partial x^2} - \frac{\partial f^2}{\partial x^1} \leq 0$$

for $u = \pm 1$ and any x^1, x^2.

In [7], it is shown that under these conditions a synthesis of Feldbaum's type is realized in the controllability region G of the object. Namely, denote by Γ^- the semitrajectory for $u \equiv -1$ and by Γ^+ the semitrajectory for $u \equiv 1$, both the semitrajectories ending at the origin. These semitrajectories are said to be 1-dimensional cells. The controllability region G is divided by $\Gamma = \Gamma^- \cup \Gamma^+$ into two 2-dimensional cells (above and below the curve Γ). Starting from any point $x_0 \in G$, it is possible to get the origin along an indicated trajectory (with no more than one switching), where $v(x) = -1$ above Γ and on Γ^-, whereas $v(x) = 1$ below Γ and on Γ^+. All indicated trajectories satisfy the Maximum Principle with respect to suitable ψ_0, $\psi(t)$. Theorem 6 implies that all indicated trajectories are time-optimal.

Example 4 In R^n, consider the controlled object

$$\dot{x}^i = u^i, \quad i = 1, \ldots, n, \tag{28}$$

where $U = M_1 \subset R^n$ is the unit ball $\{x : \|x\| \leq 1\}$. For conveniency, we consider R^n as self-conjugate space with orthonormal coordinate system in it. Hence covariant coordinates coincide with contravariant ones, and

$$\|x\|^2 = \langle x, x \rangle = (x^1)^2 + \cdots + (x^n)^2.$$

For this controlled object, we consider Lagrange's optimization problem (1), (2), (3) with the function

$$f^0(x, u) = \left(\sqrt{\|x\|}\right)^{-1} = \left((x^1)^2 + \cdots + (x^n)^2\right)^{-(1/4)}.$$

Thus the optimization problem is: Starting from a point $x_0 \in R^n \backslash M_1$, arrive to M_1 with minimal value of the integral

$$J^L = \int_{t_0}^{t_1} \frac{dt}{\sqrt{\|x(t)\|}} = \int_{t_0}^{t_1} \left((x^1(t))^2 + \cdots + (x^n(t))^2\right)^{-(1/4)} dt.$$

We introduce the function $\omega(x)$ (taking $\omega(x) \equiv 0$ on M_1) by the equality

$$-\omega(x) = 2\sqrt{\|x\|} - 2 = 2\left((x^1)^2 + \cdots + (x^n)^2\right)^{(1/4)} - 2 \quad \text{for} \quad x_0 \in R^n \setminus M_1,$$

without any discussion, why namely this function is taken. The function $\omega(x)$ is continuous in R^n and smooth in $R^n \backslash M_1$. It is easily shown that $\omega(x)$ satisfies Bellman inequality (5) in $R^n \backslash M_1$. Moreover, it is easily shown that for every $x_0 = a = (a^1, \ldots, a^n) \in R^n \backslash M_1$, there is a control $u(t) = (u^1(t), \ldots, u^n(t))$ transferring x_0 to M_1 with $J^L = -\omega(x_0)$:

$$u^i(t) = -\frac{a^i}{\|a\|} = \text{const}, \quad i = 1, \ldots, n.$$

Hence, by Theorem 2, these controls are L-optimal.

When we apply Theorem 6 (instead of Theorem 2), the reasoning takes another form. It is easily shown that the assumption $\psi_0 = 0$ is contradictory. Taking $\psi_0 = -1$, we obtain:

$$H = -f^0(x,u) + \langle \psi, u \rangle = -f^0(x,u) + \psi_1 u^1 + \cdots + \psi_n u^n,$$
$$\dot{\psi}_i = \frac{\partial f^0}{\partial x^i} = -\frac{x^i}{2\|x\|^{(5/2)}}, \quad i = 1, \ldots, n. \tag{29}$$

Let $u(t)$, $x(t)$, $0 \le t \le t_1$, be a process satisfying the Maximum Principle (*cf.* condition (E^L) in Definition 2). Denote by $p = (p^1, \ldots, p^n)$ the terminal point, *i.e.*, $p = x(t_1)$. The transversality condition means that the vector ψ is orthogonal to the sphere bd M_1 at the terminal point p, *i.e.*, $\psi(t_1) = \mu p$ where $\mu \neq 0$. Now maximum condition (16) implies $u(t_1) = p \operatorname{sign} \mu$. Thus at the terminal moment t_1, the three vectors $u(t_1)$, $x(t_1)$, $\psi(t_1)$ are collinear. By the character of Eqs. (28), (29) and the maximum condition $\langle \psi, x \rangle = \max$, this collinearity remains to be hold during all time. Consequently $u(t) = \pm p = \text{const}$.

Since the trajectory $x(t)$ arrives to the terminal point p from $R^n \backslash M_1$, we conclude that $\mu < 0$, *i.e.*, $u = -p$ during all time. Now we obtain $x(t) = -pt + \text{const}$. More detailed (taking into account that $x(t_1) = p$) we have

$$x = -pt + (p + pt_1). \tag{30}$$

These trajectories fill the n-dimensional cell $\sigma = R^n \backslash M_1$ in, and they satisfy the Maximum Principle. By Theorem 6, all these trajectories are L-optimal.

Thus, using Theorem 6, it is not necessary to know the function $\omega(x)$ and to check that it satisfies Bellman inequality. Moreover, now we can *calculate* $\omega(x)$. Indeed, by (30), the initial point $x(0) = a$ coincides with $p + pt_1$. Using this formula and the equalities

$$\|p\| = 1, \quad \|a\| = 1 + t_1, \quad \|x(t)\| = -t + 1 + t_1,$$

we obtain

$$\begin{aligned} -\omega(a) &= \int_0^{t_1} f^0(x(t), u(t))dt = \int_0^{t_1} \|x(t)\|^{-(1/2)} dt \\ &= \int_0^{t_1} (-t + 1 + t_1)^{-(1/2)} dt = (-2\sqrt{-t + 1 + t_1})_0^{t_1} \\ &= 2(\sqrt{\|a\|} - 1), \end{aligned}$$

i.e., we justified the choice of $\omega(x)$ that was made above.

We see that the sufficient condition of optimality in the form of the regular synthesis (Theorem 6) is more preferable than Dynamic Programming Method (Theorem 2).

Example 5 Consider in R^n the following controlled object with scalar control:

$$\dot{x}^i = (-2+u)x^i, \quad i=1,\ldots,n; \ -1 \le u \le 1. \tag{31}$$

As in the previous Example, the terminal set M_1 is the ball $\{x: \|x\| \le 1\}$. For this controlled object, we consider Lagrange optimization problem (1), (2), (3) with the function

$$f^0(x,u) = (3-u)(\langle x,x\rangle - 4).$$

Denote by W the ball $\{x: \|x\| \le 2\}$. We consider two n-dimensional cells $\sigma_1 = R^n \setminus W$ and $\sigma_2 = \text{int } W \setminus M_1$. The sphere bd W is the $(n-1)$-dimensional cell. Remark that $f^0(x,u)$ is positive on the cell σ_1 and negative on the cell σ_2.

We are going to show that the synthesis of optimal trajectories is given by Eq. (14) where $v(x) = -1$ on the cell σ_1 and $v(x) = 1$ on the cell σ_2.

Consider a trajectory $x(t)$, $-\infty < t \le 0$, that is going from infinity and arrives to a point $p \in$ bd M_1, *i.e.*, $x(0) = p$. According to (31), the part of this trajectory contained in σ_2 is defined by the equation $\dot{x} = -x$ (since $u = v(x) = 1$ on σ_2). Consequently $x(t) = e^{-t}p$ in σ_2. At the moment $\theta = -\ln 2$, this trajectory intersects the boundary of the ball W (since $x(\theta) = 2p$, *i.e.*, $\|x(\theta)\| = 2$). Hence for $-\infty < t < \theta$ the trajectory $x(t)$ is situated in σ_1, *i.e.*, it is defined by the equality $\dot{x} = -3x$ (since it $u = v(x) = -1$ on σ_1). The solution with the initial condition $x(\theta) = 2p$ has the form $x(t) = (1/4)e^{-3t}p$. Thus

$$x(t) = \begin{cases} (1/4)e^{-3t}p & \text{for } -\infty < t < \theta, \\ e^{-t}p & \text{for } \theta \le t \le 0. \end{cases}$$

This trajectory corresponds to the control

$$u(t) = \begin{cases} -1 & \text{for } -\infty < t < \theta, \\ 1 & \text{for } \theta \le t \le 0. \end{cases}$$

We are going to prove that this trajectory satisfies the Maximum Principle with respect to a solution $\psi(t)$ of the conjugate system.

Taking $\psi_0 = -1$ (as in the previous Example), we can write the Hamiltonian function and the conjugate system:

$$\begin{aligned} H &= -(3-u)(\langle x, x\rangle - 4) + (-2+u)\langle x, \psi\rangle, \\ \dot{\psi} &= (2-u)\psi + (3-u)\cdot 2x. \end{aligned} \tag{32}$$

Since $x(0)=p$, $u(0)=1$, the condition $H=0$ implies $\psi(0)=6p$ (taking into account the transversality condition $\psi(0)\perp \text{bd}\, M_1$ at the point $x(0)=p$). On the segment $\theta \le t \le 0$, the function $\psi(t)$ satisfies the equation $\dot{\psi} = \psi + 4x$ (*cf.* (32)) with the initial condition $\psi(0)=6p$, *i.e.*, $\psi(t) = (8e^t - 2e^{-t})p$ on this segment. It follows $\psi(\theta)=0$.

Furthermore, on the ray $-\infty < t \le \theta$ the function $\psi(t)$ satisfies the equation $\dot{\psi} = 3\psi + 8x$ (*cf.* (32)). Taking into account that $\psi(\theta)=0$, we obtain the corresponding solution:

$$\psi(t) = \begin{cases} ((64/3)e^{3t} - (1/3)e^{-3t})p & \text{for } -\infty < t < \theta, \\ (8e^t - 2e^{-t})p & \text{for } \theta \le t \le 0. \end{cases}$$

It is easily shown that the obtained functions $u(t)$, $x(t)$, $\psi(t)$ satisfy the condition $H\equiv 0$ (on the ray $-\infty < t < \theta$, and on the segment $\theta \le t \le 0$, too).

Furthermore, the maximum condition (16) means that

$$u = \text{sign}(\langle x, x+\psi\rangle - 4). \tag{33}$$

In the segment $\theta \le t \le 0$, the equality (33) takes the form $u=\text{sign}(4-e^{-2t})$, *i.e.*, $u=1$ on this segment, and the maximum condition holds. Furthermore, on the ray $-\infty < t < \theta$, the equality (33) takes the form $u=\text{sign}(1/48)(64-e^{-6t})=\text{sign}(64-e^{-6t})$, *i.e.*, $u=-1$ on this ray, and the maximum condition holds, too.

Thus the functions $u(t)$, $x(t)$, $\psi(t)$ satisfy the condition (E^L) in the Definition of regular synthesis. Denoting by l_p the ray consisting of all points $x=kp$ with $k\ge 1$, we see that the trajectory $x(t)$ moves along the ray l_p from infinity to the point p. These trajectories (taken for all $p\in \text{bd}\, M_1$) fill the set $R^n \backslash M_1$ in. By Theorem 6, all these trajectories are L-optimal.

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