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VECTOR-VALUED MEANS AND WEAKLY ALMOST PERIODIC FUNCTIONS

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ABSTRACT. A formula is set up between a vector-valued mean and scalar-valued means that enbles us translate many important results about scalar-valued means developed in [1] to vector-valued means. As applications of the theory of vector-valued means, we show that the definitions of a mean in [2] and [3] are equivalent and the space of vector-valued weakly almost periodic functions is admissible.

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Scalar-valued means have been much studied. However, little has been done on the vector-valued means. In this paper we develop the theory of vector-valued means.

In Lemma 1.4, we set up a formula between a vector-valued mean and scalar-valued means, by which we will be able to translate many important results about scalar-valued means developed in [1] to vector-valued means. We present these results in Sections 1, 2 and 3. As an application of the theory established in these sections, we investigate vector-valued weakly almost periodic functions in Section 4.

§1.Means on a Linear Subspace of $\mathcal{B}(S, X)$

Throughout this paper, S denotes a semigroup which need not have an identity, X denotes a Banach space and X^* is the dual space of X. $\mathcal{B}(S, X)$ denotes all of the bounded functions from S to X. When $X = \mathbb{C}$, we simply write $\mathcal{B}(S)$ for $\mathcal{B}(S, X)$. $\mathcal{A}$ denotes a linear subspace of $\mathcal{B}(S, X)$ containing the constant functions. $\mathcal{L}(\mathcal{A}, X)$ denotes all of the bounded linear mappings from $\mathcal{A}$ to X.

Let $f \in \mathcal{B}(S, X)$. Then the right (respectively, left) translate $R_s f$ of f by $s \in S$ is the map $R_s f(t) = f(ts)$ (respectively, $L_s f(t) = f(st)$) for all $t \in S$.

$\mathcal{A}$ is said to be right (respectively, left) translation invariant if $R_S\mathcal{A} = \{R_s f : s \in S, f \in \mathcal{A}\} \subset \mathcal{A}$ (respectively, $L_S\mathcal{A} = \{L_s f : s \in S, f \in \mathcal{A}\} \subset \mathcal{A}$). $\mathcal{A}$ is said to be translation invariant if it is both right and left translation invariant.

Definition 1.1 [2]. *A linear mapping $\mu : \mathcal{A} \to X$ is called a mean on $\mathcal{A}$ provided $\mu(f) \in \overline{co} f(S)$, for all $f \in \mathcal{A}$. Denote by $M(\mathcal{A})$ the set of all means on $\mathcal{A}$.*

If $\mathcal{A}$ is right (respectively, left) translation invariant, μ is said to be right (respectively, left) invariant if $\mu(R_s f) = \mu(f)$ (respectively, $\mu(L_s f) = \mu(f)$) for all $s \in S$ and $f \in \mathcal{A}$.

Remark 1.2. It follows from [1, 2.1.2] that Definition 1.1 will reduce to the definition of a scalar valued mean when $X = \mathbb{C}$.

Of course, the evaluation mapping $\epsilon : S \to \mathcal{L}(\mathcal{A}, X)$, defined by

$$\epsilon(s)(f) = f(s) \qquad (s \in S,\ f \in \mathcal{A})$$

is in $M(\mathcal{A})$, and if $\mu \in M(\mathcal{A})$ and $f \in \mathcal{A}$ is a constant function, then $\mu(f)$ is the constant.

The following proposition is obvious.

Proposition 1.3. *If $\mathcal{A}$ is a linear subspace of $\mathcal{B}(S, X)$ containing the constant functions, then each $\mu \in M(\mathcal{A})$ is in $\mathcal{L}(\mathcal{A}, X)$ with $\|\mu\| = 1$.*

For each $x^* \in X^*$,

$$x^*\mathcal{A} = \{x^* f = x^* \circ f : \ f \in \mathcal{A}\}$$

is a linear subspace of $\mathcal{B}(S)$.

Here we have adopted the definition in [2] of a mean on $\mathcal{A}$. [3] gives a definition of a mean in terms of a scalar–valued mean on $\overline{sp}(X^* \circ \mathcal{A}) = \overline{sp}\{x^*\mathcal{A} : \ x^* \in X^*\}$. In the next lemma, we set up a connection like this, and we will show in Theorem 1.7 that the definitions of a mean in [2] and [3] are equivalent. We will deal with other applications in §4.

Lemma 1.4. *Let $\mathcal{A}$ be a linear subspace of $\mathcal{B}(S, X)$. A mapping $\mu : \mathcal{A} \to X$ is in $M(\mathcal{A})$ if and only if, for each $x^* \in X^*$, there is a $\varphi_{\mu,x^*} \in M(x^*\mathcal{A})$ such that*

$$x^*\mu(f) = \varphi_{\mu,x^*}(x^* f) \qquad (f \in \mathcal{A}).$$

If $\mathcal{A}$ is right (left) translation invariant, then μ is right (left) invariant if and only if the φ_{μ,x^*}'s are right (left) invariant. Furthermore, the set $\varphi_\mu = \{\varphi_{\mu,x^*} : x^* \in X^*\}$ is uniquely determined by μ, i.e., $\varphi_{\mu,x^*} = \varphi_{\mu',x^*}$ for all $x^* \in X^*$ if and only if $\mu = \mu'$.

Proof. Sufficiency. First, μ is a linear mapping from $\mathcal{A}$ to X. In fact, for $f, g \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$,

$$\begin{aligned} x^*\mu(\alpha f + \beta g) &= \varphi_{\mu,x^*}(x^*(\alpha f + \beta g)) \\ &= \varphi_{\mu,x^*}(x^*(\alpha f)) + \varphi_{\mu,x^*}(x^*(\beta g)) \\ &= \alpha\varphi_{\mu,x^*}(x^* f) + \beta\varphi_{\mu,x^*}(x^* g) \\ &= \alpha x^*\mu(f) + \beta x^*\mu(g) \\ &= x^*(\alpha\mu(f) + \beta\mu(g)). \end{aligned}$$

The equality is true for all $x^* \in X^*$, therefore

$$\mu(\alpha f + \beta g) = \alpha\mu(f) + \beta\mu(g).$$

We claim that $\mu(f) \in \overline{co}f(S)$, for all $f \in \mathcal{A}$. If it is not true for some $f \in \mathcal{A}$, by the Hahn–Banach theorem there is an $x^* \in X^*$ such that

$$|x^*\mu(f)| > \sup_{s\in S}|x^*f(s)| = \|x^*f\|.$$

It follows from Remark 1.2 and Proposition 1.3 that $\varphi_{\mu,x^*} \in M(x^*\mathcal{A})$ is in $(x^*\mathcal{A})^*$ with $\|\varphi_{\mu,x^*}\| = 1$. So

$$|x^*\mu(f)| = |\varphi_{\mu,x^*}(x^*f)| \leq \|x^*f\|,$$

a contradiction.

Necessity. For each $x^* \in X^*$, define $\varphi_{\mu,x^*} \in (x^*\mathcal{A})^*$ by

$$\varphi_{\mu,x^*}(x^*f) = x^*\mu(f) \qquad (f \in \mathcal{A}).$$

φ_{μ,x^*} is well-defined on $x^*\mathcal{A}$. For, if $x^*f = 0$ for some $f \in \mathcal{A}$, then $f(S) \subset N(x^*)$, the null subspace of x^*, so $\varphi_{\mu,x^*}(x^*f) = x^*\mu(f) = 0$ since $\mu(f) \in \overline{co}f(S)$ (Definition 1.1). Clearly φ_{μ,x^*} is linear on $x^*\mathcal{A}$. Furthermore

$$\varphi_{\mu,x^*}(x^*f) = x^*\mu(f) \in x^*\overline{co}(f)(S) \subset \overline{co}x^*f(S),$$

so φ_{μ,x^*} is in $M(x^*\mathcal{A})$.

The rest of the lemma is clear.

We can furnish $\mathcal{L}(\mathcal{A}, X)$ with two topologies, both of which make $\mathcal{L}(\mathcal{A}, X)$ a locally convex topological space. One is the strong operator topology τ_s, which is the weakest topology of $\mathcal{L}(\mathcal{A}, X)$ relative to which the mapping $U \to Uf : \mathcal{L}(\mathcal{A}, X) \to X$ is continuous for each $f \in \mathcal{A}$, and the other is the weak operator topology τ_w, which is the weakest topology of $\mathcal{L}(\mathcal{A}, X)$ relative to which the mapping $U \to x^*Uf : \mathcal{L}(\mathcal{A}, X) \to \mathbb{C}$ is continuous for each $f \in \mathcal{A}$ and $x^* \in X^*$. These topologies can be relativized to $M(\mathcal{A}) \subset \mathcal{L}(\mathcal{A}, X)$.

Proposition 1.5. *Let $\mathcal{A}$ be a linear subspace of $\mathcal{B}(S, X)$. Then, for τ_s*

(1) *$M(\mathcal{A})$ is convex and closed in $\mathcal{L}(\mathcal{A}, X)$;*
(2) *$co(\epsilon(S))$ is dense in $M(\mathcal{A})$;*
(3) *if S is a topological space and $\mathcal{A} \subset \mathcal{C}(S, X)$, then $\epsilon : S \to M(\mathcal{A})$ is continuous.*

Furthermore, if the range $f(S)$ of f is relatively compact in X for each $f \in \mathcal{A}$, then $M(\mathcal{A})$ is τ_s-compact.

Proof.

(1) The convexity of $M(\mathcal{A})$ follows directly from Definition 1.1. To show that $M(\mathcal{A})$ is closed, let $\{\mu_\alpha\} \subset M(\mathcal{A})$ converge to $\mu \in \mathcal{L}(\mathcal{A}, X)$ for τ_s. Then $\mu_\alpha(f) \to \mu(f)$ for each $f \in \mathcal{A}$, and since $\mu_\alpha(f) \in \overline{co}f(S)$ for all α, $\mu(f) \in \overline{co}f(S)$. Therefore, $\mu \in M(\mathcal{A})$.
(2) Clearly, $co(\epsilon(S)) \subset M(\mathcal{A})$. If there is a $\mu \in M(\mathcal{A})$ such that $\mu \notin \overline{co}(\epsilon(S))$, the closure being taken in τ_s, then there is an $f \in \mathcal{A}$ such that $\mu(f) \notin \overline{co}(\epsilon(S)f) = \overline{co}f(S)$, which contradicts Definition 1.1.
(3) is obvious.

The proof of the compactness of $M(\mathcal{A})$, if $\mathcal{A}$ satisfies the compactness condition, is similar to that of its counterpart in the following proposition, so we omit it.

Proposition 1.6. *Let $\mathcal{A}$ be a linear subspace of $\mathcal{B}(S,X)$. Then the conclusions (1)–(3) of the previous proposition are true for τ_w. Furthermore, if $\mathcal{A}$ is such that the range $f(S)$ of f is weakly relatively compact in X for each $f \in \mathcal{A}$, then $M(\mathcal{A})$ is τ_w-compact.*

Proof. Using Lemma 1.4, we can prove (1)–(3) in much the same way that (1)–(3) of Proposition 1.5 we proved.

We now show that $M(\mathcal{A})$ is τ_w-compact when $\mathcal{A}$ satifies the weak compactness condition. For each $x^* \in X^*$, $M(x^*\mathcal{A})$ is weak* compact [1, 2.1.8]. Therefore, the product space

$$\prod := \prod\{M(x^*\mathcal{A}) : \ x^* \in X^*\}$$

is compact in the product topology.

By Lemma 1.4, the mapping $\mu \to \varphi_\mu = \{\varphi_{\mu,x^*} : x^* \in X^*\} : M(\mathcal{A}) \to \prod$ is 1–1, and it is homeomorphism when $M(\mathcal{A})$ has the topology τ_w. To show that $M(\mathcal{A})$ is τ_w-compact, it suffices to show that the image of $M(\mathcal{A})$ in $\prod$ is closed.

Let $\varphi = \{\varphi_{x^*} : x^* \in X^*\} \in \prod$ and let the image $\{\varphi_{\mu_\alpha}\}$ of $\{\mu_\alpha\}$ converge to φ in $\prod$. We show that there is a $\mu \in M(\mathcal{A})$ such that φ is the image of μ and $\mu_\alpha \to \mu$ in τ_w.

Since $f(S)$ is weakly relatively compact in X for each $f \in \mathcal{A}$, by the Krein–Smulian theorem [1, A.10] $\overline{co}f(S)$ is weakly compact in X for each $f \in \mathcal{A}$. Since $\mu_\alpha(f) \in \overline{co}f(S)$ for all α and $x^*\mu_\alpha(f) \to \varphi_{x^*}(x^*f)$ for all $x^* \in X^*$, there is a $\mu(f) \in \overline{co}f(S)$ such that $x^*\mu(f) = \varphi_{x^*}(x^*f)$ for all $x^* \in X^*$. The map $f \to \mu(f)$ is clearly linear, so $\mu \in M(\mathcal{A})$. Thus $\mu_\alpha \to \mu$ in τ_w, and the proof is complete.

The following theorem shows that the definition of a mean in [2] is equivalent to that in [3].

Theorem 1.7. *A mapping $\mu : \mathcal{A} \to X$ is in $M(\mathcal{A})$ if and only if there is a unique $\varphi_\mu \in M(\overline{sp}(X^* \circ \mathcal{A}))$ such that*

$$x^*\mu(f) = \varphi_\mu(x^*f) \qquad (x^* \in X^*, \ f \in \mathcal{A}). \tag{1.1}$$

Proof. The sufficiency comes from the sufficiency in the first statement of Lemma 1.4.

Necessity. By Lemma 1.4, if μ is in $M(\mathcal{A})$, then for each $x^* \in X^*$ there is a φ_{μ,x^*} in $M(x^*\mathcal{A})$ such that

$$x^*\mu(f) = \varphi_{\mu,x^*}(x^*f) \qquad (f \in \mathcal{A}).$$

We show first that φ_{μ,x^*} is independent of $x^* \in X^*$, i.e., if x_1^*, $x_2^* \in X^*$ and f_1, $f_2 \in \mathcal{A}$ are such that $x_1^*f_1 = x_2^*f_2$, then $\varphi_{\mu,x_1^*}(x_1^*f_1) = \varphi_{\mu,x_2^*}(x_2^*f_2)$.

Since $\mu \in M(\mathcal{A})$, by Proposition 1.6 (2) there is a net $\{\sum_{s\in S}\lambda_\alpha(s)\epsilon(s)\}$ converging to μ for τ_w; here each $\lambda_\alpha : S \to [0,1]$ has finite support and satisfies $\sum_{s\in S}\lambda_\alpha(s) = 1$. Next, $x_1^*(\sum_{s\in S}\lambda_\alpha(s)f_1(s)) = x_2^*(\sum_{s\in S}\lambda_\alpha(s)f_2(s))$ because $x_1^*f_1 = x_2^*f_2$, so

$$\varphi_{\mu,x_1^*}(x_1^* f_1) = x_1^*\mu(f_1) = \lim_\alpha x_1^* \sum_{s\in S} \lambda_\alpha(s) f_1(s)$$
$$= \lim_\alpha x_2^* \sum_{s\in S} \lambda_\alpha(s) f_2(s) = x_2^*\mu(f_2) = \varphi_{\mu,x_2^*}(x_2^* f_2).$$

Therefore we can define φ_μ for $\sum_{i=1}^m \alpha_i x_i^* f_i \in sp(X^* \circ \mathcal{A})$ by

$$\varphi_\mu(\sum_{i=1}^m \alpha_i x_i^* f_i) = \sum_{i=1}^m \alpha_i \varphi_{\mu,x_i^*}(x_i^* f_i).$$

It is easy to see that φ_μ is in $M(sp(X^* \circ \mathcal{A}))$. Therefore φ_μ has a unique extension to $\overline{sp}(X^* \circ \mathcal{A})$ and satisfies (1.1).

The uniqueness is clear. The proof is finished.

By Theorem 1.7, we can write φ_μ for φ_{μ,x^*} in Lemma 1.4.

§2. Introversion and Semigroups of Vector–Valued Means

Definition 2.1. *Let $\mathcal{A}$ be a translation invariant linear subspace of $\mathcal{B}(S,X)$. For a linear map μ from $\mathcal{A}$ to X, define the left introversion operator $T_\mu : \mathcal{A} \to \mathcal{B}(S,X)$ by*

$$T_\mu f(s) = \mu(L_s f) \qquad (f \in \mathcal{A},\ s \in S)$$

and analogously define the right introversion operator $U_\mu : A \to \mathcal{B}(S,X)$ by

$$U_\mu f(s) = \mu(R_s f) \qquad (f \in \mathcal{A},\ s \in S).$$

If $T_\mu\mathcal{A} \subset \mathcal{A}$ for all $\mu \in M(\mathcal{A})$, we will say that $\mathcal{A}$ is left introverted; we will say that A is right introverted if $U_\mu\mathcal{A} \subset \mathcal{A}$. $\mathcal{A}$ is introverted if it is both left and right introverted.

A semitopological semigroup S is a semigroup and a Hausdorff topological space in such a way that multiplication is separately continuous, i.e., the maps $s \to ts$ and $s \to st$ from S into S are continuous for all $t \in S$. $\mathcal{C}(S,X)$ denotes the Banach space of all continuous members of $\mathcal{B}(S,X)$.

Example 2.2. $\mathcal{C}(S,X)$ is introverted if S is a compact semitopological semigroup.

For $\mu \in M(\mathcal{C}(S,X))$ and $f \in \mathcal{C}(S,X)$, we must show that $T_\mu f$ and $U_\mu f$ are continuous.

Let $g \in \mathcal{C}(S)$ and let $x \in X$. $g(\cdot)x \in \mathcal{C}(S,X)$. Theorem 1.7 implies that $\mu(g(\cdot)x) = \varphi_\mu(g)x$ and $T_\mu(g(\cdot)x) = T_{\varphi_\mu}(g)x$. Therefore $T_\mu(g(\cdot)x) \in \mathcal{C}(S,X)$ since $T_{\varphi_\mu}(g) \in \mathcal{C}(S)$ [1, 2.2.5]. Note the fact that $\mathcal{C}(S,X) = \overline{sp}\{g(\cdot)x : g \in \mathcal{C}(S),\ x \in X\}$ since S is compact. For $\epsilon > 0$ there is $p(\cdot) = \sum_{i=1}^n f_i(\cdot)x_i$, where $f_i \in \mathcal{C}(S)$ and $x_i \in X$, $i = 1, 2, \cdots, n$, such that

$$\|f - p\| < \epsilon.$$

Now $p \in \mathcal{C}(S,X)$ and

$$\|T_\mu f - T_\mu p\| = \max_{s\in S} \|\mu(L_s(f-p))\| \le \|f - p\| < \epsilon.$$

Therefore $T_\mu f \in \mathcal{C}(S,X)$.

Similarly $U_\mu f \in \mathcal{C}(S,X)$. The proof is finished.

Proposition 2.3. *Let $\mathcal{A}$ be a translation invariant linear subspace of $\mathcal{B}(S, X)$ containing the constant functions and let $\epsilon : S \to M(\mathcal{A})$ be the evaluation mapping. Then*

(1) *for each $\mu \in M(\mathcal{A})$, $T_\mu : \mathcal{A} \to \mathcal{B}(S, X)$ is a bounded linear transformation with $\|T_\mu\| \le \|\mu\|$;*

(2) *the mapping $\mu \to T_\mu : M(\mathcal{A}) \to \mathcal{L}(\mathcal{A}, \mathcal{B}(S, X))$ is a bounded transformation;*

(3) *if $\mu \in M(\mathcal{A})$, then $T_\mu(x) = x$, $x \in X$;*

(4) *for all $s \in S$ and $\mu \in M(\mathcal{A})$*

$$T_\mu L_s = L_s T_\mu$$

$$T_\mu R_s = T_{R_s^* \mu}$$

$$T_{\epsilon(s)} = R_s,$$

where $R_s^* : M(\mathcal{A}) \to M(\mathcal{A})$ is the adjoint of R_s;

(5) *if $f \in \mathcal{A}$, then $\{T_\mu f : \mu \in M(\mathcal{A})\}$ is the closure in $\mathcal{B}(S, X)$ of $co(R_S f)$ in the topology of pointwise convergence on S.*

The proof of the proposition above is like that for [1, 2.2.3], so we omit it.

Definition 2.4. *Let $\mathcal{A}$ be a translation invariant linear subspace of $\mathcal{B}(S, X)$ containing the constant functions, and define*

$$Z_T = \{\nu \in \mathcal{L}(\mathcal{A}, X) : \ T_\nu \mathcal{A} \subset \mathcal{A}\}$$

and

$$Z_U = \{\mu \in \mathcal{L}(\mathcal{A}, X) : \ U_\mu \mathcal{A} \subset \mathcal{A}\}.$$

If $\mu \in \mathcal{L}(\mathcal{A}, X)$ and $\nu \in Z_T$, define $\mu\nu$: $\mathcal{A} \to X$ by

$$\mu\nu(f) = \mu(T_\nu f) \qquad (f \in \mathcal{A}).$$

If $\mu \in Z_U$ and $\nu \in \mathcal{L}(\mathcal{A}, X)$, define $\mu * \nu$: $\mathcal{A} \to X$ by

$$\mu * \nu(f) = \nu(U_\mu f) \qquad (f \in \mathcal{A}).$$

Definition 2.5. *An admissible subspace $\mathcal{A}$ of $\mathcal{B}(S, X)$ is a norm closed, translation invariant, left introverted subspace of $\mathcal{B}(S, X)$ containing the constant functions. In the case that $X = \mathbb{C}$, an admissible subspace $\mathcal{A} \subset \mathcal{B}(S)$ is also required to be conjugate closed.*

Let S be a semigroup. Define $\rho_t : S \to S$ and $\lambda_t : S \to S$ by

$$\rho_t = st, \quad \lambda_t = ts \qquad (s \in S).$$

S is called a right topological semigroup if it is a topological space and ρ_t is continuous for all $t \in S$. Set

$$\Lambda(S) = \{s \in S : \lambda_s \text{ is continuous}\}.$$

An affine semigroup S is a semigroup and a convex subset of a vector space in such a way that ρ_t and λ_t are affine mappings for each $t \in S$. The requirement that ρ_t and λ_t be affine means that if r, $s \in S$ and a, $b \in [0, 1]$ with $a + b = 1$ then

$$(ar + bs)t = art + bst \text{ and } t(ar + bs) = atr + bts,$$

where (+) denotes vector addition.

The following lemma summarizes the properties of the operation $(\mu, \nu) \to \mu\nu$. The proof is similar to that of [1, 2.2.9]. We omit the statements of the corresponding properties of the operation $(\mu, \nu) \to \mu * \nu$.

Lemma 2.6. *Let $\mathcal{A}$ be as in Definition 2.4 and let $\epsilon : \mathcal{A} \to X$ be the evaluation mapping. Then*

(1) *Z_T is a linear subspace of $\mathcal{L}(\mathcal{A}, X)$ containing $\epsilon(S)$;*

(2) *$\mu\nu \in \mathcal{L}(\mathcal{A}, X)$ for all $\mu \in \mathcal{L}(\mathcal{A}, X)$ and $\nu \in Z_T$;*

(3) *if $\mu \in \mathcal{L}(\mathcal{A}, X)$, $\nu \in Z_T$ and $s \in S$, we have*

$$T_{\mu\nu} = T_\mu \circ T_\nu,$$

$$\varepsilon(s)\nu = L_s^*\nu,$$

$$\mu\varepsilon(s) = R_s^*\mu, \text{ and}$$

$$\|\mu\nu\| \leq \|\mu\|\|\nu\|,$$

where $L_s^* : M(\mathcal{A}) \to M(\mathcal{A})$ is the adjoint of L_s;

(4) *Z_T is a right topological semigroup.*

The following result is essentially a consequence of the preceding lemma and Propositions 1.5 and 1.6.

Theorem 2.7.

(1) *If $\mathcal{A}$ is an admissible subspace of $\mathcal{B}(S, X)$, then for τ_s or τ_w, and multiplication $(\mu, \nu) \to \mu\nu$, $M(\mathcal{A})$ is a right topological affine subsemigroup of $\mathcal{L}(\mathcal{A}, X)$, $co(\epsilon(S)) \subset \Lambda(M(\mathcal{A}))$ and $\epsilon : S \to M(\mathcal{A})$ is a homomorphism.*

(2) *If we also assume that $f(S)$ is (weakly) relatively compact for all $f \in \mathcal{A}$, then $M(\mathcal{A})$ is also compact for (τ_w) τ_s.*

Let S be a compact semitopological semigroup. By Example 2.2, $\mathcal{C}(S, X)$ is introverted. Hence $\mu\nu$, $\mu * \nu \in M(\mathcal{C}(S, X))$; indeed, they are equal.

Proposition 2.8. *Let S be a compact semitopological semigroup and let $\mathcal{A} = \mathcal{C}(S, X)$. Then*

(1) $\mu\nu = \mu * \nu$ for all $\mu, \nu \in M(\mathcal{A})$;

(2) for τ_s and multiplication $(\mu, \nu) \to \mu\nu$, $M(\mathcal{A})$ is a compact semitopological affine semigroup;

(3) if S is also a topological semigroup, so is $M(\mathcal{A})$ in τ_s.

Proof. (1). Note that $\varphi_\mu\varphi_\nu = \varphi_\mu * \varphi_\nu$ [1, 2.2.12 (a)]. Similar to the proof of Example 2.2, we have, for $g \in \mathcal{C}(S)$ and $x \in X$,

$$\mu\nu(g(\cdot)x) = \mu(T_\nu g(\cdot)x) = \varphi_\mu(T_{\varphi_\nu} g)x = \varphi_\mu\varphi_\nu(g)x = \varphi_\mu * \varphi_\nu(g)x = \mu * \nu(g(\cdot)x).$$

Therefore

$$\mu\nu(f) = \mu * \nu(f) \qquad (f \in \mathcal{C}(S, X)),$$

i.e., $\mu\nu = \mu * \nu$.

(2) is a consequence of (1) and Theorem 2.7 (1).

To verify (3), we need to show that if $\mu_\alpha \to \mu$ and $\nu_\alpha \to \nu$ for τ_s then $\mu_\alpha\nu_\alpha \to \mu\nu$ for τ_s. Note that $\varphi_{\mu_\alpha}\varphi_{\nu_\alpha}(g) \to \varphi_\mu\varphi_\nu(g)$ for every $g \in \mathcal{C}(S)$ [1, 2.2.12 (c)]. Now, for $x \in X$,

$$\mu_\alpha\nu_\alpha(g(\cdot)x) = \varphi_{\mu_\alpha}\varphi_{\nu_\alpha}(g)x \to \varphi_\mu\varphi_\nu(g)x = \mu\nu(g(\cdot)x).$$

Again using the fact that $\mathcal{C}(S, X) = \overline{sp}\{g(\cdot)x : g \in \mathcal{C}(S), x \in X\}$, we have $\mu_\alpha\nu_\alpha(f) \to \mu\nu(f)$ for every $f \in \mathcal{C}(S, X)$.

§3. Invariant Vector–Valued Means

S denotes a semigroup which need not have an identity and $\mathcal{A}$ denotes a linear subspace of $\mathcal{B}(S, X)$ containing the constant functions. Let $LIM(\mathcal{A})$ ($RIM(\mathcal{A})$) denotes the set of left (right) invariant means on $\mathcal{A}$. $\mathcal{A}$ is said to be left (right) amenable if $LIM(\mathcal{A}) \neq \phi$ ($RIM(\mathcal{A}) \neq \phi$). If $\mathcal{A}$ is translation invariant , we set

$$IM(\mathcal{A}) = LIM(\mathcal{A}) \cap RIM(\mathcal{A})$$

and call members of $IM(\mathcal{A})$ invariant means. $\mathcal{A}$ is said to be amenable if $IM(\mathcal{A}) \neq \phi$.

As in the scalar case, we have the following proposition, whose proof is similar to that of [1, 2.3.5]; so we omit it.

Proposition 3.1. *Let $\mathcal{A}$ be an admissible subspace of $\mathcal{B}(S, X)$ and let $\epsilon : S \to \mathcal{L}(\mathcal{A}, X)$ be the evaluation mapping.*

(1) *$LIM(\mathcal{A})$ is the set of right zeros of $M(\mathcal{A})$; hence if $\mathcal{A}$ is left amenable , then $LIM(\mathcal{A})$ is a closed ideal of $M(\mathcal{A})$ contained in every right ideal.*
(2) *If $\mathcal{A}$ is right amenable, then $RIM(\mathcal{A})$ is a closed left idéal of $M(\mathcal{A})$.*

Corollary 3.2. *Let $\mathcal{A}$ be an admissible subspace of $\mathcal{B}(S, X)$. If $\mathcal{A}$ is left and right amenable, then it is amenable.*

Proof. If $\mu \in LIM(\mathcal{A})$ and $\nu \in RIM(\mathcal{A})$, then $\mu\nu \in IM(\mathcal{A})$.

Corollary 3.3. *Let $\mathcal{A}$ be an admissible right introverted subspace of $\mathcal{B}(S, X)$ such that $\mu\nu = \mu * \nu$ for all μ, $\nu \in M(\mathcal{A})$. Then $\mathcal{A}$ has at most one invariant mean.*

Proof. By the proposition and its right introverted analog, if μ, $\nu \in IM(\mathcal{A})$, then $\nu = \mu\nu = \mu * \nu = \mu$.

Theorem 3.4. *Let $\mathcal{A}$ be an admissible subspace of $\mathcal{B}(S, X)$ such that, for each $f \in \mathcal{A}$, the range $f(S)$ of f is relatively weakly compact. Let $K(f)$ denote the closure in $\mathcal{B}(S, X)$ of $co(R_S f)$ for the pointwise topology. The following assertions are equivalent:*

(1) *$\mathcal{A}$ is left amenable;*

(2) *for each $f \in \mathcal{A}$, $K(f)$ contains a constant function;*

(3) *for each $f \in \mathcal{A}$, and $s \in S$, $0 \in K(f - L_s f)$.*

Furthermore, if (1) holds then, for each $f \in \mathcal{A}$, $\{\mu(f) : \mu \in LIM(\mathcal{A})\}$ is the set of constant functions in $K(f)$.

Proof. We omit the proofs that (1)$\Rightarrow$ (2) $\Rightarrow$ (3) which do not use weak compactenss hypothesis. Here we show that (3) $\Rightarrow$ (1).

For each $f \in \mathcal{A}$ and $s \in S$, let

$$M(f, s) = \{\mu \in M(A) : \ T_\mu(f - L_s f) = 0\}.$$

The sets $M(f, s)$ are τ_w–closed, and therefore τ_w–compact. For, let $\{\mu_\alpha\} \subset M(f, s)$ converge to $\mu \in M(\mathcal{A})$. We want to show that $\mu \in M(f, s)$, i.e.,

$$T_\mu(f - L_s f) = 0.$$

Note that

$$T_\mu(f - L_s f)(t) = \mu(L_t f - L_{ts} f) \qquad (t \in S)$$

and $\mu_\alpha(L_t f - L_{ts} f) = T_{\mu_\alpha}(f - L_s f)(t) = 0$ for all α. Since $\mu_\alpha(L_t f - L_{ts} f) \to \mu(L_t f - L_{ts} f)$ weakly, $\mu(L_t f - L_{ts} f) = 0$. That is, $T_\mu(f - L_s f) = 0$.

As in the proof of [1, 2.3.11], we can show that the family $\{M(f, s) : f \in \mathcal{A},\ s \in S\}$ has the finite intersection property. By Proposition 1.6 $M(\mathcal{A})$ is τ_w–compact. So

$$\bigcap\{M(f, s) : f \in \mathcal{A}, s \in S\} \neq \phi.$$

Let μ be any member of this intersection, then $\mu^2 \in LIM(\mathcal{A})$.

Let S be a group and let $\mathcal{A}$ be a linear subspace of $\mathcal{B}(S, X)$. For each $f \in A$ define $\tilde{f}$: $S \to X$ by

$$\tilde{f}(s) = f(s^{-1}) \qquad (s \in S),$$

and set

$$\tilde{\mathcal{A}} = \{\tilde{f} : f \in \mathcal{A}\}.$$

If $\mu \in M(\mathcal{A})$, define $\tilde{\mu} \in M(\tilde{\mathcal{A}})$ by

$$\tilde{\mu}(\tilde{f}) = \mu(f) \qquad (f \in \mathcal{A}).$$

If $\tilde{\mathcal{A}} = \mathcal{A}$ and $\tilde{\mu} = \mu$, then μ is said to be inversion invariant.

Theorem 3.5. *Let G be a compact Hausdorff topological group. Then $\mathcal{C}(G, X)$ has a unique invariant mean μ. Furthermore μ is inversion invariant.*

Proof. The mean μ can be expressed as

$$\mu(f) = \int_G f d\nu \qquad (f \in \mathcal{C}(G, X)),$$

where ν is normalized Haar measure on G; the properties of μ follows from those of ν.

The scalar version of the next theorem is [1, 2.3.14]; a similar result has appeared in [3], but there S is required to have an identity. A small modification of the proof of [1, 2.3.14] yields a proof of the present theorem.

Theorem 3.6. *Let S be a compact Hausdorff semitopological semigroup. Then the following assertions hold:*

(1) *$\mathcal{C}(S,X)$ is left (respectively right) amenable if and only if S has a unique minimal right (respectively, left) ideal;*

(2) *$\mathcal{C}(S,X)$ is amenable if and only if the minimal ideal of S is a compact topological group.*

§4. Vector–Valued Weakly Almost Periodic Functions

Let S be a semitopological semigroup; we do not assume S has an identity. Let $\mathcal{WAP}(S,X)$ consist of those members f of $\mathcal{C}(S,X)$ for which the right orbit $R_S f = \{R_s f : s \in S\}$ is weakly relatively compact in $\mathcal{C}(S,X)$.

With a proof similar to that for [1, 4.2.5], one sees that the space $\mathcal{WAP}(S,X)$ is a closed translation invariant subspace of $\mathcal{C}(S,X)$. When $X = \mathbb{C}$, $\mathcal{WAP}(S,X)$ is just $\mathcal{WAP}(S)$, the C^*-algebra of weakly almost periodic functions on S. We note that

$$x^* \circ \mathcal{WAP}(S,X) = \mathcal{WAP}(S) \qquad (x^* \in X^*,\ x^* \neq 0).$$

Recall that $\epsilon : S \to \mathcal{L}(\mathcal{A},X)$ is the evaluation mapping $\epsilon(s)f = f(s), f \in \mathcal{WAP}(S,X)$. When $X = \mathbb{C}$ we denote this mapping by ϵ'.

Let $aS^{\mathcal{WAP}}$ denote the w^* closure in $\mathcal{WAP}(S)^*$ of $co\epsilon'(S)$; $aS^{\mathcal{WAP}}$ is a compact affine semitopological semigroup [1, 4.2.11].

Theorem 4.1. *Let S be a semitopological semigroup and let $\mathcal{A} = \mathcal{WAP}(S,X)$. The following assertions hold:*

(1) *$\mathcal{A}$ is an admissible subspace of $\mathcal{B}(S,X)$;*

(2) *for τ_w and multiplication $(\mu,\nu) \to \mu\nu$, $M(\mathcal{A})$ is an affine semitopological semigroup;*

(3) *if $f(S)$ is weakly relatively compact in X for each $f \in \mathcal{A}$, then $M(\mathcal{A})$ is τ_w-compact; in this case $\mathcal{A}$ is left amenable if and only if $\mathcal{WAP}(S)$ is left amenable.*

Proof. (1) Since $\mathcal{A}$ is a closed translation invariant subspace of $\mathcal{C}(S,X)$, to show that $\mathcal{A}$ is admissible we need to show that $\mathcal{A}$ is left introverted, i.e., if $f \in \mathcal{A}$ then $T_\mu f \in \mathcal{A}$ for all $\mu \in M(\mathcal{A})$.

Define $V : M(\mathcal{A}) \to \mathcal{B}(S,X)$ by

$$V(\mu) = T_\mu f \qquad (\mu \in M(\mathcal{A})).$$

By Proposition 2.3 (5)

$$V(M(\mathcal{A})) = \overline{co}(R_S f), \tag{4.1}$$

the closure being taken in the pointwise topology. Since $f \in \mathcal{A}$, $co(R_S f)$ is weakly relatively compact in $\mathcal{A}$; in view of (4.1) this implies that $V(M(\mathcal{A}))$ is the weak closure in $\mathcal{A}$ of $co(R_S f)$. So $T_\mu f \in \mathcal{A}$ for all $\mu \in M(\mathcal{A})$.

(2) By Theorem 2.7 (1), for τ_w and multiplication $(\mu, \nu) \to \mu\nu$, $M(\mathcal{A})$ is a right topological affine semigroup. It follows from Theorem 1.7 that the mapping $\Pi : \mu \to \varphi_\mu$ is a τ_w-w^* homeomorphism of $M(\mathcal{A})$ into $aS^{\mathcal{WAP}}$. Since $x^*\nu(f) = \varphi_\nu(x^* f)$ for $f \in \mathcal{A}$ and $x^* \in X^*$, $x^*(T_\nu f) = T_{\varphi_\nu}(x^* f)$. It follows that $\varphi_{\mu\nu} = \varphi_\mu \varphi_\nu$. Since $\Pi(\mu\nu) = \varphi_{\mu\nu}$, Π is a homomorphism too. So $M(\mathcal{A})$ is an affine semitopological semigroup because $aS^{\mathcal{WAP}}$ is.

(3) When $\mathcal{A}$ satisfies the compactness condition, the τ_w-compactness of $M(\mathcal{A})$ is a consequence of Theorem 2.7 (2). In this case, $M(\mathcal{A}) \cong aS^{\mathcal{WAP}}$. So we get the last statement.

The proof is complete.

Remark 4.2. For $f \in \mathcal{WAP}(S, X)$, in general $f(S) \subset X$ is not weakly relatively compact. However, if S admits an identity, it follows from the double limit property (e.g., [2, Theorem 3]) that $f(S)$ is weakly relatively compact. Of course, if X is reflexive then $f(S)$ is weakly relatively compact.

Theorem 4.3. *For a compact semitopological semigrooup S, $\mathcal{WAPS}, X = \mathcal{C}(S, X)$.*

The theorem holds because the facts of $\mathcal{C}(S, X) = \overline{sp}\{f(\cdot)x : f \in \mathcal{C}(s), x \in X\}$ and $\mathcal{WAP}(S) = \mathcal{C}(S)$ [1, 4.2.9].

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