Hindawi Publishing Corporation
The Scientific World Journal
Volume 2014, Article ID 461060, 6 pages
http://dx.doi.org/10.1155/2014/461060

Research Article

(M, N)-Soft Intersection BL-Algebras and Their Congruences

Xueling Ma[1] **and Hee Sik Kim**[2]

[1] *Department of Mathematics, Hubei Minzu University, Enshi, Hubei 445000, China*
[2] *Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, Republic of Korea*

Correspondence should be addressed to Hee Sik Kim; heekim@hanyang.ac.kr

Received 7 February 2014; Accepted 1 March 2014; Published 1 April 2014

Academic Editor: Jianming Zhan

The purpose of this paper is to give a foundation for providing a new soft algebraic tool in considering many problems containing uncertainties. In order to provide these new soft algebraic structures, we discuss a new soft set-(M, N)-soft intersection set, which is a generalization of soft intersection sets. We introduce the concepts of (M, N)-SI filters of BL-algebras and establish some characterizations. Especially, (M, N)-soft congruences in BL-algebras are concerned.

1. Introduction

It is well known that certain information processing, especially inferences based on certain information, is based on classical two-valued logic. In making inference levels, it is natural and necessary to attempt to establish some rational logic system as the logical foundation for uncertain information processing. BL-algebra has been introduced by Hájek as the algebraic structures for his Basic Logic [1]. A well-known example of a BL-algebra is the interval $[0, 1]$ endowed with the structure induced by a continuous t-norm. In fact, the MV-algebras, Gödel algebras, and product algebras are the most known classes of BL-algebras. BL-algebras are further discussed by many researchers; see [2–12].

We note that the complexities of modeling uncertain data in economics, engineering, environmental science, sociology, information sciences, and many other fields cannot be successfully dealt with by classical methods. Based on this reason, Molodtsov [13] proposed a completely new approach for modeling vagueness and uncertainty, which is called soft set theory. We note that soft set theory emphasizes a balanced coverage of both theory and practice. Nowadays, it has promoted a breath of the discipline of information sciences, intelligent systems, expert and decision support systems, knowledge systems and decision making, and so on. For example, see [14–24]. In particular, Çağman et al., Sezgin et al., and Jun et al. applied soft intersection theory to groups [25], near-rings [26], and BL-algebras [27], respectively.

In this paper, we organize the recent paper as follows. In Section 2, we recall some concepts and results of BL-algebras and soft sets. In Section 3, we investigate some characterizations of (M, N)-SI filters of BL-algebras. In particular, some important properties of (M, N)-soft congruences of BL-algebras are discussed in Section 4.

2. Preliminaries

Recall that an algebra $L = (L, \leq, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL-*algebra* [1] if it is a bounded lattice such that the following conditions are satisfied:

(i) $(L, \odot, 1)$ is a commutative monoid;

(ii) $\odot$ and $\rightarrow$ form an adjoin pair; that is, $z \leq x \rightarrow y$ if and only if $x \odot z \leq y$ for all $x, y, z \in L$;

(iii) $x \wedge y = x \odot (x \rightarrow y)$;

(iv) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

In what follows, L is a BL-algebra unless otherwise specified.

In any BL-algebra L, the following statements are true (see [1, 5, 6]):

(a_1) $x \leq y \Leftrightarrow x \rightarrow y = 1$;

(a_2) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z)$;

(a_3) $x \odot y \leq x \wedge y$;

(a_4) $x \to y \leq (z \to x) \to (z \to y), x \to y \leq (y \to z) \to (x \to z)$;

(a_5) $x \to x' = x'' \to x$;

(a_6) $x \vee x' = 1 \Rightarrow x \wedge x' = 0$;

(a_7) $(x \to y) \odot (y \to z) \leq x \to z$;

(a_8) $x \leq y \Rightarrow x \to z \geq y \to z$;

(a_9) $x \leq y \Rightarrow z \to x \leq z \to y$,

where $x' = x \to 0$.

A nonempty subset A of L is called a *filter* of L if it satisfies the following conditions:

(I1) $1 \in A$,

(I2) $\forall x \in A, \forall y \in L, x \to y \in A \Rightarrow y \in A$.

It is easy to check that a nonempty subset A of L is a filter of L if and only if it satisfies

(I3) $\forall x, y \in L, x \odot y \in A$,

(I4) $\forall x \in A, \forall y \in L, x \leq y \Rightarrow y \in A$ (see [6]).

From now on, we let L be a BL-algebra, U an initial universe, E a set of parameters, and $P(U)$ the power set of U and $A, B, C \subseteq E$.

Definition 1 (see [13, 16]). A soft set f_A over U is a set defined by $f_A : E \to P(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$. Here f_A is also called an *approximate function*. A soft set over U can be represented by the set of ordered pairs $f_A = \{(x, f_A(x)) \mid x \in E, f_A(x) \in P(U)\}$. It is clear to see that a soft set is a parameterized family of subsets of U. Note that the set of all soft sets over U will be denoted by $S(U)$.

Definition 2 (see [16]). Let $f_A, f_B \in S(U)$.

(1) f_A is said to be a *soft subset* of f_B and denoted by $f_A \tilde{\subseteq} f_B$ if $f_A(x) \subseteq f_B(x)$, for all $x \in E$. f_A and f_B are said to be *soft equal*, denoted by $f_A = f_B$, if $f_A \tilde{\subseteq} f_B$ and $f_A \tilde{\supseteq} f_B$.

(2) The union of f_A and f_B, denoted by $f_A \tilde{\cup} f_B$, is defined as $f_A \tilde{\cup} f_B = f_{A \cup B}$, where $f_{A \cup B}(x) = f_A(x) \cup f_B(x)$, for all $x \in E$.

(3) The intersection of f_A and f_B, denoted by $f_A \tilde{\cap} f_B$, is defined as $f_A \tilde{\cap} f_B = f_{A \cap B}$, where $f_{A \cap B}(x) = f_A(x) \cap f_B(x)$, for all $x \in E$.

Definition 3 (see [27]). A soft set f_L over U is called an SI-*filter* of L over U if it satisfies

(S_1) $f_L(x) \subseteq f_L(1)$ for any $x \in L$,

(S_2) $f_L(x \to y) \cap f_L(x) \subseteq f_L(y)$ for all $x, y \in L$.

3. (M, N)-SI Filters

In this section, we introduce the concept of (M, N)-SI filters in BL-algebras and investigate some characterizations. From now on, we let $\emptyset \subseteq M \subset N \subseteq U$.

Definition 4. A soft set f_L over U is called an (M, N)-*soft intersection filter* (briefly, (M, N)-SI *filter*) of L over U if it satisfies

(SI_1) $f_L(x) \cap N \subseteq f_L(1) \cup M$ for all $x \in L$,

(SI_2) $f_L(x \to y) \cap f_L(x) \cap N \subseteq f_L(y) \cup M$ for all $x, y \in L$.

Remark 5. If f_L is an (M, N)-SI filter of L over U, then f_L is an $(\emptyset, U)$-SI filter of L over U. Hence every SI-filter of L is an (M, N)-SI filter of L, but the converse need not be true in general. See the following example.

Example 6. Assume that $U = S_3$, the symmetric 3-group is the universal set, and let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$. We define $x \wedge y := \min\{x, y\}$, $x \vee y := \max\{x, y\}$ and $\odot$ and $\to$ as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

(1)

It is clear that $(L, \wedge, \vee, \odot, \to, 1)$ is a BL-algebra. Let $M = \{(13), (123)\}$ and $N = \{(1), (12), (13), (123)\}$. Define a soft set f_L over U by $f_L(1) = \{(1), (12), (123)\}$, $f_L(b) = \{(1), (12), (13), (123)\}$ and $f_L(a) = f_L(0) = \{(1), (12)\}$. Then we can easily check that f_L is an (M, N)-SI filter of L over U, but it is not SI-filter of L over U since $f_L(b) \nsubseteq f_L(1)$.

The following proposition is obvious.

Proposition 7. *If a soft set f_L over U is an (M, N)-SI filter of L over U, then*

$$(f_S(1) \cap N) \cup M \supseteq (f_S(x) \cap N) \cup M \qquad \forall x \in S. \tag{2}$$

Define an ordered relation "$\tilde{\subseteq}_{(M,N)}$" on $S(U)$ as follows: for any $f_L, g_L \in S(U)$, $\emptyset \subseteq M \subset N \subseteq U$, we define $f_L \tilde{\subseteq}_{(M,N)} g_L \Leftrightarrow f_L \cap N \tilde{\subseteq} g_L \cup M$. And we define a relation "$=_{(M,N)}$" as follows: $f_L =_{(M,N)} g_L \Leftrightarrow f_L \tilde{\subseteq}_{(M,N)} g_L$ and $g_L \tilde{\subseteq}_{(M,N)} f_L$. Using this notion we state Definition 4 as follows.

Definition 8. A soft set f_L over U is called an (M, N)-*soft intersection filter* (briefly, (M, N)-SI *filter*) of L over U if it satisfies

(SI_1') $f_L(x) \tilde{\subseteq}_{(M,N)} f_L(1)$ for all $x \in L$,

(SI_2') $f_L(x \to y) \cap f_L(x) \tilde{\subseteq}_{(M,N)} f_L(y)$ for all $x, y \in L$.

Proposition 9. *If f_L is an (M, N)-SI filter of L over U, then $f_L^* = \{x \in L \mid (f_L(x) \cap N) \cup M = (f_L(1) \cap N) \cup M\}$ is a filter of L.*

Proof. Assume that f_L is an (M, N)-SI filter of L over U. Then it is clear that $1 \in f_L^*$. For any $x, x \to y \in f_L^*$, $(f_L(x) \cap N) \cup M = (f_L(x \to y) \cap N) \cup M = (f_L(1) \cap N) \cup M$. By

Proposition 7, we have $(f_L(y) \cap N) \cup M \subseteq (f_L(1) \cap N) \cup M$. Since f_L is an (M,N)-SI filter of L over U, we have

$$\begin{aligned}(f_L(y) \cap N) \cup M &= ((f_L(y) \cup M) \cap N) \cup M \\ &\supseteq (f_L(x) \cap f_L(x \longrightarrow y) \cap N) \cup M \\ &= ((f_L(y) \cap N) \cup M) \\ &\cap ((f_L(x \longrightarrow y) \cap N) \cup M) \\ &= (f_L(1) \cap N) \cup M.\end{aligned} \quad (3)$$

Hence, $(f_L(y) \cap N) \cup M = (f_L(1) \cap N) \cup M$, which implies $y \in f_L^*$. This shows that f_L^* is a filter of L. □

Proposition 10. *If a soft set f_L over U is an (M,N)-SI filter of L, then for any $x, y, z \in L$,*

(1) $x \leq y \Rightarrow f_L(x) \tilde{\subseteq}_{(M,N)} f_L(y)$,

(2) $f_L(x \to y) = f_L(1) \Rightarrow f_L(x) \tilde{\subseteq}_{(M,N)} f_L(y)$,

(3) $f_L(x \odot y) =_{(M,N)} f_L(x) \cap f_L(y) =_{(M,N)} f_L(x \wedge y)$,

(4) $f_L(0) =_{(M,N)} f_L(x) \cap f_L(x')$,

(5) $f_L(x \to y) \cap f_L(y \to z) \tilde{\subseteq}_{(M,N)} f_L(x \to z)$,

(6) $f_L(x) \cap f_L(y) \tilde{\subseteq}_{(M,N)} f_L(x \odot z \to y \odot z)$,

(7) $f_L(x \to y) \tilde{\subseteq}_{(M,N)} f_L((y \to z) \to (x \to z))$,

(8) $f_L(x \to y) \tilde{\subseteq}_{(M,N)} f_L((z \to x) \to (z \to y))$.

Proof. (1) Let $x, y \in L$ be such that $x \leq y$. Then $x \to y = 1$, and hence

$$\begin{aligned}(f_L(x) \cap N) &= (f_L(x) \cap N) \cap (f_L(1) \cup M) \\ &= (f_L(y) \cap N) \cap (f_L(x \longrightarrow y) \cup M) \\ &\subseteq (f_L(x) \cap f_L(x \longrightarrow y) \cap N) \cup M \\ &\subseteq f_L(y) \cup M,\end{aligned} \quad (4)$$

which implies $f_L(x) \tilde{\subseteq}_{(M,N)} f_L(y)$.

(2) Let $x, y \in L$ be such that $f_L(x \to y) = f_L(1)$. Then,

$$\begin{aligned}f_L(x) \cap N &= (f_L(x) \cap N) \cap (f_L(1) \cup M) \\ &= (f_L(x) \cap N) \cap (f_L(x \longrightarrow y) \cup M) \\ &\subseteq (f_L(x) \cap f_L(x \longrightarrow y) \cap N) \cup M \\ &\subseteq f_L(y) \cup M;\end{aligned} \quad (5)$$

that is, $f_L(x) \tilde{\subseteq}_{(M,N)} f_L(y)$.

(3) By (a_3), we have $x \odot y \leq x \wedge y$ for all $x, y \in L$. By (1), $f_L(x \odot y) \tilde{\subseteq}_{(M,N)} f_L(x) \cap f_L(y)$. Since $x \leq y \to x \odot y$, we obtain $f_L(x) \tilde{\subseteq}_{(M,N)} f_L(y \to (x \odot y))$. It follows from (SI_2) that $f_L(x) \cap f_L(y) \tilde{\subseteq}_{(M,N)} f_L(y \to (x \odot y)) \cap f_L(y) \subseteq f_L(x \odot y)$. Hence, $f_L(x \odot y) =_{(M,N)} f_L(x) \cap f_L(y)$.

Since $y \leq x \to y$ and $x \odot (x \to y) \leq x \wedge y$, we have $f_L(y) \tilde{\subseteq}_{(M,N)} f_L(x \to y)$ and $f_L(x \odot (x \to y)) \tilde{\subseteq}_{(M,N)} f_L(x \wedge y)$. Hence we have

$f_L(x) \cap f_L(y) \tilde{\subseteq}_{(M,N)} f_L(x) \cap f_L(x \to y) =_{(M,N)} f_L(x \odot (x \to y)) \tilde{\subseteq}_{(M,N)} f_L(x \wedge y) \tilde{\subseteq}_{(M,N)} f_L(x) \cap f_L(y)$, which implies $f_L(x) \cap f_L(y) =_{(M,N)} f_L(x \wedge y)$. Thus $f_L(x \odot y) =_{(M,N)} f_L(x) \cap f_L(y) =_{(M,N)} f_L(x \wedge y)$.

(4) It is a consequence of (3), since $x \odot x' = 0$.

(5) By (a_4).

(6) By (a_7).

(7) By (a_8).

(8) By (a_9). □

By Definition 4 and Proposition 10, we can deduce the following result.

Proposition 11. *A soft set f_L over U is an (M,N)-SI filter of L over U if and only if it satisfies*

$$(SI_3)\ x \longrightarrow (y \longrightarrow z) = 1 \Longrightarrow f_L(x) \cap f_L(y) \tilde{\subseteq}_{(M,N)} f_L(z). \quad (6)$$

Proposition 12. *A soft set f_L over U is an (M,N)-SI filter of L over U if and only if it satisfies*

(SI_4) $\forall x, y \in L, x \leq y \Rightarrow f_L(x) \tilde{\subseteq}_{(M,N)} f_L(y)$,

(SI_5) $\forall x, y \in L, f_L(x \odot y) =_{(M,N)} f_L(x) \cap f_L(y)$.

Proof. ($\Rightarrow$) By Proposition 10 (1) and (3).

($\Leftarrow$) Let $x, y \in L$. Since $x \leq 1$, by (SI_3), we have $f_L(x) \tilde{\subseteq}_{(M,N)} f_L(1)$. Hence (SI_1') holds. Since $x \odot (x \to y) \leq y$, by (SI_3) and (SI_4), we have $f_L(x) \cap f_L(x \to y) =_{(M,N)} f_L(x \odot (x \to y)) \tilde{\subseteq}_{(M,N)} f_L(y)$; that is, (SI_2') holds. Therefore, f_L is an (M,N)-SI filter of L over U. □

4. (M,N)-Soft Congruences

In this section, we investigate (M,N)-soft congruences, (M,N)-soft congruences classes, and quotient soft BL-algebras.

Definition 13. A soft relation θ from $f_L \times f_L$ to $P(U \times U)$ is called an (M,N)-*congruence* in L over $U \times U$ if it satisfies

(C_1) $\theta(1,1) =_{(M,N)} \theta(x,x), \forall x \in L$,

(C_2) $\theta(x,y) =_{(M,N)} \theta(y,x), \forall x \in L$,

(C_3) $\theta(x,y) \cap \theta(y,z) \tilde{\subseteq}_{(M,N)} \theta(x,z), \forall x, y, z \in L$,

(C_4) $\theta(x,y) \tilde{\subseteq}_{(M,N)} \theta(x \odot z, y \odot z), \forall x, y, z \in L$,

(C_5) $\theta(x,y) \tilde{\subseteq}_{(M,N)} \theta(x \to z, y \to z) \cap \theta(z \to x, z \to y), \forall x, y, z \in L$.

Definition 14. Let θ be an (M,N)-congruence in BL-algebra L over $U \times U$ and $x \in L$. Define θ^x in L as $\theta^x(y) = \theta(x,y), \forall y \in L$. The set θ^x is called an (M,N)-*congruence class* of x by θ in L. The set $L/\theta = \{\theta^x \mid x \in L\}$ is called a *quotient soft set* by θ.

Lemma 15. *If θ is an (M,N)-congruence in L over $U \times U$, then $\theta(x,y) \tilde{\subseteq}_{(M,N)} \theta(1,1), \forall x, y \in L$.*

Proof. By (C_1) and (C_3), we have $\theta(1,1) = \theta(x,x) \tilde{\supseteq}_{(M,N)} \theta(x,y) \cap \theta(y,x) = \theta(x,y)$. □

Lemma 16. *If θ is an (M,N)-congruence in L over $U\times U$, then θ^1 is an (M,N)-SI filter of L over U.*

Proof. For any $x\in L$, we have

$$\theta^1(1)=\theta(1,1)\,\widetilde{\supseteq}_{(M,N)}\theta(1,x)=\theta^1(x). \tag{7}$$

This proves that (SI_1') holds.

For any $x,y\in L$, by (C_3) and (C_5), we obtain

$$\begin{gathered}\theta(1,y)\,\widetilde{\supseteq}_{(M,N)}\theta(1,x\longrightarrow y)\cap\theta(x\longrightarrow y,y),\\ \theta(x\longrightarrow y,y)=\theta(x\longrightarrow y,1\longrightarrow y)\,\widetilde{\supseteq}_{(M,N)}\theta(x,1).\end{gathered} \tag{8}$$

It follows that

$$\begin{aligned}\theta(1,y)\,\widetilde{\supseteq}_{(M,N)}&\theta(1,x\longrightarrow y)\cap\theta(x,1)\\ &=\theta(1,x)\cap\theta(1,x\longrightarrow y);\end{aligned} \tag{9}$$

that is, $\theta^1(y)\widetilde{\supseteq}_{(M,N)}\theta^1(x)\cap\theta^1(x\to y)$. This proves that (SI_2') holds. Thus, θ^1 is an (M,N)-SI filter of L over U. □

Lemma 17. *Let f_L be an (M,N)-SI filter of L over U. Then $\theta(x,y)=f_L(x\to y)\cap f_L(y\to x)$ is an (M,N)-soft congruence in L.*

Proof. For any $x,y,z\in L$, we have the following.

(C_1) Consider

$$\begin{aligned}\theta_f(1,1)&=f_L(1\longrightarrow 1)\cap f_L(1\longrightarrow 1)\\ &=f_L(1)=f_L(x\longrightarrow x)\cap f_L(x\longrightarrow x)=\theta_f(x,x).\end{aligned} \tag{10}$$

This proves that (C_1) holds.

(C_2) It is clear that (C_2) holds.

(C_3) By Proposition 10(5), we have

$$\begin{aligned}&\theta_f(x,y)\cap\theta_f(y,z)\\ &\quad=(f_L(x\longrightarrow y)\cap f_L(y\longrightarrow x))\\ &\quad\cap(f_L(y\longrightarrow z)\cap f_L(z\longrightarrow y))\\ &\quad=(f_L(x\longrightarrow y)\cap f_L(y\longrightarrow z))\\ &\quad\cap(f_L(y\longrightarrow x)\cap f_L(z\longrightarrow y))\\ &\quad\widetilde{\subseteq}_{(M,N)}f_L(x\longrightarrow z)\cap f_L(z\longrightarrow x)\\ &\quad=\theta_f(x,z).\end{aligned} \tag{11}$$

Thus (C_3) holds.

(C_4) Since $x\to y\le(x\odot z)\to(y\odot z)$ and $y\to x\le(y\odot z)\to(x\odot z)$, we have

$$\begin{gathered}f_L(x\longrightarrow y)\,\widetilde{\subseteq}_{(M,N)}f_L((x\odot z)\longrightarrow(y\odot z)),\\ f_L(y\longrightarrow z)\,\widetilde{\subseteq}_{(M,N)}f_L((y\odot z)\longrightarrow(x\odot z)),\end{gathered} \tag{12}$$

Thus, we have

$$\begin{aligned}&f_L(x\longrightarrow y)\cap f_L(y\to x)\\ &\qquad\widetilde{\subseteq}_{(M,N)}f_L((x\odot z)\longrightarrow(y\odot z))\\ &\qquad\cap f_L((y\odot z)\longrightarrow(x\odot z)).\end{aligned} \tag{13}$$

which implies

$$\theta_f(x,y)\,\widetilde{\subseteq}_{(M,N)}\theta_f(x\odot z,y\odot z). \tag{14}$$

This implies that (C_4) holds.

(C_5) Finally, we prove condition (C_5):

$$\begin{aligned}&\theta_f(x\longrightarrow z,y\longrightarrow z)\cap\theta_f(z\longrightarrow x,z\longrightarrow y)\\ &\quad=f_L((x\longrightarrow z)\longrightarrow(y\longrightarrow z))\\ &\quad\cap f_L((y\longrightarrow z)\longrightarrow(x\longrightarrow z))\\ &\quad\cap f_L((z\longrightarrow x)\longrightarrow(z\longrightarrow y))\\ &\quad\cap f_L((z\longrightarrow y)\longrightarrow(z\longrightarrow x))\\ &\quad\widetilde{\supseteq}_{(M,N)}f_L(y\longrightarrow x)\cap f_L(x\longrightarrow y)\\ &\quad=\theta_f(x,y).\end{aligned} \tag{15}$$

Thus, (C_5) holds. Therefore θ_f is an (M,N)-soft congruence in L. □

Let f_L be an (M,N)-SI filter of L over U and $x\in L$. In the following, let f^x denote the (M,N)-congruence class of x by θ_f in L and let L/f be the quotient soft set by θ_f.

Lemma 18. *If f_L is an (M,N)-SI filter of L over U, then $f^x=_{(M,N)}f^y$ if and only if $f_L(x\to y)=_{(M,N)}f_L(y\to x)=_{(M,N)}f_L(1)$ for all $x,y\in L$.*

Proof. If f_L is an (M,N)-SI filter of L over U, then $f^\mu(\nu)=\theta_f^\mu(\nu)=\theta_f(\mu,\nu)=f_L(\mu\to\nu)\cap f_L(\nu\to\mu)$; that is, $f^\mu(\nu)=f_L(\mu\to\nu)\cap f_L(\nu\to\mu)$ for all $x,y\in L$. If $f^x=_{(M,N)}f^y$, then $f^x(x)=_{(M,N)}f^y(x)$, and hence $f_L(x\to x)=f_L(1)=_{(M,N)}f_L(y\to x)\cap f_L(x\to y)$. Thus, $f_L(y\to x)=_{(M,N)}f_L(x\to y)=_{(M,N)}f_L(1)$.

Conversely, assume the given condition holds. By Proposition 10, we have $f_L(x\to z)\widetilde{\supseteq}_{(M,N)}f_L(x\to y)\cap f_L(y\to z)$ and $f_L(y\to z)\widetilde{\supseteq}_{(M,N)}f_L(y\to x)\cap f_L(x\to z)$.If $f_L(y\to x)=_{(M,N)}f_L(x\to y)=_{(M,N)}f_L(1)$, then $f_L(x\to z)\supseteq_{(M,N)}f_L(y\to z)$ and $f_L(y\to z)\supseteq_{(M,N)}f_L(x\to z)$. Thus $f_L(x\to z)=_{(M,N)}f_L(y\to z)$. Similarly, we can prove that $f_L(z\to x)=_{(M,N)}f_L(z\to y)$. This implies that

$$\begin{aligned}f^x(z)&=f_L(x\longrightarrow z)\cap f_L(z\longrightarrow x)\\ &=_{(M,N)}f_L(y\longrightarrow z)\cap f_L(z\longrightarrow y)=f_L^y(z),\end{aligned} \tag{16}$$

for all $z\in L$. Hence, $f^x=_{(M,N)}f^y$. □

We denote $f_{f(1)}$ by $f_{f(1)}:=\{x\in L\mid f(x)=_{(M,N)}f(1)\}$.

Corollary 19. *If f is an (M,N)-SI filter of L over U, then $f^x =_{(M,N)} f^y$ if and only if $x \sim_{f_{f(1)}} y$, where $x \sim_{f_{f(1)}} y$ if and only if $x \to y \in f_{f(1)}$ and $y \to x \in f_{f(1)}$.*

Let f be an (M,N)-SI filter of L over U. For any $f^x, f^y \in L/f$, we define

$$f^x \vee f^y =_{(M,N)} f^{x\vee y}, \qquad f^x \wedge f^y =_{(M,N)} f^{x\wedge y},$$
$$f^x \odot f^y =_{(M,N)} f^{x\odot y}, \qquad f^x \longrightarrow f^y =_{(M,N)} f^{x\to y}. \tag{17}$$

Theorem 20. *If f is an (M,N)-SI filter of L over U, then $L/f = (L/f, \wedge, \vee, ', \to, f^0, f^1)$ is a BL-algebra.*

Proof. We claim that the above operations on L/f are well defined. In fact, if $f^x =_{(M,N)} f^y$ and $f^a =_{(M,N)} f^b$, by Corollary 19, we have $x \sim_{f_{(f(1))}} y$ and $a \sim_{f_{f(1)}} b$, and so $x \vee a \sim_{f_{f(1)}} y \vee b$. Thus $f^{x\vee a} =_{(M,N)} f^{y\vee b}$. Similarly, we prove $f^{x\wedge a} =_{(M,N)} f^{y\wedge b}$, $f^{x\odot a} =_{(M,N)} f^{y\odot b}$, and $f^{x\to a} =_{(M,N)} f^{y\to b}$. Then it is easy to see that L/f is a BL-algebra. Especially, we prove the divisibility in L/f as follows. Define a lattice ordered relation "$\preccurlyeq_{(M,N)}$" on L/f as follows:

$$f^x \preccurlyeq_{(M,N)} f^y \Longleftrightarrow f^x \vee f^y =_{(M,N)} f^1. \tag{18}$$

By Corollary 19, we have $f_L(x \to y) =_{(M,N)} f_L(1)$. If $f^x, f^y, f^z \in L/f$, then

$$\begin{aligned} f^x \odot f^y \preccurlyeq_{(M,N)} f^z &\Longleftrightarrow f^{x\odot y} \preccurlyeq_{(M,N)} f^z \\ &\Longleftrightarrow f_L((x \odot y) \longrightarrow z) =_{(M,N)} f_L(1) \\ &\Longleftrightarrow f_L(x \to (y \longrightarrow z)) =_{(M,N)} f_L(1) \\ &\Longleftrightarrow f^x \preccurlyeq_{(M,N)} f^{y\to z} \\ &\Longleftrightarrow f^x \preccurlyeq_{(M,N)} f^y \longrightarrow f^z. \end{aligned} \tag{19}$$

□

Theorem 21. *If f_L is an (M,N)-SI filter of L over U, then $L/f \cong L/f_{f(1)}$.*

Proof. Define $\varphi : L \to L/f$ by $\varphi(x) = f^x$ for all $x \in L$. For any $x, y \in L$, we have

$$\begin{aligned} \varphi(x \vee y) &= f^{x\vee y} =_{(M,N)} f^x \vee f^y = \varphi(x) \vee \varphi(y), \\ \varphi(x \wedge y) &= f^{x\wedge y} =_{(M,N)} f^x \wedge f^y = \varphi(x) \wedge \varphi(y), \\ \varphi(x \odot y) &= f^{x\odot y} =_{(M,N)} f^x \odot f^y = \varphi(x) \odot \varphi(y), \\ \varphi(x \longrightarrow y) &= f^{x\to y} =_{(M,N)} f^x \longrightarrow f^y = \varphi(x) \longrightarrow \varphi(y). \end{aligned} \tag{20}$$

Hence, φ is an epic. Moreover, we have

$$\begin{aligned} x \in \operatorname{Ker}\varphi &\Longleftrightarrow \varphi(x) = f^1 \\ &\Longleftrightarrow f^x =_{(M,N)} f^1 \Longleftrightarrow x \sim_{f_{f(1)}} 1 \Longleftrightarrow x \in f_{f(1)}, \end{aligned} \tag{21}$$

which shows that $L/f \cong L/f_{f(1)}$. □

5. Conclusions

As a generalization of soft intersection filters of BL-algebras, we introduce the concept of (M,N)-SI (implicative) filters of BL-algebras. We investigate their characterizations. In particular, we describe (M,N)-soft congruences in BL-algebras.

To extend this work, one can further investigate (M,N)-SI prime (semiprime) filters of BL-algebras. Maybe one can apply this idea to decision making, data analysis, and knowledge based systems.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] P. Hájek, *Metamathematics of Fuzzy Logic*, Kluwer Academic, Dodrecht, The Netherlands, 1998.

[2] Y. B. Jun and J. M. Ko, "Deductive systems of *BL*-algebras," *Honam Mathematical Journal*, vol. 28, no. 1, pp. 45–56, 2006.

[3] L. Liu and K. Li, "Fuzzy implicative and Boolean filters of *BL*-algebras," *Information Sciences*, vol. 171, no. 1-3, pp. 61–71, 2005.

[4] A. B. Saeid and S. Motamed, "Some results in *BL*-algebras," *Mathematical Logic Quarterly*, vol. 55, no. 6, pp. 649–658, 2009.

[5] E. Turunen, "*BL*-algebras of basic fuzzy logic," *Mathware & Soft Computing*, vol. 6, pp. 49–61, 1999.

[6] E. Turunen, "Boolean deductive systems of *BL*-algebras," *Archive for Mathematical Logic*, vol. 40, no. 6, pp. 467–473, 2001.

[7] E. Turunen, *Mathematics Behind Fuzzy Logic*, Physica, Heidelberg, Germany, 1999.

[8] W. Wang and X.-L. Xin, "On fuzzy filters of pseudo *BL*-algebras," *Fuzzy Sets and Systems*, vol. 162, no. 1, pp. 27–38, 2011.

[9] J. Zhan, W. A. Dudek, and Y. B. Jun, "Interval valued $(\epsilon, \epsilon \vee_q)$-fuzzy filters of pseudo *BL*-algebras," *Soft Computing*, vol. 13, no. 1, pp. 13–21, 2009.

[10] J. Zhan and Y. Xu, "Some types of generalized fuzzy filters of *BL*-algebras," *Computers and Mathematics with Applications*, vol. 56, no. 6, pp. 1604–1616, 2008.

[11] J. Zhang, G. Wang, and M. Hu, "Topology on the set of *BL*-semantics for *BL*-algebras," *Soft Computing*, vol. 12, no. 6, pp. 585–591, 2008.

[12] X.-H. Zhang and X.-S. Fan, "Pseudo-*BL* algebras and pseudo-effect algebras," *Fuzzy Sets and Systems*, vol. 159, no. 1, pp. 95–106, 2008.

[13] D. Molodtsov, "Soft set theory—first results," *Computers and Mathematics with Applications*, vol. 37, no. 4-5, pp. 19–31, 1999.

[14] M. I. Ali, F. Feng, X. Liu, W. K. Min, and M. Shabir, "On some new operations in soft set theory," *Computers and Mathematics with Applications*, vol. 57, no. 9, pp. 1547–1553, 2009.

[15] M. I. Ali, M. Shabir, and M. Naz, "Algebraic structures of soft sets associated with new operations," *Computers and Mathematics with Applications*, vol. 61, no. 9, pp. 2647–2654, 2011.

[16] N. Çağman and S. Enginoğlu, "Soft matrix theory and its decision making," *Computers and Mathematics with Applications*, vol. 59, no. 10, pp. 3308–3314, 2010.

[17] N. Çağman and S. Enginolu, "Soft set theory and uni-int decision making," *European Journal of Operational Research*, vol. 207, no. 2, pp. 848–855, 2010.

[18] F. Feng, Y. B. Jun, X. Liu, and L. Li, "An adjustable approach to fuzzy soft set based decision making," *Journal of Computational and Applied Mathematics*, vol. 234, no. 1, pp. 10–20, 2010.

[19] F. Feng and Y. Li, "Soft subsets and soft product operations," *Information Sciences*, vol. 232, pp. 44–57, 2013.

[20] P. K. Maji, R. Biswas, and A. R. Roy, "Soft set theory," *Computers and Mathematics with Applications*, vol. 45, no. 4-5, pp. 555–562, 2003.

[21] P. K. Maji, A. R. Roy, and R. Biswas, "An application of soft sets in a decision making problem," *Computers and Mathematics with Applications*, vol. 44, no. 8-9, pp. 1077–1083, 2002.

[22] K. Y. Qin, D. Meng, Z. Pei, and Y. Xu, "Combination of interval set and soft set," *International Journal of Computational Intelligence Systems*, vol. 6, no. 2, pp. 370–380, 2013.

[23] M. Shabir and A. Ali, "On soft ternary semigroups," *Annals of Fuzzy Mathematics and Informatics*, vol. 3, no. 1, pp. 39–359, 2012.

[24] J. Zhan and Y. B. Jun, "Soft *BL*-algebras based on fuzzy sets," *Computers and Mathematics with Applications*, vol. 59, no. 6, pp. 2037–2046, 2010.

[25] N. Çağman, F. Citak, and H. Aktas, "Soft-int group and its applications to group theory," *Neural Computing and Applications*, vol. 21, no. 1, supplement, pp. 151–158, 2012.

[26] A. Sezgin, A. O. Atagun, and N. N. Çağman, "Soft intersection near-rings with its applications," *Neural Computing and Applications*, vol. 21, no. 1, supplement, pp. 221–229, 2012.

[27] Y. B. Jun, S. Z. Song, and K. J. Lee, *Deductive systems of BL-algebras based onintersection-soft soft set*. In press.