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Method of Interior Boundaries in a Mixed Problem of Acoustic Scattering

P.A. KRUTITSKII*

Department of Mathematics, Faculty of Physics, Moscow State University, Moscow 119899, Russia

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The mixed problem for the Helmholtz equation in the exterior of several bodies (obstacles) is studied in 2 and 3 dimensions. The Dirichlet boundary condition is given on some obstacles and the impedance boundary condition is specified on the rest. The problem is investigated by a special modification of the boundary integral equation method. This modification can be called 'Method of interior boundaries', because additional boundaries are introduced inside scattering bodies, where impedance boundary condition is given. The solution of the problem is obtained in the form of potentials on the whole boundary. The density in the potentials satisfies the uniquely solvable Fredholm equation of the second kind and can be computed by standard codes. In fact our method holds for any positive wave numbers. The Neumann, Dirichlet, impedance problems and mixed Dirichlet–Neumann problem are particular cases of our problem.

Keywords: Acoustic scattering; Mixed problem; Boundary integral equation

1. INTRODUCTION

We study mixed problem for the propagative Helmholtz equation in the exterior of several bodies (obstacles) in 2 and 3 dimensions. The Dirichlet boundary condition is given on some bodies and the impedance boundary condition is specified on the rest. Similar problems model, for example, scattering acoustic waves by several obstacles and have numerous applications in different fields of physics,

* Fax: 7-095-932-8820. E-mail: krutitsk@math.phys.msu.su.

engineering and industry. The Dirichlet, Neumann, impedance and mixed Dirichlet–Neumann problems are particular cases of our problem. The aim of the present paper is to suggest a new approach to reduction of the mixed problem for the propagative Helmholtz equation to the uniquely solvable Fredholm integral equation of the second kind. This equation is very useful in applications, because its numerical solution can be obtained by standard codes. To derive this equation we put additional boundaries inside obstacles with impedance boundary condition and specify appropriate boundary conditions on the additional boundaries. The modified problem with additional boundaries has no more than one solution. We look for a solution of the problem in the form of single layer potential on additional boundaries and on the obstacles with impedance boundary condition. According to [2,12,17], a linear combination of single and double layer potentials is taken on the obstacles with Dirichlet boundary condition. Substituting the solution in the form of potentials to the boundary condition we obtain integral equation on the whole boundary. Next we verify that obtained integral equation is uniquely solvable Fredholm equation of the second kind.

Let us compare our approach with 2 classical methods. In [4,5,21,22] it was suggested to put infinite number of point sources inside obstacles with the impedance boundary condition. This method enables to prove formal solvability theorem, but it was not widely used in applications, since it is very hard to take into account infinite number of point sources when finding numerical solution. In our approach we exchange point sources for distributed sources in the form of additonal boundaries. In [12,17] it was proposed to look for a solution of the problem in the form of a linear combination of single and double layer potentials on the obstacles with impedance boundary condition. The hypersingular integral equation has been obtained on the surface of these obstacles. The numerical analysis of hypersingular integral equations requires special approaches [3,11,13–15,18–20], and it is much more complicated than numerical treatment of uniquely solvable Fredholm equation obtained in our method. In addition, the normal derivative of the double layer potential may not exist, while the classical solution exists. Therefore the solution can not be represented as a sum of single and double layer potentials on obstacles with impedance boundary condition in certain cases.

Our method holds for any wave number $k \in (0, k_0]$, where k_0 is an arbitrary fixed positive number. In fact, our method holds for all k, which may be used for computations in practical problems, since k_0 can be taken enough large, i.e. as large as it is necessary. In addition, the case of sufficiently large k is not interesting in diffraction theory, since if $k \to \infty$, then the diffraction is absent and waves are subject to the laws of geometry optics such as reflection and refraction.

The problems on scattering waves by a finite number of 2-D non-closed screens (open arcs) were reduced to the uniquely solvable Fredholm integral equations in [6–10].

2. FORMULATION OF THE PROBLEM

Let $x = (x_1, \dots, x_m) \in R^m$ for $m = 2$ or $m = 3$, and Δ is a Laplacian in R^m. We consider exterior open multiply connected domain $\mathcal{D} \subset R^m$ with the boundary $\Gamma = \Gamma^1 \cup \Gamma^2$, where

$$\Gamma^1 = \bigcup_{n=1}^{N_1} \Gamma_n^1 \in C^{2,\lambda}, \quad \Gamma^2 = \bigcup_{n=1}^{N_2} \Gamma_n^2 \in C^{2,0}, \quad \lambda \in (0, 1],$$

and $\Gamma_1^1, \dots, \Gamma_{N_1}^1$, $\Gamma_1^2, \dots, \Gamma_{N_2}^2$ are simple closed surfaces if $m = 3$ or curves if $m = 2$ without common points. Each surface (curve) Γ_n^j bounds interior single connected open domain $\mathcal{D}_n^j$ ($n = 1, \dots, N_j$; $j = 1, 2$). Let $\mathbf{n}_x$ be a unit normal vector to Γ at $x \in \Gamma$. The vector $\mathbf{n}_x$ is an outward normal regarding to $\mathcal{D}$. Consider Γ_n^j as a double-sided surface (curve). By $(\Gamma_n^j)^-$ we denote that side of Γ_n^j which we observe when facing towards the normal's tips. The opposite side will be called $(\Gamma_n^j)^+$. Accordingly, $\Gamma^\pm = (\Gamma^1)^\pm \cup (\Gamma^2)^\pm$, $(\Gamma^j)^\pm = \bigcup_{n=1}^{N_j} (\Gamma_n^j)^\pm, j = 1,2$.

We say that the function $W(x)$ defined in $\overline{\mathcal{D}}$ belongs to the smoothness class $\mathbf{K}$ if

(1) $W(x) \in C^0(\overline{\mathcal{D}}) \cap C^2(\mathcal{D})$,
(2) there exists a uniform for all $x \in (\Gamma^2)^+$ limit of $(\mathbf{n}_x, \nabla_{\bar{x}} W(\bar{x}))$ as $\bar{x} \in \mathcal{D}$ tends to $x \in (\Gamma^2)^+$ along the normal $\mathbf{n}_x$,
(3) $\nabla W(x)$ can be continuously extended on $(\Gamma^1)^+$ from the domain $\mathcal{D}$.

Let us formulate the exterior mixed problem for the propagative Helmholtz equation in the domain $\mathcal{D} \subset R^m$ with $m = 3$ or $m = 2$.

PROBLEM **V** *To find a function $W(x)$ of the class* **K**, *so that $W(x)$ satisfies the Helmholtz equation in $\mathcal{D}$*

$$\Delta W(x) + k^2 W(x) = 0, \quad k = \mathrm{Re}\, k = const > 0, \tag{2.1}$$

satisfies the boundary conditions

$$W(x)|_{x\in(\Gamma^1)^+} = f(x)|_{x\in\Gamma^1}, \tag{2.2a}$$

$$\left(\frac{\partial W(x)}{\partial \mathbf{n}_x} + g(x)W(x)\right)\Bigg|_{x\in(\Gamma^2)^+} = f(x)|_{x\in\Gamma^2} \tag{2.2b}$$

and meets the radiation conditions at infinity

$$W = \mathrm{O}(|x|^{(1-m)/2}), \quad \frac{\partial W}{\partial |x|} - ikW = \mathrm{o}(|x|^{(1-m)/2}) \tag{2.3}$$

as $|x| = \sqrt{x_1^2 + \cdots + x_m^2} \to \infty$. *The functions $g(x)$ and $f(x)$ are given, and* $\mathrm{Im}\, g(x) \leq 0$ *for any* $x \in \Gamma^2$.

All conditions of the problem must be satisfied in the classical sense. By $\partial W(x)/\partial \mathbf{n}_x$ on $(\Gamma^2)^+$ we mean the limit ensured in the point (2) of the definition of the smoothness class **K**. The mixed Dirichlet–Neumann problem is a particular case of the Problem **V** if $g(x) \equiv 0$. Problem **V** transforms to the Neumann or Dirichlet problem if, in addition, $\Gamma^1 = \emptyset$ or $\Gamma^2 = \emptyset$ respectively.

The theorem holds.

THEOREM 1 *There is no more than one solution of the Problem* **V**.

Proof Let $W_0(x)$ be an arbitrary solution of the homogeneous Problem **V**. Our aim is to show that $W_0(x) \equiv 0$ in $\mathcal{D}$.

By C_r we denote a ball (circle if $m = 2$) of the large radius r with the center in the origin. By $\overline{W}_0(x)$ we denote a function which is complex conjugate to $W_0(x)$. Clearly, $\overline{W}_0(x)$ belongs to the class **K**. We envelope $\Gamma_1^1, \ldots, \Gamma_{N_1}^1, \Gamma_1^2, \ldots, \Gamma_{N_2}^2$, by closed equidistant surfaces (contours) [24] lying in domain $\mathcal{D}$ and write energy identity for the domain bounded by these surfaces (contours) and C_r. Then we tend surfaces (contours) to Γ^+ and r to infinity. Using smoothness of $W_0(x)$ ensured by the class **K** we get

$$\begin{aligned}
&\lim_{r\to\infty}(\|\nabla W_0\|^2_{L_2(C_r\cap\mathcal{D})} - k^2\|W_0\|^2_{L_2(C_r\cap\mathcal{D})}) \\
&= \int_{\Gamma^+} \overline{W}_0(x)\frac{\partial W_0(x)}{\partial \mathbf{n}_x}\,\mathrm{d}s + \lim_{r\to\infty}\int_{\partial C_r} \overline{W}_0(x)\frac{\partial W_0(x)}{\partial r}\,\mathrm{d}s \\
&= -\int_{(\Gamma^2)^+} |W_0(x)|^2 g(x)\,\mathrm{d}s + \mathrm{i}k\lim_{r\to\infty}\int_{\partial C_r}|W_0(x)|^2\,\mathrm{d}s, \qquad (2.4)
\end{aligned}$$

where the conditions (2.3) and (2.2) were used. By $\int\cdots\mathrm{d}s$ we denote the surface (curvelinear) integral of the 1-st kind. We recall that $\operatorname{Im} g(x)\le 0$ and take the imaginary part in (2.4), then we obtain

$$\int_{(\Gamma^2)^+} |W_0(x)|^2|\operatorname{Im} g(x)|\,\mathrm{d}s + k\lim_{r\to\infty}\int_{\partial C_r}|W_0(x)|^2\,\mathrm{d}s = 0.$$

Since $k=\operatorname{Re}k>0$, we have

$$\lim_{r\to\infty}\int_{\partial C_r}|W_0(x)|^2\,\mathrm{d}s = 0,$$

and it follows from the Rellich lemma [2,25] that $W_0(x)\equiv 0$ for $x\in\mathcal{D}$. Hence the homogeneous Problem **V** has only a trivial solution, and the theorem is proved due to the linearity of the Problem **V**.

3. THE MODIFIED PROBLEM

In this section we consider the modification of the Problem **V** which will be called $\mathbf{V}_0$. Recall that in our notations $m=3$ and $m=2$ correspond to 3-D and 2-D cases respectively, and $\partial\mathcal{D}^2_n=\Gamma^2_n$ $(n=1,\ldots,N_2)$, $\partial\mathcal{D}^1_n=\Gamma^1_n$ $(n=1,\ldots,N_1)$. In each domain $\mathcal{D}^2_n$ we consider a simple closed surface if $m=3$ or curve if $m=2$ of class $C^{2,0}$ and denote it γ_n $(n=1,\ldots,N_2)$. Suppose that γ_n bounds the interior open single connected domain $\mathcal{D}^*_n\subset\mathcal{D}^2_n$. The surfaces (curves) γ_n are chosen in such a way that for any k from the set $\mathcal{I}\subset(0,\infty)$ the following

Dirichlet problem in $\mathcal{D}_n^*$:

$$\begin{cases} u(x) \in C^0(\overline{\mathcal{D}_n^*}) \cap C^2(\mathcal{D}_n^*), \\ \Delta u(x) + k^2 u(x) = 0, \quad x \in \mathcal{D}_n^*, \\ u(x)|_{x\in\gamma_n} = 0 \end{cases} \tag{3.1}$$

has only the trivial solution ($n = 1, \ldots, N_2$).

Clearly, surfaces (curves) γ_n can be chosen in different ways. For example, let

$$k \in \mathcal{I} = (0, k_0], \tag{3.2}$$

where k_0 is an arbitrary fixed positive number. For this set $\mathcal{I}$ as γ_n we can take an arbitrary sphere if $m = 3$ or an arbitrary circumference if $m = 2$ lying in $\mathcal{D}_n^2$ with the radius r satisfying the estimation

$$r < \frac{\pi}{k_0} \quad \text{if } m = 3, \qquad r < \frac{2.40475}{k_0} \quad \text{if } m = 2. \tag{3.3}$$

Let us show that the problem (3.1) has only the trivial solution for any k in (3.2). We consider the spectral problem in $\mathcal{D}_n^*$

$$\begin{cases} U(x) \in C^0(\overline{\mathcal{D}_n^*}) \cap C^2(\mathcal{D}_n^*), \\ \Delta U(x) + \lambda^2 U(x) = 0, \quad x \in \mathcal{D}_n^*, \\ U(x)|_{x\in\gamma_n} = 0. \end{cases} \tag{3.4}$$

The eigenvalues of (3.4) are positive and can be numbered is ascending order [1, p. 298]. The first eigenvalue of the ball is $\lambda_1 = \pi/r$. The first eigenvalue of the circle is $\lambda_1 = c/r$, where $c \cong 2.4048$ is the least positive root of the equation $J_0(z) = 0$. Here $J_0(z)$ is the Bessel function of index zero [1, p. 301; 24]. If $\lambda < \lambda_1(r)$ then the problem (3.4) has only a trivial solution. Consequently, the inequality $k_0 < \lambda_1(r)$ ensures that the problem (3.1) has only the trivial solution for any k in (3.2). This inequality leads to (3.3).

Instead of ball (circle) we can take as $\mathcal{D}_n^*$ an arbitrary single connected domain in $\mathcal{D}_n$, which diameter d satisfies the estimation

$$d < \ln\left(1 + \frac{1}{2k_0^2}\right), \tag{3.5}$$

then the problem (3.1) has only the trivial solution for any k in (3.2). This statement results from [2, Section 3.4, Lemma 3.26].

Thus, below we suppose that the surfaces (curves) $\gamma_1, \dots, \gamma_{N_2}$ are chosen in the following way.

$$\left\{\begin{array}{l}\text{Each } \gamma_n\ (n = 1, \dots, N_2) \text{ is a simple closed surface if } m = 3\\ \text{or curve if } m = 2, \text{such that } \gamma_n \subset \mathcal{D}_n^2, \gamma_n \text{ is of class } C^{2,0} \text{ and}\\ \text{the problem (3.1) has only a trivial solution for any } k\\ \text{belonging to the set } \mathcal{I} \subset (0, \infty).\end{array}\right. \tag{3.6}$$

Consider also one more assumption, where $\gamma_1, \dots, \gamma_{N_2}$ are taken constructively.

$$\left\{\begin{array}{l}\text{Let } k_0 \text{ be an arbitrary fixed positive number, and each } \gamma_n\\ (n = 1, \dots, N_2) \text{ is subject to one of two conditions:}\\ (1)\ \gamma_n \text{ is a sphere if } m = 3 \text{ or circumference if } m = 2,\\ \quad\ \gamma_n \subset \mathcal{D}_n^2, \text{ and the radius of } \gamma_n \text{ meets inequality (3.3);}\\ (2)\ \gamma_n \text{ is a simple closed surface if } m = 3 \text{ or curve if } m = 2,\\ \quad\ \gamma_n \text{ is of class } C^{2,0}, \gamma_n \subset \mathcal{D}_n^2, \text{ and the diameter of } \gamma_n \text{ meets}\\ \quad\ \text{inequality (3.5).}\end{array}\right. \tag{3.7}$$

As indicated above, *if* (3.7) *holds, then* (3.6) *is satisfied with* $\mathcal{I} = (0, k_0]$.

We put $\gamma = \bigcup_{n=1}^{N_2} \gamma_n$ and introduce the unit normal vector $\mathbf{n}_x$ to γ at $x \in \gamma$. If $x \in \gamma_n$, then the vector $\mathbf{n}_x$ is an outward normal regarding to the domain $\mathcal{D}_n^*$ bounded by γ_n. Consider γ_n as a double-sided surface (curve). By γ_n^- we denote that side of γ_n which we observe when facing towards the normal's tips. The opposite side of γ_n will be called γ_n^+. Set $\gamma^\pm = \bigcup_{n=1}^{N_2} \gamma_n^\pm$ and $\mathcal{D}^0 = \bigcup_{n=1}^{N_2} (\mathcal{D}_n^2 \backslash (\mathcal{D}_n^* \cup \gamma_n))$, $\mathcal{D}^1 = \bigcup_{n=1}^{N_1} \mathcal{D}_n^1$.

We say that the function $W(x)$ defined in $R^m \backslash \mathcal{D}^1$ belongs to the class of smoothness $\mathbf{K}_0$ if

(1) $W(x) \in C^0(R^m \backslash \mathcal{D}^1) \cap C^2(R^m \backslash (\mathcal{D}^1 \cup \Gamma \cup \gamma))$,
(2) the points (2) and (3) of the definition of the class $\mathbf{K}$ hold,
(3) there exists the uniform for all $x \in (\Gamma^2)^- \cup \gamma^-$ limit of $(\mathbf{n}_x, \nabla_{\bar{x}} W(\bar{x}))$ as $\bar{x} \in \mathcal{D}^0$ tends to $x \in (\Gamma^2)^- \cup \gamma^-$ along the normal $\mathbf{n}_x$.

Clearly, any function of class $\mathbf{K}_0$ belongs to the class $\mathbf{K}$ that is $\mathbf{K}_0 \subset \mathbf{K}$.
Now we formulate the modified problem, which we call $\mathbf{V}_0$.

PROBLEM $\mathbf{V}_0$ *In assumption that condition* (3.6) *holds we must find a function* $W(x)$ *of the class* $\mathbf{K}_0$, *so that* $W(x)$ *satisfies the Helmholtz equation* (2.1) *in* $R^m \backslash (\mathcal{D}^1 \cup \Gamma \cup \gamma)$, *meets the radiation conditions* (2.3), *satisfies the boundary conditions* (2.2) *and the additional homogeneous boundary condition on* γ^-

$$\left(\frac{\partial W(x)}{\partial \mathbf{n}_x} - \mathrm{i} W(x) \right) \Bigg|_{x \in \gamma^-} = 0. \tag{3.8}$$

By $\partial W(x)/\partial \mathbf{n}_x$ *on* γ^- *we mean the limit ensured in the point* (3) *of the definition of the smoothness class* $\mathbf{K}_0$. *All conditions of the Problem* $\mathbf{V}_0$ *must be satisfied in the classical sense.*

Clearly, *any solution of the Problem* $\mathbf{V}_0$ *is a solution of the Problem* $\mathbf{V}$.
Let us prove the uniqueness theorem.

THEOREM 2 *If condition* (3.6) *holds, then for any* $k \in \mathcal{I}$ *the Problem* $\mathbf{V}_0$ *has no more than one solution.*

Proof Let $W_0(x)$ be a solution of the homogeneous problem $\mathbf{V}_0$. Our aim is to show that $W_0(x) \equiv 0$. As noted above, $W_0(x)$ satisfies the homogeneous problem $\mathbf{V}$. According to Theorem 1:

$$W_0(x) \equiv 0, \quad x \in \mathcal{D}. \tag{3.9}$$

It follows from the definition of the class $\mathbf{K}_0$ that $W_0(x)$ is continuous across Γ^2. So, $W_0(x)$ satisfies the following homogeneous boundary value problem in $\mathcal{D}^0$:

$$\Delta W_0(x) + k^2 W_0(x) = 0, \quad x \in \mathcal{D}^0 = \bigcup_{n=1}^{N_2} (\mathcal{D}_n^2 \backslash (\mathcal{D}_n^* \cup \gamma_n)), \tag{3.10a}$$

$$W_0(x)|_{x \in \Gamma^2} = 0, \tag{3.10b}$$

$$\left(\frac{\partial W_0(x)}{\partial \mathbf{n}_x} - \mathrm{i} W_0(x) \right) \Bigg|_{x \in \gamma^-} = 0. \tag{3.10c}$$

We construct equidistant surfaces (curves) in $\mathcal{D}^0$ for boundaries Γ^2 and γ, write energy equalities in domains bounded by these surfaces (curves) and tend these surfaces (curves) to the boundaries [24]. Using the smoothness properties, ensured by the class $\mathbf{K}_0$, we obtain

$$\begin{aligned}
&-\|\nabla W_0\|^2_{L_2(\mathcal{D}^0)} + k^2\|W_0\|_{L_2(\mathcal{D}^0)} \\
&= \int_{(\Gamma^2)^- \cup \gamma^-} \overline{W}_0(x)\frac{\partial W_0(x)}{\partial \mathbf{n}_x}\,\mathrm{d}s \\
&= \int_{\gamma^-} \overline{W}_0(x)\frac{\partial W_0(x)}{\partial \mathbf{n}_x}\,\mathrm{d}s \\
&= \mathrm{i}\int_{\gamma} |W_0(x)|^2\,\mathrm{d}s,
\end{aligned}$$

where we applied the boundary conditions (3.10b) and (3.10c). Taking the imaginary part in the latter identity we have

$$\int_{\gamma} |W_0(x)|^2\,\mathrm{d}s = 0,$$

therefore

$$W_0(x)|_{x\in\gamma} = 0, \tag{3.11}$$

and thanks to (3.10c)

$$\left.\frac{\partial W_0(x)}{\partial \mathbf{n}_x}\right|_{x\in\gamma^-} = 0. \tag{3.12}$$

The function $W_0(x)$ is continuous across γ since $W_0(x)\in\mathbf{K}_0$. Taking into account (3.11) we observe that the function $W_0(x)$ satisfies the following Dirichlet problem in each domain $\mathcal{D}_n^*$ ($n = 1,\dots,N_2$):

$$\begin{aligned}
\Delta W_0(x) + k^2 W_0(x) = 0, \quad x \in \mathcal{D}_n^*, \\
W_0(x)|_{x\in\gamma_n} = 0.
\end{aligned}$$

It follows from the condition (3.6) that

$$W_0(x) \equiv 0, \quad x \in \mathcal{D}_n^* \;\; (n = 1,\dots,N_2), \tag{3.13}$$

therefore

$$\left.\frac{\partial W_0(x)}{\partial \mathbf{n}_x}\right|_{x\in\gamma^+} = 0. \tag{3.14}$$

Joining (3.11), (3.12) and (3.14) we obtain that the matching conditions hold

$$W_0(x)|_{x\in\gamma^+} = W_0(x)|_{x\in\gamma^-}, \quad \left.\frac{\partial W_0(x)}{\partial \mathbf{n}_x}\right|_{x\in\gamma^+} = \left.\frac{\partial W_0(x)}{\partial \mathbf{n}_x}\right|_{x\in\gamma^-}. \tag{3.15}$$

Recall that $W_0(x)$ is twice continuously differentiable and obeys the Helmholtz equation in $\left(\bigcup_{n=1}^{N_2} \mathcal{D}_n^2\right)\backslash\gamma$. Thanks to matching conditions (3.15), the function $W_0(x)$ can be analytically continued across γ, because γ is a set of removable singularities for $W_0(x)$. In other words, it can be shown with the help of (3.15) and the 3rd Green's formula [2,24] that $W_0(x) \in C^2(\mathcal{D}_n^2)$ $(n=1,\dots,N_2)$, and $W_0(x)$ satisfies the Helmholtz equation (2.1) everywhere in $\mathcal{D}_n^2$, in particular, on γ_n. As shown in (3.13), $W_0(x)$ is identically equal to zero in the subdomain of $\mathcal{D}_n^2$, because $\mathcal{D}_n^* \subset \mathcal{D}_n^2$. At the same time, $W_0(x)$ is analytic in $\mathcal{D}_n^2$ as a solution of the Helmholtz equation [23, Chapter 4, Section 4.4]. According to the method of analytic continuation we can prove that

$$W_0(x) \equiv 0, \quad x \in \mathcal{D}_n^2 \ (n = 1,\dots,N_2). \tag{3.16}$$

Remark Indeed, we can show that $W_0(x^0)=0$ for any $x^0 \in \mathcal{D}_n^2$. To prove this we connect x^0 and a fixed interior point of $\mathcal{D}_n^*$ by an arc lying in $\mathcal{D}_n^2$. Then we cover the arc by a finite number of balls (circles if $m=2$) which lie in $\mathcal{D}_n^2$. The center of the first ball (circle) is the mentioned interior point of $\mathcal{D}_n^*$. Then balls (circles) go to x^0, so that the center of each ball (circle) is contained in the previous one and belongs to the arc. The last ball (circle) contains x^0. Since $W_0(x)$ is analytic in $\mathcal{D}_n^2$ as a solution of the Helmholtz equation [23, Chapter 4, Section 4.4], we expand it in the convergence Taylor series in each ball (circle). The coefficients of the Taylor series coincide with derivatives of $W_0(x)$ in the origin. The Taylor expansion in the first ball (circle) is identically equal to zero, because $W_0(x) \equiv 0$ in the vicinity of its origin thanks to (3.13), and so all derivatives of $W_0(x)$ in the origin as well as Taylor coefficients are equal to zero. Then we sequentially show that expansions in all other balls (circles) are also equal to zero, since the vicinity of the origin of each ball (circle) lies in the previous ball (circle), where $W_0(x)$ is identically equal to zero. Consequently, $W_0(x^0) = 0$ for any $x^0 \in \mathcal{D}_n^2$.

Using (3.9), (3.16) and smoothness of $W_0(x)$ ensured by the class $\mathbf{K}_0$ we obtain

$$W_0(x) \equiv 0 \quad \text{in } R^m \backslash \mathcal{D}^1 \quad (m = 2 \text{ or } m = 3).$$

Thus, the homogeneous problem $\mathbf{V}_0$ has only a trivial solution. Consequently, the inhomogeneous problem $\mathbf{V}_0$ has no more than one solution. The theorem is proved.

4. INTEGRAL EQUATIONS AND THE SOLUTION THE PROBLEM

In the present section we obtain the solution of the Problem $\mathbf{V}_0$ in the form of potentials which density obeys the uniquely solvable Fredholm equation of the second kind on the total boundary $\Gamma \cup \gamma$. As noted above this solution of the Problem $\mathbf{V}_0$ is also a solution of the Problem $\mathbf{V}$.

To prove existence theorem we impose the additional conditions to the functions in (2.2):

$$g(x) \in C^0(\Gamma^2), \tag{4.1a}$$

$$f(x) \in C^{1,\lambda}(\Gamma^1) \cap C^0(\Gamma^2), \quad \lambda \in (0, 1]. \tag{4.1b}$$

We look for a solution of the Problem $\mathbf{V}_0$ in the form:

$$W[\mu](x) = \int_{\Gamma^2 \cup \gamma} \mu(y) \Phi_k(x, y)\, \mathrm{d}s_y + \int_{\Gamma^1} \mu(y) \left(\frac{\partial}{\partial \mathbf{n}_y} - \mathrm{i} \right) \Phi_k(x, y)\, \mathrm{d}s_y, \tag{4.2}$$

where $y=(y_1,\ldots,y_m)\in\Gamma\cup\gamma$ and $\Phi_k(x,y)$ is a fundamental solution of the Helmholtz equation (2.1) in R^m, so that

$$\Phi_k(x,y)=\begin{cases}\dfrac{\mathrm{i}}{4}\mathcal{H}_0^{(1)}(k|x-y|) & \text{if } m=2,\\[2mm] \dfrac{1}{4\pi}\dfrac{\exp(\mathrm{i}k|x-y|)}{|x-y|} & \text{if } m=3.\end{cases}$$

By $\mathcal{H}_0^{(1)}(z)$ we denote the Hankel function of the 1st kind and index zero [16,24]

$$\mathcal{H}_0^{(1)}(z)=\frac{\sqrt{2}\exp(\mathrm{i}z-\mathrm{i}\pi/4)}{\pi\sqrt{z}}\int_0^\infty \exp(-t)t^{-1/2}\left(1+\frac{\mathrm{i}t}{2z}\right)^{-1/2}\mathrm{d}t.$$

We look for the density $\mu(x)$ of the potential (4.2) in $C^{1,\omega}(\Gamma^1)\cap C^0(\Gamma^2\cup\gamma)$, where the Hölder exponent $\omega\in(0,1)$. According to the properties of potentials [2,6,24,25] the function (4.2) belongs to the class $\mathbf{K}_0$ and satisfies all conditions of the Problem $\mathbf{V}_0$ except for the boundary conditions on Γ^+ and γ^-. To satisfy the boundary conditions we substitute (4.2) into (2.2) and (3.8), use the limit formulas for normal derivatives of a single layer potential [2,6,24,25] and arrive at the following integral equations of the second kind for the density $\mu(x)$:

$$-\frac{1}{2}\mu(x)+\int_{\Gamma^1}\mu(y)\left(\frac{\partial}{\partial\mathbf{n}_y}-\mathrm{i}\right)\Phi_k(x,y)\,\mathrm{d}s_y$$
$$+\int_{\Gamma^2\cup\gamma}\mu(y)\Phi_k(x,y)\,\mathrm{d}s_y=f(x),\quad x\in\Gamma^1, \tag{4.3a}$$

$$\frac{1}{2}\mu(x)+\int_{\Gamma^2\cup\gamma}\mu(y)\left(\frac{\partial}{\partial\mathbf{n}_x}+g(x)\right)\Phi_k(x,y)\,\mathrm{d}s_y$$
$$+\int_{\Gamma^1}\mu(y)\left(\frac{\partial}{\partial\mathbf{n}_x}+g(x)\right)\left(\frac{\partial}{\partial\mathbf{n}_y}-\mathrm{i}\right)\Phi_k(x,y)\,\mathrm{d}s_y=f(x),\quad x\in\Gamma^2, \tag{4.3b}$$

$$-\frac{1}{2}\mu(x)+\int_{\Gamma^2\cup\gamma}\mu(y)\left(\frac{\partial}{\partial\mathbf{n}_x}-\mathrm{i}\right)\Phi_k(x,y)\,\mathrm{d}s_y$$
$$+\int_{\Gamma^1}\mu(y)\left(\frac{\partial}{\partial\mathbf{n}_x}-\mathrm{i}\right)\left(\frac{\partial}{\partial\mathbf{n}_y}-\mathrm{i}\right)\Phi_k(x,y)\,\mathrm{d}s_y=0,\quad x\in\gamma. \tag{4.4}$$

Set

$$\delta(y,\Gamma^1)=\begin{cases}0 & \text{if } y\notin\Gamma^1,\\ 1 & \text{if } y\in\Gamma^1.\end{cases}$$

Equations (4.3) and (4.4) can be written in the form of one equation of the second kind on the whole boundary $\Gamma\cup\gamma$:

$$\frac{1}{2}\mu(x)+\int_{\Gamma\cup\gamma}\mu(y)\Omega_k(x,y)\,\mathrm{d}s_y=f_0(x),\quad x\in\Gamma\cup\gamma, \tag{4.5}$$

where

$$\Omega_k(x,y)$$
$$=\begin{cases}-\left(\delta(y,\Gamma^1)\left(\dfrac{\partial}{\partial\mathbf{n}_y}-\mathrm{i}\right)+(1-\delta(y,\Gamma^1))\right)\Phi_k(x,y) & \text{if } x\in\Gamma^1,\\ \left(\dfrac{\partial}{\partial\mathbf{n}_x}+g(x)\right)\left[\left((1-\delta(y,\Gamma^1))+\delta(y,\Gamma^1)\left(\dfrac{\partial}{\partial\mathbf{n}_y}-\mathrm{i}\right)\right)\Phi_k(x,y)\right] & \text{if } x\in\Gamma^2,\\ -\left(\dfrac{\partial}{\partial\mathbf{n}_x}-\mathrm{i}\right)\left[\left((1-\delta(y,\Gamma^1))+\delta(y,\Gamma^1)\left(\dfrac{\partial}{\partial\mathbf{n}_y}-\mathrm{i}\right)\right)\Phi_k(x,y)\right] & \text{if } x\in\gamma,\end{cases}$$

$$f_0(x)=\begin{cases}f(x) & \text{if } x\in\Gamma,\\ 0 & \text{if } x\in\gamma.\end{cases} \tag{4.6}$$

Since $\Gamma^1\in C^{2,\lambda}$, $\Gamma^2\cup\gamma\in C^{2,0}$, the kernel in the integral equation (4.5) has a weak singularity, and the integral term in (4.5) is continuous on $\Gamma\cup\gamma$ in x (see [2,24]). Therefore, the integral operator in (4.5) maps $C^0(\Gamma\cup\gamma)$ into itself. Moreover, (4.5) is a Fredholm integral equation in $C^0(\Gamma\cup\gamma)$, because its kernel has a weak singularity [24].

Let us show that any solution of integral equation (4.5) in $C^0(\Gamma\cup\gamma)$ automatically belongs to $C^{1,\omega}(\Gamma^1)\cap C^0(\Gamma^2\cup\gamma)$ with $\omega\in(0,1]$. Indeed, let $\mu(x)$ be an arbitrary solution of the integral equation (4.5) in $C^0(\Gamma \cup \gamma)$, i.e. $\mu(x)$ obeys (4.3) and (4.4). The second integral term in (4.3a) is infinitely differentiable in x, since it does not have singularity if $x = y$. According to [2, Theorem 2.15], the first integral in (4.3a) is a Hölder function in x on Γ^1. It follows from the identity (4.3a) for $\mu(x)$ that $\mu(x)$ is a Hölder function in x on Γ^1 also (here we take into account condition (4.1b)). Using [2, Theorem 2.22] we verify that the first integral term in (4.3a) belongs to $C^{1,\omega_0}(\Gamma^1)$ in x for some $\omega_0 \in (0,1)$. Proceeding from the identity (4.3a) for $\mu(x)$ and taking into account (4.1) we obtain: $\mu(x) \in C^{1,\omega}(\Gamma^1)$, where $\omega = \min\{\omega_0, \lambda\}$, $0 < \omega < 1$.

So, any solution of Eq. (4.5) in $C^0(\Gamma \cup \gamma)$ automatically belongs to $C^{1,\omega}(\Gamma^1) \cap C^0(\Gamma^2 \cup \gamma)$ with $\omega \in (0,1)$. The potential $W[\mu](x)$ belongs to the class $\mathbf{K}_0$ and satisfies all conditions of the Problem $\mathbf{V}_0$. We arrive at

LEMMA *Let conditions* (3.6) *and* (4.1) *hold. If* $\mu(x) \in C^0(\Gamma \cup \gamma)$ *obeys Fredholm equation* (4.5), *then*

(1) $\mu(x) \in C^{1,\omega}(\Gamma^1) \cap C^0(\Gamma^2 \cup \gamma)$ *for some* $\omega \in (0,1)$;
(2) *the potential* (4.2) *is a solution of the Problem* $\mathbf{V}_0$.

Remark The lemma is true for any $k \geq 0$ in a 3-D case and for any $k > 0$ in a 2-D case, i.e. we do not require that $k \in \mathcal{I}$ in lemma.

Thus, below we look for a solution of Eq. (4.5) in $C^0(\Gamma \cup \gamma)$.

Assuming that condition (3.6) holds, we will show that the homogeneous Fredholm equation (4.5) has only the trivial solution for any $k \in \mathcal{I}$. Let $\mu^0(x)$ be a solution of the homogeneous equation (4.5), then it obeys homogeneous equations (4.3) and (4.4). We substitute $\mu^0(x)$ in (4.2) and consider a function $W[\mu^0](x)$. On the basis of the lemma, $W[\mu^0](x)$ is a solution of the homogeneous problem $\mathbf{V}_0$. According to the Theorem 2, since condition (3.6) holds, this problem has only a trivial solution for any $k \in \mathcal{I}$, and we obtain

$$W[\mu^0](x) \equiv 0 \quad \text{in } R^m \backslash \mathcal{D}^1 \quad (m = 2 \text{ or } m = 3).$$

Recall, $W[\mu^0](x)$ belongs to the class $\mathbf{K}_0$, and so $W[\mu^0](x) \in C^0(\overline{R^m \backslash \mathcal{D}^1}) \cap C^2(R^m \backslash (\mathcal{D}^1 \cup \Gamma \cup \gamma))$. Using the jump formulas [2,6, 24,25] for the normal derivatives of the single layer potential on Γ^2

and γ we obtain

$$\frac{\partial}{\partial \mathbf{n}_x} W[\mu^0](x)\bigg|_{x\in(\Gamma^2)^+} - \frac{\partial}{\partial \mathbf{n}_x} W[\mu^0](x)\bigg|_{x\in(\Gamma^2)^-} = \mu^0(x)|_{x\in\Gamma^2} = 0,$$

$$\frac{\partial}{\partial \mathbf{n}_x} W[\mu^0](x)\bigg|_{x\in\gamma^+} - \frac{\partial}{\partial \mathbf{n}_x} W[\mu^0](x)\bigg|_{x\in\gamma^-} = \mu^0(x)|_{x\in\gamma} = 0.$$

Consequently, $\mu^0(x)\equiv 0$ for $x\in\Gamma^2\cup\gamma$ and $\mu^0(x)$ satisfies the following homogeneous equation on Γ^1:

$$-\frac{1}{2}\mu^0(x) + \int_{\Gamma^1} \mu^0(y)\left(\frac{\partial}{\partial \mathbf{n}_y} - i\right)\Phi_k(x,y)\,\mathrm{d}s_y = 0, \quad x\in\Gamma^1. \tag{4.7}$$

To study this equation using the Fredholm alternative we consider the adjoint integral equation

$$-\frac{1}{2}\rho^0(x) + \int_{\Gamma^1} \rho^0(y)\left(\frac{\partial}{\partial \mathbf{n}_x} + i\right)\bar{\Phi}_k(x,y)\,\mathrm{d}s_y = 0, \quad x\in\Gamma^1. \tag{4.8}$$

where

$$\bar{\Phi}_k(x,y) = \begin{cases} -\dfrac{\mathrm{i}}{4}\mathcal{H}_0^{(2)}(k|x-y|) & \text{if } m=2, \\ \dfrac{1}{4\pi}\dfrac{\exp(-\mathrm{i}k|x-y|)}{|x-y|} & \text{if } m=3 \end{cases}$$

is a fundamental solution for Eq. (2.1). Below suppose that $\rho^0(x)$ is an arbitrary solution of the homogeneous equation (4.8) in $C^0(\Gamma^1)$. Then the single layer potential

$$v(x) = v[\rho^0](x) = \int_{\Gamma^1} \rho^0(y)\bar{\Phi}_k(x,y)\,\mathrm{d}s_y \in C^0(R^m)\cap C^2(R^m\backslash\Gamma^1)$$

obeys the Helmholtz equation (2.1) in $R^m\backslash\Gamma^1$ and meets the radiation conditions at infinity:

$$v = \mathrm{O}(|x|^{(1-m)/2}),\ \frac{\partial v}{\partial|x|} + \mathrm{i}kv = \mathrm{o}(|x|^{(1-m)/2}), \quad |x|\to\infty. \tag{4.9}$$

In addition, $\partial v/\partial \mathbf{n}_x$ exists on $(\Gamma^1)^+$ and $(\Gamma^1)^-$ as a uniform (regarding to $x \in \Gamma^1$) limit in the normal direction. The potential $v(x)$ satisfies the interior impedance problem in $\mathcal{D}_n^1$ $(n = 1, \ldots, N_1)$:

$$\begin{aligned} &\Delta v + k^2 v = 0 \quad \text{in } \mathcal{D}_n^1, \\ &\frac{\partial v}{\partial \mathbf{n}_x} + \mathrm{i}v = 0 \quad \text{on } (\Gamma_n^1)^-, \end{aligned} \tag{4.10}$$

where (4.10) holds since (4.8) is true. Consider the energy equality for Eq. (2.1) in $\mathcal{D}_n^1$ $(n = 1, \ldots, N_1)$:

$$-\|\nabla v\|^2_{L_2(\mathcal{D}_n^1)} + k^2 \|v\|^2_{L_2(\mathcal{D}_n^1)} = \int_{(\Gamma_n^1)^-} \bar{v}(x) \frac{\partial v(x)}{\partial \mathbf{n}_x} \,\mathrm{d}s = -\mathrm{i} \int_{(\Gamma_n^1)^-} |v(x)|^2 \,\mathrm{d}s. \tag{4.11}$$

This equality can be derived by the technique of equidistant surfaces (curves) [24]. Taking the imaginary part in (4.11) we obtain

$$\int_{(\Gamma_n^1)^-} |v(x)|^2 \,\mathrm{d}s = 0 \quad (n = 1, \ldots, N_1).$$

Consequently

$$v(x)|_{(\Gamma_n^1)^-} = v(x)|_{\Gamma_n^1} = v(x)|_{(\Gamma_n^1)^+} = 0 \quad (n = 1, \ldots, N_1), \tag{4.12}$$

because the single layer potential is continuous across Γ^1. From (4.10) we have

$$\left.\frac{\partial v}{\partial \mathbf{n}_x}\right|_{(\Gamma_n^1)^-} = 0 \quad (n = 1, \ldots, N_1). \tag{4.13}$$

Using (4.12) we verify that $v(x)$ obeys the following homogeneous Dirichlet problem for the Eq. (2.1) in the exterior domain $R^m \backslash \mathcal{D}^1$:

$$\begin{gathered} \Delta v + k^2 v = 0 \quad \text{in } R^m \backslash \mathcal{D}^1, \\ v|_{(\Gamma^1)^+} = 0, \\ v = \mathrm{O}(|x|^{(1-m)/2}), \quad \frac{\partial v}{\partial |x|} + \mathrm{i}kv = \mathrm{o}(|x|^{(1-m)/2}), \quad |x| \to \infty. \end{gathered}$$

One can prove with the help of energy equalities and the Rellich lemma [2] that this problem has only the trivial solution $v(x)\equiv 0$ in $R^m\backslash\mathcal{D}^1$. Thanks to existing a uniform normal derivative of $v(x)$ on $(\Gamma^1)^+$ we obtain

$$\left.\frac{\partial v}{\partial \mathbf{n}_x}\right|_{(\Gamma^1)^+} = 0. \tag{4.14}$$

Using (4.13), (4.14) and the jump relation for the single layer potential, we obtain

$$\left.\frac{\partial v}{\partial \mathbf{n}_x}\right|_{(\Gamma^1)^+} - \left.\frac{\partial v}{\partial \mathbf{n}_x}\right|_{(\Gamma^1)^-} = \rho^0(x) = 0, \quad x\in\Gamma^1.$$

Hence, $\rho^0(x)\equiv 0$ on Γ^1 and homogeneous equation (4.8) has only a trivial solution. Proceeding from the Fredholm alternative, the adjoint homogeneous equation (4.7) has only a trivial solution also, i.e. $\mu^0(x)\equiv 0$ on Γ^1. Therefore $\mu^0(x)\equiv 0$ on $\Gamma\cup\gamma$.

Thus, assuming that condition (3.6) holds, we have proved that the homogeneous Fredholm integral equation (4.5) has only a trivial solution for any $k\in\mathcal{I}$. According to Fredholm alternative the inhomogeneous equation (4.5) is uniquely solvable in these assumptions for any $f_0(x)\in C^0(\Gamma\cup\gamma)$. We arrive at

THEOREM 3 *Let conditions* (3.6) *and* (4.1a) *hold. If $k\in\mathcal{I}$, then the Fredholm integral equation* (4.5) *has a unique solution $\mu(x)\in C^0(\Gamma\cup\gamma)$ for any $f_0(x)\in C^0(\Gamma\cup\gamma)$, in particular, for any $f(x)\in C^0(\Gamma)$ in* (4.6). *If, in addition, condition* (4.1b) *holds, then the solution $\mu(x)$ belongs to $C^{1,\omega}(\Gamma^1)\cap C^0(\Gamma^2\cup\gamma)$ with some $\omega\in(0,1)$.*

Recall that condition (3.6) follows from (3.7).

COROLLARY 1 *Let conditions* (3.7) *and* (4.1a) *hold. If $k\in(0,k_0]$, then the Fredholm equation* (4.5) *has a unique solution $\mu(x)\in C^0(\Gamma\cup\gamma)$ for any $f_0(x)\in C^0(\Gamma\cup\gamma)$, in particular, for any $f(x)\in C^0(\Gamma)$ in* (4.6). *If, in addition, condition* (4.1b) *holds, then the solution $\mu(x)$ belongs to $C^{1,\omega}(\Gamma^1)\cap C^0(\Gamma^2\cup\gamma)$ with some $\omega\in(0,1)$.*

The last statement of Theorem 3 and Corollary 1 follows from the lemma. From Theorem 3 and lemma we obtain the solvability theorem for the Problem $\mathbf{V}_0$.

THEOREM 4 *If conditions* (3.6) *and* (4.1) *hold, then for any* $k \in \mathcal{I}$ *the solution of the Problem* $\mathbf{V}_0$ *exists and is given by a potential* (4.2), *where* $\mu(x)$ *is a unique solution of the Fredholm integral equation* (4.5), *ensured by the Theorem* 3.

As noted above, any solution of the Problem $\mathbf{V}_0$ satisfies the Problem $\mathbf{V}$. Therefore the solution of the Problem $\mathbf{V}_0$ constructed in the Theorem 4 satisfies the Problem $\mathbf{V}$.

THEOREM 5 *If conditions* (3.6) *and* (4.1) *hold, then for any* $k \in \mathcal{I}$ *the solution of the problem* $\mathbf{V}$ *exists and is given by the potential* (4.2), *where* $\mu(x)$ *is a unique solution of the Fredholm integral equation* (4.5), *ensured by the Theorem* 3.

COROLLARY 2 *Let assumptions* (3.7) *and* (4.1) *hold. Then for any* $k \in (0, k_0]$ *the solution of the Problem* $\mathbf{V}$ *is given by* (4.2), *where* $\mu(x)$ *is a unique solution of the Fredholm integral equation* (4.5) *ensured by the Corollary* 1.

Remark Consider a 3-D case ($m = 3$). Suppose that in addition to the condition $\operatorname{Im} g(x) \leq 0$, $x \in \Gamma^2$, we have $\operatorname{Re} g(x) \geq 0$, $x \in \Gamma^2$. Then Theorem 1 holds for $k \geq 0$, and Corollaries 1 and 2 hold for $k \in [0, k_0]$. This case includes 3-D Neumann problem as well as 3-D mixed Dirichlet–Neumann problem.

Note that condition (3.7) can be always satisfied.

The Theorem 5 and the Corollary 2 are the main results of the present paper. Basing on the method of Fredholm integral equations, we proved the solvability of the Problem $\mathbf{V}$ for $\Gamma^1 \in C^{2,\lambda}$, $\Gamma^2 \in C^{2,0}$, $g(x) \in C^0(\Gamma^2)$, $f(x) \in C^{1,\lambda}(\Gamma^1) \cap C^0(\Gamma^2)$. In fact, our proof is valid for any positive k, since k_0 can be taken as large as necessary. The method of hypersingular integral equations of Γ^2 presented in [2,12,17] does not enable do so, since the normal derivative of the double layer potential used in this method may not exist under our conditions. Recall that in the method of a hypersingular integral equation we look for a solution of the Problem $\mathbf{V}$ on Γ^2 as a sum of a single and double layer potentials.

The basic idea of our method is such that we introduce the interior boundary inside interior domains (scatterers) bounded by Γ^2 and reduce the Problem **V** to the uniquely solvable Fredholm equation on the whole boundary. The solution of the problem is represented in the form of potentials on the whole boundary. From physical stand-point the single layer potential defined on interior boundaries can be considered as distributed sources placed inside interior domains (scatterers) instead of infinite number of point sources used in [4,5,21,22].

The advantage of our approach is so that the uniquely solvable Fredholm integral equation (4.5) can be computed by standard codes, i.e. by discretization and inversion of a matrix. Since our method holds for any $k \in (0, k_0]$, where k_0 is an arbitrary positive number, the Problem **V** can be computed for different k without any changes in a computational scheme.

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