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Research Article

Oscillation Criteria of Third-Order Nonlinear Impulsive Differential Equations with Delay

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This paper deals with the oscillation of third-order nonlinear impulsive equations with delay. The results in this paper improve and extend some results for the equations without impulses. Some examples are given to illustrate the main results.

1. Introduction

In this paper, we are concerned with oscillation of the third-order nonlinear impulsive equations with delay

$$\begin{gathered} x'''(t) + f(t, x(t), x(t-\sigma)) = 0, \quad t \ge t_0, \ t \ne \tau_k, \\ x(\tau_k^+) = a_k x(\tau_k), \qquad x'(\tau_k^+) = b_k x'(\tau_k), \\ x''(\tau_k^+) = c_k x''(\tau_k), \quad k = 1, 2, \ldots, \end{gathered} \tag{1}$$

where $\sigma > 0$ is the delay, $\{\tau_k\}$ is the sequence of impulsive moments which satisfies $0 \le t_0 < \tau_1 < \cdots < \tau_k < \cdots$ and $\lim_{k\to\infty}\tau_k = \infty$, $\tau_{k+1} - \tau_k \ge \sigma$.

Throughout this paper, we will assume that the following assumptions are satisfied:

(H1) $f(t, u, v)$ is continuous in $[t_0 - \sigma, \infty) \times \mathbb{R} \times \mathbb{R}$, $uf(t, u, v) > 0$ for $uv > 0$;

(H2) $f(t, u, v)/\varphi(v) \ge p(t)$ for $v \ne 0$, where $p(t)$ is continuous in $[t_0 - \sigma, \infty)$, $p(t) \ge 0 (\not\equiv 0)$, $\varphi(x)/x \ge \mu > 0$ for all $x \ne 0$;

(H3) a_k, b_k, and c_k are positive constants.

Our attention is restricted to those solutions of (1) which exist on half line $[t_0, \infty)$ and satisfy $\sup\{|x(t)| : t > T\} > 0$ for any $T \ge t_x$. For the general theory of impulsive differential equations with/without delay, we refer the readers to monographs or papers [1–4]. A solution of (1) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is nonoscillatory. It is well known that there is a drastic difference in the behavior of solutions between differential equations with impulses and those without impulses. Some differential equations are nonoscillatory, but they may become oscillatory if some proper impulse controls are added to them, see [5] and Example 13 in Section 4. In the recent years, the oscillation theory and asymptotic behavior of impulsive differential equations and their applications have been and still are receiving intensive attention. For contribution, we refer to the recent survey paper by Agarwal et al. [6] and the references cited therein. But to the best of the authors' knowledge, it seems that little has been done for oscillation of third-order impulsive differential equations [7].

When $a_k = b_k = c_k = 1$, (1) reduces third-order delay equation with/without delay, which oscillatory theory has been studied by many researchers, see [8–12].

Our aim in this paper is to establish some new sufficient conditions which ensure that the solutions of (1) oscillate or converge to a finite limit as t tends to infinity. In particular, we extend some results in [9, 11] to impulsive delay differential equations. The results in this paper are more general compared by those obtained by Mao and Wan [7] and improve some of the results in [7] (see Example 13 in Section 4). The new results will be proved by making use of the techniques used in [9, 11].

The paper is organized as follows. In Section 2, we prove some lemmas which play important roles in the proof of the main results. In Section 3, some new sufficient conditions which guarantee that the solution of (1) oscillates or converges to a finite limit are established. In Section 4, two examples are given to illustrate the main results.

2. Preliminary Results

In this section, we state and prove some lemmas which we will need in the proofs of the main results. First of all, we introduce the following notations: $\mathbb{R}^+$ and $\mathbb{N}$ are the sets of real numbers and positive integer numbers, respectively, PC^1 is defined by

$$\begin{aligned} &\mathrm{PC}^1(\mathbb{R}^+,\mathbb{R}) \\ &\quad = \{x : \mathbb{R}^+ \longrightarrow \mathbb{R} : x(t) \text{ is differentiable for } \\ &\qquad t \geq 0 \text{ and } t \neq \tau_k, x(\tau_k^+) \text{ and } x'(\tau_k^+) \text{ exist, } \\ &\qquad \text{and } x(\tau_k^-) = x(\tau_k),\ x'(\tau_k^-) = x'(\tau_k)\}. \end{aligned} \tag{2}$$

The following lemma is from Lakshmikantham et al. [3, Page 32, Theorem 1.4.1].

Lemma 1. *Assume that*

(i) $\{\tau_k\}_{k\in\mathbb{N}}$ *is the impulse moments sequence with* $0 \leq t_0 < \tau_1 < \cdots < \tau_k < \cdots$, $\lim_{k\to\infty}\tau_k = \infty$;

(ii) $m \in \mathrm{PC}^1(\mathbb{R}^+,\mathbb{R})$, *and for* $t \geq t_0$, $k \in \mathbb{N}$, *it holds that*

$$\begin{aligned} m'(t) &\leq u(t)m(t) + v(t), \qquad t \neq \tau_k, \\ m(\tau_k^+) &\leq d_k m(\tau_k) + e_k, \end{aligned} \tag{3}$$

where $u, v \in C(\mathbb{R}^+,\mathbb{R})$, $d_k \geq 0$, *and* e_k *are real constants. Then,*

$$\begin{aligned} m(t) \leq{}& m(t_0)\prod_{t_0<\tau_k<t} d_k \exp\left(\int_{t_0}^t u(s)\mathrm{d}s\right) \\ &+ \int_{t_0}^t \prod_{s<\tau_k<t} d_k \exp\left(\int_s^t u(\sigma)\mathrm{d}\sigma\right) v(s)\mathrm{d}s \\ &+ \sum_{t_0<\tau_k<t}\prod_{\tau_k<\tau_j<t} d_j \exp\left(\int_{\tau_k}^t u(s)\mathrm{d}s\right) e_k. \end{aligned} \tag{4}$$

Motivated by the ideas of Chen and Feng [5], we present the following key lemma which determines the sign of $x'(t)$ and $x''(t)$ of the nonoscillation solution $x(t)$ of (1).

Lemma 2. *Suppose that* $x(t)$ *is an eventually positive solution of* (1), *and*

$$\lim_{t\to\infty}\int_{t_0}^t \prod_{t_0<\tau_k<s}\frac{b_k}{a_k}\,\mathrm{d}s = \infty, \tag{5}$$

$$\lim_{t\to\infty}\int_{t_0}^t \prod_{t_0<\tau_k<s}\frac{c_k}{b_k}\,\mathrm{d}s = \infty. \tag{6}$$

Then, it holds that one of the following two cases for sufficiently large T:

(i) $x''(\tau_k^+) > 0$, $x''(t) > 0$ *and* $x'(\tau_k^+) > 0$, $x'(t) > 0$,

(ii) $x''(\tau_k^+) > 0$, $x''(t) > 0$ *and* $x'(\tau_k^+) < 0$, $x'(t) < 0$,

with $t \in (\tau_k, \tau_{k+1}]$ *and* $\tau_k \geq T$.

Proof. Assume that $x(t)$ is an eventually positive solution of (1). We may assume that there exists $t_1 > t_0$ such that $x(t) > 0$ and $x(t-\sigma) > 0$ for $t \geq t_1$. First, we assert that $x''(\tau_k) > 0$ for any $k \in \mathbb{N}$. Suppose not, there exists some $\tau_j \geq t_1$ such that $x''(\tau_j) \leq 0$. By

$$x'''(t) = -f(t, x(t), x(t-\sigma)) \leq 0, \not\equiv 0, \tag{7}$$

we have $x''(t)$ monotonically decreasing in $(\tau_i, \tau_{i+1}]$, $i = j, j+1, \ldots$. Thus, $x''(\tau_i^+) = c_i x''(\tau_i) < 0$, $i = j+1, j+2, \ldots$. Consider the impulsive differential inequalities

$$\begin{aligned} x'''(t) &< 0, \quad t \geq \tau_{j+1},\ t \neq \tau_k, \\ x''(\tau_k^+) &\leq c_k x''(\tau_k), \quad k = j+1, j+2, \ldots. \end{aligned} \tag{8}$$

By Lemma 1, we have

$$x''(t) \leq x''(\tau_{j+1}^+)\prod_{\tau_{j+1}<\tau_k<t} c_k := -\alpha \prod_{\tau_{j+1}<\tau_k<t} c_k < 0. \tag{9}$$

There are two cases of the sign of $x'(\tau_k)$.

Case 1. If there exists some $\tau_n \geq \tau_{j+1}$ such that $x'(\tau_n) \leq 0$, since $x''(t) < 0$, then $x'(\tau_{n+1}) < x'(\tau_n^+) \leq 0$ and $x'(\tau_{n+1}^+) := \beta = b_{n+1}x'(\tau_{n+1}) < 0$. By induction it easily show $x'(\tau_k) < 0$, and hence $x'(\tau_k^+) \leq b_k x'(\tau_k) < 0$ for $k = n+1, n+2, \ldots$. So, we obtain the following impulsive differential inequalities:

$$\begin{aligned} x''(t) &\leq 0, \quad t \geq \tau_{n+1},\ t \neq \tau_k, \\ x'(\tau_k^+) &\leq b_k^* x'(\tau_k), \quad k = n+1, n+2, \ldots, \end{aligned} \tag{10}$$

which follows from Lemma 1 that

$$x'(t) \leq x'(\tau_{n+1}^+)\prod_{\tau_{n+1}<\tau_k<t} b_k = -\beta\prod_{\tau_{n+1}<\tau_k<t} b_k. \tag{11}$$

From (11) and applying Lemma 1, noting that $x(t) > 0$ for $t > t_1$ and $x(\tau_k^+) \leq a_k x(\tau_k)$ for any $k \in \mathbb{N}$, we have

$$\begin{aligned} x(t) &\leq x(\tau_{n+1}^+)\prod_{\tau_{n+1}<\tau_k<t} a_k - \beta\int_{\tau_{n+1}}^t \prod_{s<\tau_k<t} a_k \prod_{\tau_{n+1}<\tau_k<s} b_k \mathrm{d}s \\ &= \prod_{\tau_{n+1}<\tau_k<t} a_k \left(x(\tau_{n+1}^+) - \beta\int_{\tau_{n+1}}^t \prod_{\tau_{n+1}<\tau_k<s}\frac{b_k}{a_k}\mathrm{d}s\right). \end{aligned} \tag{12}$$

Thus, by (5) we have $x(t) < 0$ for t sufficiently large which is a contradiction.

Case 2. If $x'(\tau_k) > 0$ for any $k \geq j$, noting that $x''(t) < 0$ for $t \in (\tau_{j+1}, \tau_{j+2}]$, we have $x'(t) > x'(\tau_{j+1}) > 0$. By induction, we get that $x'(t) > 0$ for any $t \in (\tau_k, \tau_{k+1}]$, $k = j+1, j+2, \ldots$. So the following impulsive differential inequalities hold:

$$x''(t) \le -\alpha \prod_{\tau_{j+1}<\tau_k<t} c_k, \quad t > \tau_{j+1},\ t \ne \tau_k,$$
$$x'(\tau_k^+) \le b_k x(\tau_k), \quad k = j+1, j+2, \ldots. \tag{13}$$

According to Lemma 1, we get

$$\begin{aligned} x'(t) &\le x'(\tau_{j+1}^+) \prod_{\tau_{j+1}<\tau_k<t} b_k - \alpha \int_{\tau_{j+1}}^{t} \prod_{s<\tau_k<t} b_k \prod_{\tau_{j+1}<\tau_k<s} c_k \, ds \\ &= \prod_{\tau_{j+1}<\tau_k<t} b_k \left(x'(\tau_{j+1}^+) - \alpha \int_{\tau_{j+1}}^{t} \prod_{\tau_{j+1}<\tau_k<s} \frac{c_k}{b_k} ds \right). \end{aligned} \tag{14}$$

Hence, the condition (6) implies that $x'(t) < 0$ when t is sufficiently large, which contradicts to $x'(t) > 0$ for $t > \tau_{j+1}$ again. In terms of the above discussion, we see that $x''(\tau_k) > 0$ for any $\tau_k > T$ with sufficiently large T. Consequently, noting that $x'''(t) < 0$ for any $t \in (\tau_k, \tau_{k+1}]$, we have $x''(t) > x''(\tau_{k+1}) > 0$.

Next, if there exists a $\tau_j > T$ such that $x'(\tau_j) \ge 0$, then $x'(\tau_j^+) = b_j(x(\tau_j^+) \ge 0$, $x'(\tau_{j+1}) > x'(\tau_j^+) \ge 0$. Therefore, by induction, we have $x'(t) > x'(\tau_i^+) > 0$ for $t \in (\tau_i, \tau_{i+1}]$, $i = j+1, j+2, \ldots$. So case (i) is satisfied. Otherwise, if $x'(\tau_k) < 0$ for all $\tau_k \ge T$, then $x'(\tau_k^+) = b_k x(\tau_k^+) < 0$. Thus, for $t \in (\tau_k, \tau_{k+1}]$, using $x''(t) > 0$, we have $x'(t) < x'(\tau_{k+1}) < 0$; hence, case (ii) is satisfied. This completes the proof. □

Remark 3. Suppose that $x(t)$ is an eventually negative solution of (1). If (5) and (6) hold, one can prove it holds that one of the following two cases in a similar way as Lemma 2:

(i) $x''(\tau_k^+) < 0$, $x''(t) < 0$ and $x'(\tau_k^+) < 0$, $x'(t) < 0$,

(ii) $x''(\tau_k^+) < 0$, $x''(t) < 0$ and $x'(\tau_k^+) > 0$, $x'(t) > 0$, with $t \in (\tau_k, \tau_{k+1}]$ and $\tau_k \ge T$.

Lemma 4. *Let $x(t)$ be a piecewise continuous function on $\cup_{k\in\mathbb{N}}(\tau_k, \tau_{k+1}]$, which is continuous at $t \ne \tau_k$ and is left continuous at $t = \tau_k$. If*

(1) *$x(t) \ge 0$ (≤ 0) for $t \ge t_0$;*

(2) *$x(t)$ is monotone nonincreasing (monotone nondecreasing) on $(\tau_k, \tau_{k+1}]$ $(\tau_k \ge T)$ for T large enough;*

(3) *$\sum_{k=1}^{\infty}[x(\tau_k^+) - x(\tau_k)]$ converges,*

then $\lim_{t\to\infty} x(t) = a \ge 0$ (≤ 0).

The proof of Lemma 4 is similar to that of [13, Theorem 5], and hence is omitted.

Lemma 5. *Assume that $x(t)$ is a solution of* (1) *which satisfies case (ii) in Lemma 2. In addition, if*

$$\sum_{k=1}^{\infty} |a_k - 1| < \infty, \qquad \prod_{k=1}^{\infty} a_k \text{ is bounded}, \tag{15}$$

then $\lim_{t\to\infty} x(t)$ *exists (finite).*

Proof. First, we claim that $\sum_{k=1}^{\infty}[x(\tau_k^+) - x(\tau_k)]$ is convergence. In fact, since $x(t)$ is decreasing on $(\tau_k, \tau_{k+1}]$ ($\tau_k \ge T$, and T is defined in Lemma 2), then

$$x(\tau_{k+1}) \le x(\tau_k^+), \qquad x(\tau_{k+1}^+) \le a_{k+1} x(\tau_{k+1}) \le a_{k+1} x(\tau_k^+). \tag{16}$$

Obviously, by induction, we can get

$$\begin{aligned} x(\tau_{k+n}) &\le a_{k+n-1} \cdots a_{k+1} x(\tau_k^+), \\ x(\tau_{k+n}^+) &\le a_{k+n} \cdots a_{k+1} x(\tau_k^+). \end{aligned} \tag{17}$$

Since $\prod_{k=1}^{\infty} a_k$ is bounded, we conclude that $\{x(\tau_k)\}$ is bounded, which follows that there exists $M_1 > 0$ such that

$$|x(\tau_k^+) - x(\tau_k)| = |a_k - 1| x(\tau_k) \le M_1 |a_k - 1|. \tag{18}$$

Hence, $\sum_{k=1}^{\infty}[x(\tau_k^+) - x(\tau_k)]$ is convergence since $\sum_{k=1}^{\infty}|a_k - 1|$ is convergence, which follows from Lemma 4 that $\lim_{t\to\infty} x(t)$ exists. The proof is complete. □

3. Main Results

In this section, we establish some sufficient conditions which guarantee that every solution $x(t)$ of (1) either oscillates or has a finite limit. Occasionally, we will make the additional assumption

$$\limsup_{t\to\infty} \int_{\eta}^{t} \int_{u}^{\infty} \int_{s}^{\infty} p(\theta)\, d\theta\, ds\, du = \infty, \tag{19}$$

where here it is understood that

$$\int_{t_0}^{\infty} p(t)\, dt < \infty, \qquad \int_{t_0}^{\infty} \int_{u}^{\infty} p(\theta)\, d\theta\, du < \infty. \tag{20}$$

Now we are ready to state and prove the main results in this paper. The results will be proved by making use of the technique in [11].

Theorem 6. *Assume that* (5), (6), *and* (19) *hold, and $x(t)$ is a solution of* (1). *Furthermore, assume that $a_k \le 1$, $b_k \ge 1$, and $c_k \le 1$ for $k \in \mathbb{N}$. If there exists a positive differentiable function r such that*

$$\lim_{t\to\infty} \int_{t_0}^{t} [p(s) r(s) - A(s, \tau_{k(s)})] \prod_{t_0<\tau_k<s} \frac{1}{c_k} \prod_{t_0<\tau_k+\sigma<s} a_k\, ds = \infty, \tag{21}$$

where $k(s) = \max\{k : \tau_k < s\}$, and

$$A(s, \tau_{k(s)}) = \begin{cases} \dfrac{c_{k(s)}[r'(s)]^2}{4\mu r(s)(s - \tau_{k(s)})}, & s \in (\tau_k, \tau_k + \sigma], \\ \dfrac{[r'(s)]^2}{4\mu r(s)(s - \tau_{k(s)} - ((c_{k(s)} - 1)/c_{k(s)})\sigma)}, & s \in (\tau_k + \sigma, \tau_{k+1}]. \end{cases} \tag{22}$$

Then $x(t)$ is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of (1), without loss of generality, we may assume that $x(t) > 0$ eventually ($x(t) < 0$ eventually can be achieved in the similar way). By Lemma 2, either case (i) or case (ii) in Lemma 2 holds. Assume that $x(t)$ satisfies case (i), then $x''(t) > 0$, $x'(t) > 0$ for $t \in (\tau_j, \tau_{j+1}]$, $\tau_j \geq T$ (T is defined in Lemma 2). Define the Riccati transformation u by

$$u(t) = \frac{r(t)\,x''(t)}{\varphi(x(t-\sigma))}, \quad t \geq \tau_j,\ t \neq \tau_k. \tag{23}$$

Thus, $u(t) > 0$ for $t \in (\tau_k, \tau_{k+1}]$, $k = j, j+1, \ldots$, and

$$\begin{aligned} u'(t) &= \frac{r(t)}{\varphi(x(t-\sigma))}x'''(t) \\ &\quad + \frac{r'(t)\,\varphi(x(t-\sigma)) - r(t)\,\varphi'(x(t-\sigma))\,x'(t-\sigma)}{\varphi^2(x(t-\sigma))} \\ &\quad \times x''(t) \\ &\leq -p(t)\,r(t) + \frac{r'(t)}{r(t)}u(t) \\ &\quad - \frac{r(t)\,\mu x'(t-\sigma)}{\varphi^2(x(t-\sigma))}x''(t). \end{aligned} \tag{24}$$

If $t \in (\tau_k, \tau_k + \sigma] \subset (\tau_k, \tau_{k+1}]$, namely, $t - \sigma \leq \tau_k < t$, $x''(t)$ is decreasing in $(t-\sigma, \tau_k]$ and $(\tau_k, t]$, respectively. In view of the following

$$x''(t-\sigma) > x''(\tau_k) \geq \frac{x''(\tau_k^+)}{c_k} > \frac{x''(t)}{c_k}, \tag{25}$$

we have

$$\begin{aligned} x'(t-\sigma) &= x'(\tau_k - \sigma) + \int_{\tau_k-\sigma}^{t-\sigma} x''(s)\,\mathrm{d}s \\ &> x''(t-\sigma)(t-\tau_k) > \frac{x''(t)}{c_k}(t-\tau_k). \end{aligned} \tag{26}$$

Thus,

$$\begin{aligned} u'(t) &\leq -p(t)\,r(t) + \frac{r'(t)}{r(t)}u(t) - \frac{\mu r(t)(t-\tau_k)}{c_k\varphi^2(x(t-\sigma))}\left[x''(t)\right]^2 \\ &= -p(t)\,r(t) + \frac{r'(t)}{r(t)}u(t) - \frac{\mu(t-\tau_k)}{c_k r(t)}[u(t)]^2 \\ &= -\left(\sqrt{\frac{\mu(t-\tau_k)}{c_k r(t)}}u(t) - \sqrt{\frac{c_k}{4\mu r(t)(t-\tau_k)}}r'(t)\right)^2 \\ &\quad - \left[p(t)\,r(t) - \frac{c_k\left[r'(t)\right]^2}{4\mu r(t)(t-\tau_k)}\right] \\ &\leq -\left[p(t)\,r(t) - \frac{c_k\left[r'(t)\right]^2}{4\mu r(t)(t-\tau_k)}\right]. \end{aligned} \tag{27}$$

If $t \in (\tau_k+\sigma, \tau_{k+1}] \subset (\tau_k, \tau_{k+1}]$, that is, $\tau_k < t-\sigma < t \leq \tau_{k+1}$, then

$$\begin{aligned} x'(t-\sigma) &= x'(\tau_k - \sigma) + \int_{\tau_k-\sigma}^{\tau_k} x''(s)\,\mathrm{d}s + \int_{\tau_k}^{t-\sigma} x''(s)\,\mathrm{d}s \\ &\geq x''(\tau_k)\,\sigma + x''(t-\sigma)(t-\tau_k-\sigma) \\ &\geq \frac{x''(t)}{c_k}\sigma + x''(t)(t-\tau_k-\sigma) \\ &\geq x''(t)\left(t-\tau_k-\frac{c_k-1}{c_k}\sigma\right). \end{aligned} \tag{28}$$

Similarly, we have

$$u'(t) \leq -\left[p(t)\,r(t) - \frac{\left[r'(t)\right]^2}{4\mu r(t)(t-\tau_k-((c_k-1)/c_k)\sigma)}\right]. \tag{29}$$

Thus, we obtain

$$u'(t) \leq -\left[p(t)\,r(t) - A(t, \tau_{k(t)})\right] \quad \text{for } t \in (\tau_k, \tau_{k+1}]. \tag{30}$$

On the other hand,

$$u(\tau_k^+) = \frac{r(\tau_k)\,x''(\tau_k^+)}{\varphi(x(\tau_k-\sigma))} \leq c_k u(\tau_k). \tag{31}$$

Observing that $\varphi(u) \geq \mu u$, we have

$$\begin{aligned} u(\tau_k^+ + \sigma) &= \frac{r(\tau_k+\sigma)\,x''(\tau_k+\sigma)}{\varphi(x(\tau_k^+))} \\ &\leq \frac{r(\tau_k+\sigma)\,x''(\tau_k+\sigma)}{\varphi(a_k x(\tau_k))} \leq \frac{u(\tau_k+\sigma)}{\mu a_k}. \end{aligned} \tag{32}$$

Applying Lemma 1, it follows from (30), (31), and (32) that

$$\begin{aligned} u(t) &\leq u(\tau_j^+)\prod_{\tau_j<\tau_k<t} c_k \prod_{\tau_j<\tau_k+\sigma<t}\frac{1}{\mu a_k} \\ &\quad - \int_{\tau_j}^{t}\left[p(s)\,r(s) - A(s, \tau_{k(s)})\right] \\ &\quad \cdot \prod_{s<\tau_k<t} c_k \prod_{s<\tau_k+\sigma<t}\frac{1}{\mu a_k}\mathrm{d}s \\ &\leq \prod_{\tau_j<\tau_k<t} c_k \prod_{\tau_j<\tau_k+\sigma<t}\frac{1}{\mu a_k} \\ &\quad \times \left(u(\tau_j^+) - \mu\int_{\tau_j}^{t}\left[p(s)\,r(s) - A(s, \tau_{k(s)})\right]\right. \\ &\quad \left. \cdot \prod_{\tau_j<\tau_k<s}\frac{1}{c_k}\prod_{\tau_j<\tau_k+\sigma<s} a_k\,\mathrm{d}s\right), \end{aligned} \tag{33}$$

which yields $u(t) < 0$ for all large t. This is contrary to $u(t) > 0$, and so, case (i) in Lemma 2 is not possible.

If $x(t)$ satisfies the case (ii) in Lemma 2, that is, $x''(\tau_k^+) > 0$, $x''(t) > 0$ and $x'(\tau_k^+) < 0$, $x'(t) < 0$, which proves that the solution $x(t)$ is positive and decreasing. Integrating (1) from s to t $(t \geq s \geq T)$, we obtain

$$x''(t) - \sum_{s<\tau_k<t} (c_k - 1) x''(\tau_k) - x''(s) + \int_s^t p(\theta) \varphi(x(\theta - \sigma)) d\theta \leq 0. \tag{34}$$

Noting $c_k \leq 1$ and $x''(t) > 0$, then it holds that

$$x''(t) - x''(s) + \int_s^t p(\theta) \varphi(x(\theta - \sigma)) d\theta \leq 0, \tag{35}$$

which leads to

$$-x''(s) + \int_s^t p(\theta) \varphi(x(\theta - \sigma)) d\theta \leq 0, \tag{36}$$

and hence

$$-x''(s) + \int_s^\infty p(\theta) \varphi(x(\theta - \sigma)) d\theta \leq 0. \tag{37}$$

Integrating the above inequality again from u to t $(t \geq u \geq T)$, one has

$$-x'(t) + \sum_{s<\tau_k<t} (b_k - 1) x'(\tau_k) + x'(u) + \int_u^t \int_s^\infty p(\theta) \varphi(x(\theta - \sigma)) d\theta\, ds \leq 0. \tag{38}$$

Using $x'(t) < 0$ and $b_k \geq 1$, we have

$$x'(u) + \int_u^\infty \int_s^\infty p(\theta) \varphi(x(\theta - \sigma)) d\theta\, ds \leq 0. \tag{39}$$

Now, we integrate the last inequality from η to t $(t \geq \eta \geq T)$ to obtain

$$x(t) - \sum_{\eta<\tau_k<t} (a_k - 1) x(\tau_k) - x(\eta) + \int_\eta^t \int_u^\infty \int_s^\infty p(\theta) \varphi(x(\theta - \sigma)) d\theta\, ds\, du \leq 0. \tag{40}$$

Since $a_k \leq 1$ $(k \in \mathbb{N})$ and $x(t)$ is decreasing, then for $t \in (\tau_k, \tau_{k+1}]$, $x(\tau_{k+1}) \leq x(t) \leq x(\tau_k^+) = a_k x(\tau_k) \leq x(\tau_k)$, $k \in \mathbb{N}$. Thus, we get

$$\mu x(\eta) \int_\eta^t \int_u^\infty \int_s^\infty p(\theta) d\theta\, ds\, du \leq -x(t) + x(\eta) \leq x(\eta), \tag{41}$$

and then,

$$\int_\eta^t \int_u^\infty \int_s^\infty p(\theta) d\theta\, ds\, du \leq \frac{1}{\mu}, \tag{42}$$

which contradicts the condition (19). The proof is complete. □

Replace the condition (19) with (15), we may obtain the following asymptotic results.

Theorem 7. *Assume that* (5), (6), *and* (15) *hold, and $x(t)$ is a solution of* (1). *If there exists a positive differentiable function r such that* (21) *hold, then $x(t)$ is either oscillatory or has a finite limit.*

Proof. By the proof of Theorem 6, we know the case (i) in Lemma 2 is not possible, too, since the condition (19) is not required to prove it. So it suffices to show if there is a solution satisfying case (ii) in Lemma 2, that is, if

$$x''(\tau_k^+) > 0, \quad x''(t) > 0, \quad x'(\tau_k^+) < 0, \quad x'(t) < 0 \tag{43}$$

with $t \in (\tau_k, \tau_{k+1}]$ and $\tau_k \geq T$. then $\lim_{t\to\infty} x(t)$ exists. This is obtained by applying Lemma 5 which leads to $\lim_{t\to\infty} x(t)$ exists. The proof is complete. □

Corollary 8. *In addition to the assumption of Theorem 7, assume that*

$$\lim_{t\to\infty} \int_{t_0}^t \prod_{t_0<\tau_k<s} \frac{1}{c_k} p(s)\, ds = \infty. \tag{44}$$

Then, solution $x(t)$ of (1) *either oscillates or satisfies $\lim_{t\to\infty} x(t) = 0$.*

Proof. By the proof of Theorem 7, $\lim_{t\to\infty} x(t)$ exists, and we define it by $\lim_{t\to\infty} x(t) = \gamma \geq 0$. We now show $\gamma = 0$. If not, then $\gamma > 0$. So, $\lim_{t\to\infty} \varphi(x(t - \sigma)) = \varphi(\gamma) =: \kappa > 0$. Hence, there exists $\tau_j > T$ such that $\varphi(x(t-\sigma)) > \kappa/2$ for $t > \tau_j$. Then

$$\begin{aligned} x'''(t) &= -f(t, x(t), x(t - \sigma)) \\ &\leq -p(t) \varphi(x(t - \sigma)) \leq -\frac{\kappa}{2} p(t), \quad t \geq \tau_j, \end{aligned} \tag{45}$$

and note that $x''(\tau_k^+) \leq c_k x''(\tau_k)$ since $x''(t) > 0$, which imply that

$$\begin{aligned} x''(t) &\leq x''(\tau_j^+) \prod_{\tau_j<\tau_k<t} c_k - \frac{\kappa}{2} \int_{\tau_j}^t \prod_{s<\tau_k<t} c_k p(s)\, ds \\ &\leq \prod_{\tau_j<\tau_k<t} c_k \left[x''(\tau_j^+) - \frac{\kappa}{2} \int_{\tau_j}^t \prod_{\tau_j<\tau_k<s} \frac{1}{c_k} p(s)\, ds \right]. \end{aligned} \tag{46}$$

Thus, in virtue of (44) it holds that $x''(t) < 0$ and contradicts $x''(t) > 0$ for t large enough, the proof is complete. □

Remark 9. Theorem 6 and Corollary 8 extend the results in [11, Theorem 3.1] and [9, Corollary 1], respectively. In fact, when $a_k = b_k = c_k = 1$ for $k \in \mathbb{N}$ which implies that the impulses in (1) disappear. In such a case, (5) and (6) hold naturally, and (21) and (44) are reduced to

$$\begin{gathered} \lim_{t\to\infty} \int_{t_0}^t \left[p(s) r(s) - \frac{[r'(s)]^2}{4\mu r(s)(s - T)} \right] ds = \infty, \\ \lim_{t\to\infty} \int_{t_0}^t p(s)\, ds = \infty, \end{gathered} \tag{47}$$

which are similar to those in [11, Theorem 3.1] and [9, Corollary 1], respectively.

Next, we present some new oscillation results for (1), by using an integral averaging condition of Kamenev's type.

Theorem 10. *Assume* (5), (6), *and* (19) *hold. Furthermore,* $a_k \le 1/\mu \le 1$, $b_k \ge 1$ *and* $c_k \le 1$, $k \in \mathbb{N}$. *If there exists a positive differentiable function r such that*

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_T^t (t-s)^m \left[p(s) r(s) - A\left(s, \tau_{k(s)}\right)\right] \mathrm{d}s = \infty, \tag{48}$$

where $A(s, \tau_{k(s)})$ *is defined by* (21) *and* $m \ge 1$. *Then every solution of* (1) *is oscillatory.*

Proof. We choose T large enough such that Lemma 2 holds. By Lemma 2 there are two possible cases. First, if the case (i) holds, proceeding as in the proof of Theorem 6, we will end up with (32). By (30), we have

$$p(t) r(t) - A\left(t, \tau_{k(t)}\right) \le -u'(t), \quad t \ge T,\ t \ne \tau_k. \tag{49}$$

If $t \in (\tau_k + \sigma, \tau_{k+1}] \subset (\tau_k, \tau_{k+1}]$, for $\tau_j \ge T$, we obtain

$$\begin{aligned} &\int_{\tau_j}^t (t-s)^m \left[p(s) r(s) - A\left(s, \tau_{k(s)}\right)\right] \mathrm{d}s \\ &\quad \le -\int_{\tau_j}^t (t-s)^m u'(s) \,\mathrm{d}s. \end{aligned} \tag{50}$$

An integration by parts of the right-hand side leads to

$$\begin{aligned} &\int_{\tau_j}^t (t-s)^m u'(s) \,\mathrm{d}s \\ &= \left(\int_{\tau_j}^{\tau_j+\sigma} + \int_{\tau_j+\sigma}^{\tau_{j+1}} + \cdots + \int_{\tau_k}^{\tau_k+\sigma} + \int_{\tau_k+\sigma}^{t}\right)(t-s)^m u'(s) \,\mathrm{d}s \\ &= \int_{\tau_j}^t \frac{u(s)}{m} (t-s)^{m-1} \mathrm{d}s \\ &\quad + \sum_{i=j}^{k} \left[t - (\tau_i + \sigma)\right]^m \left[u(\tau_i + \sigma) - u(\tau_i^+ + \sigma)\right] \\ &\quad + \sum_{i=j+1}^{k} (t-\tau_i)^m \left[u(\tau_i) - u(\tau_i^+)\right] - \left(t-\tau_j\right)^m u\left(\tau_j^+\right). \end{aligned} \tag{51}$$

Take into account (31), (32), $a_k \le 1/\mu$, and $c_k \le 1$, we have

$$\begin{aligned} &\int_{\tau_j}^t (t-s)^m u'(s) \,\mathrm{d}s \\ &\quad \ge \sum_{i=j+1}^{k} (t-\tau_i)^m (1 - c_i) u(\tau_i) - \left(t-\tau_j\right)^m u\left(\tau_j^+\right) \\ &\quad \ge -\left(t-\tau_j\right)^m u\left(\tau_j^+\right). \end{aligned} \tag{52}$$

If $t \in (\tau_k, \tau_k + \sigma]$, similarly we also get

$$\int_{\tau_j}^t (t-s)^m u'(s) \,\mathrm{d}s \ge -\left(t-\tau_j\right)^m u\left(\tau_j^+\right). \tag{53}$$

So, it yields

$$\int_{\tau_j}^t (t-s)^m \left[p(s) r(s) - A\left(s, \tau_{k(s)}\right)\right] \mathrm{d}s \le \left(t-\tau_j\right)^m u\left(\tau_j^+\right), \tag{54}$$

which follows that

$$\begin{aligned} &\frac{1}{t^m} \int_{\tau_j}^t (t-s)^m \left[p(s) r(s) - A\left(s, \tau_{k(s)}\right)\right] \mathrm{d}s \\ &\quad \le \left(\frac{t-\tau_j}{t}\right)^m u\left(\tau_j^+\right). \end{aligned} \tag{55}$$

Hence,

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{\tau_j}^t (t-s)^m \left[p(s) r(s) - A\left(s, \tau_{k(s)}\right)\right] \mathrm{d}s \le u\left(\tau_j^+\right), \tag{56}$$

which is a contradiction of (48).

If case (ii) holds, then as a manner with case (ii) in Theorem 6, it is not possible, too. The proof is complete. □

Corollary 11. *Assume* (19) *holds and* $a_k = b_k = c_k = 1$, *for* $k \in \mathbb{N}$. *If there exists a positive differential function r such that*

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_T^t (t-s)^m \left[p(s) r(s) - \frac{\left[r'(s)\right]^2}{4\mu r(s)(s-T)}\right] \mathrm{d}s = \infty, \tag{57}$$

where T is large enough such that Lemma 2 holds. Then every solution of (1) *is oscillatory.*

Remark 12. Corollary 11 is an extension of [11, Theorem 3.2] into impulsive case. Especially, let $r(t) \equiv 1$ in (48), it reduces to

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_T^t (t-s)^m p(s) \,\mathrm{d}s = \infty, \tag{58}$$

naturally, which can be considered as the extension of Kamenev-type oscillation criteria for third-order impulsive differential equations with delay (see [8, 14, 15]).

4. Examples

Example 13. Consider the third-order impulsive differential equation with delay

$$\begin{gathered} x'''(t) + \left(1 + \alpha x^2(t)\right) x(t-\sigma) = 0, \quad t > t_0,\ t \ne \tau_k, \\ x\left(\tau_k^+\right) = a_k x\left(\tau_k\right), \qquad x'\left(\tau_k^+\right) = b_k x'\left(\tau_k\right), \\ x''\left(\tau_k^+\right) = c_k x''\left(\tau_k\right), \quad k \in \mathbb{N}, \end{gathered} \tag{59}$$

where $\sigma > 0$, $\alpha \ge 0$ are constants, $\tau_k - \tau_{k-1} > \sigma$ for any $k \in \mathbb{N}$.

When $a_k = b_k = c_k = 1$ for any $k \in \mathbb{N}$, the impulses in (59) disappear, by [16, Theorem 4], (59) is nonoscillatory if $\sigma < e/3$ and $\alpha = 0$. However, we may change its oscillation by proper impulsive control. In fact, let $\sigma < e/3$ and $\alpha = 0$ and $\tau_k = t_0 + k\sigma$ $(k \in \mathbb{N})$ in (59); choose $\varphi(x) = x$, $p(t) \equiv 1$, and $r(t) = 1$; a simple calculation leads to

$$\begin{aligned}
&\int_{t_0}^{\tau_n} \prod_{t_0<\tau_k<s} \frac{1}{c_k} \prod_{t_0<\tau_k+\sigma<s} a_k p(s)\,\mathrm{d}s \\
&= \left[\int_{t_0}^{\tau_1} + \left(\int_{\tau_1}^{\tau_1+\sigma} + \int_{\tau_1+\sigma}^{\tau_2}\right) + \cdots + \left(\int_{\tau_{n-1}}^{\tau_{n-1}+\sigma} + \int_{\tau_{n-1}+\sigma}^{\tau_n}\right)\right] \\
&\quad \cdot \prod_{t_0<\tau_k<s} \frac{1}{c_k} \prod_{t_0<\tau_k+\sigma<s} a_k \mathrm{d}s \\
&= (\tau_1 - t_0) + \left[\frac{\sigma}{c_1} + \frac{a_1}{c_1}(\tau_2 - \tau_1 - \sigma)\right] + \cdots + \frac{a_1 a_2 \cdots a_{n-2}\sigma}{c_1 c_2 \cdots c_{n-1}} \\
&\quad + \frac{a_1 a_2 \cdots a_{n-1}}{c_1 c_2 \cdots c_{n-1}}(\tau_n - \tau_{n-1} - \sigma) \\
&= \sigma\left(1 + \frac{1}{c_1} + \frac{1}{c_2} + \cdots + \frac{1}{c_{n-1}}\right).
\end{aligned} \tag{60}$$

Then, let

$$a_k = c_k = \frac{k}{k+1}, \quad b_k = 1, \quad k \in \mathbb{N}. \tag{61}$$

Obviously, (5), (6), and (19) hold, and

$$\begin{aligned}
&\int_{t_0}^{\tau_n} \prod_{t_0<\tau_k<s} \frac{1}{c_k} \prod_{t_0<\tau_k+\sigma<s} a_k p(s)\,\mathrm{d}s \\
&\quad = \sigma\left(1 + \frac{2}{1} + \frac{3}{2} + \cdots + \frac{n}{n-1}\right) \longrightarrow \infty \quad (n \longrightarrow \infty).
\end{aligned} \tag{62}$$

Thus, (21) is also satisfied. By Theorem 6, every solution of (59) is oscillatory.

If we let

$$a_k = 1 + \frac{1}{k^2}, \quad b_k = c_k = \frac{k+1}{k}, \quad k \in \mathbb{N}. \tag{63}$$

In this case, it is easily to verify (5), (6), (15), (44), and (21) hold. By Corollary 8, every solution of (59) is either oscillatory or tends to zero.

Remark 14. It is easy to verify that in [7, Theorems 1, 2, and 3], cannot be applied to (59). On the other hand, Theorem 7 is not applicable for the condition (61) since $\sum_{n=1}^{\infty} |a_k - 1|$ does not convergence.

Example 15. Consider the third-order impulsive differential equation with delay

$$\begin{aligned}
&x'''(t) + e^t \cosh\left(|x(t)|^{\alpha-1} x(t)\right) x(t-1) = 0, \\
&\qquad t \ge 0, \quad t \ne 2k, \\
&x(\tau_k^+) = a_k x(\tau_k), \qquad x'(\tau_k^+) = b_k x'(\tau_k), \\
&x''(\tau_k^+) = c_k x''(\tau_k), \quad \tau_k = 2k, \ k \in \mathbb{N},
\end{aligned} \tag{64}$$

where $\alpha > 0$, $a_k = 1$, $b_k = c_k = k/(k+1)$, $k \in \mathbb{N}$.

Let $\varphi(x) = x$, $p(t) = e^t$, it is easy to verify that (5), (6), and (19) hold. Choose $r(t) \equiv 1$ and $m = 1$, we have

$$\frac{1}{t}\int_T^t (t-s)\, e^s \mathrm{d}s = \frac{e^t}{t} + \left(\frac{T-1}{t} - 1\right) e^T \rightarrow \infty \quad (t \rightarrow \infty). \tag{65}$$

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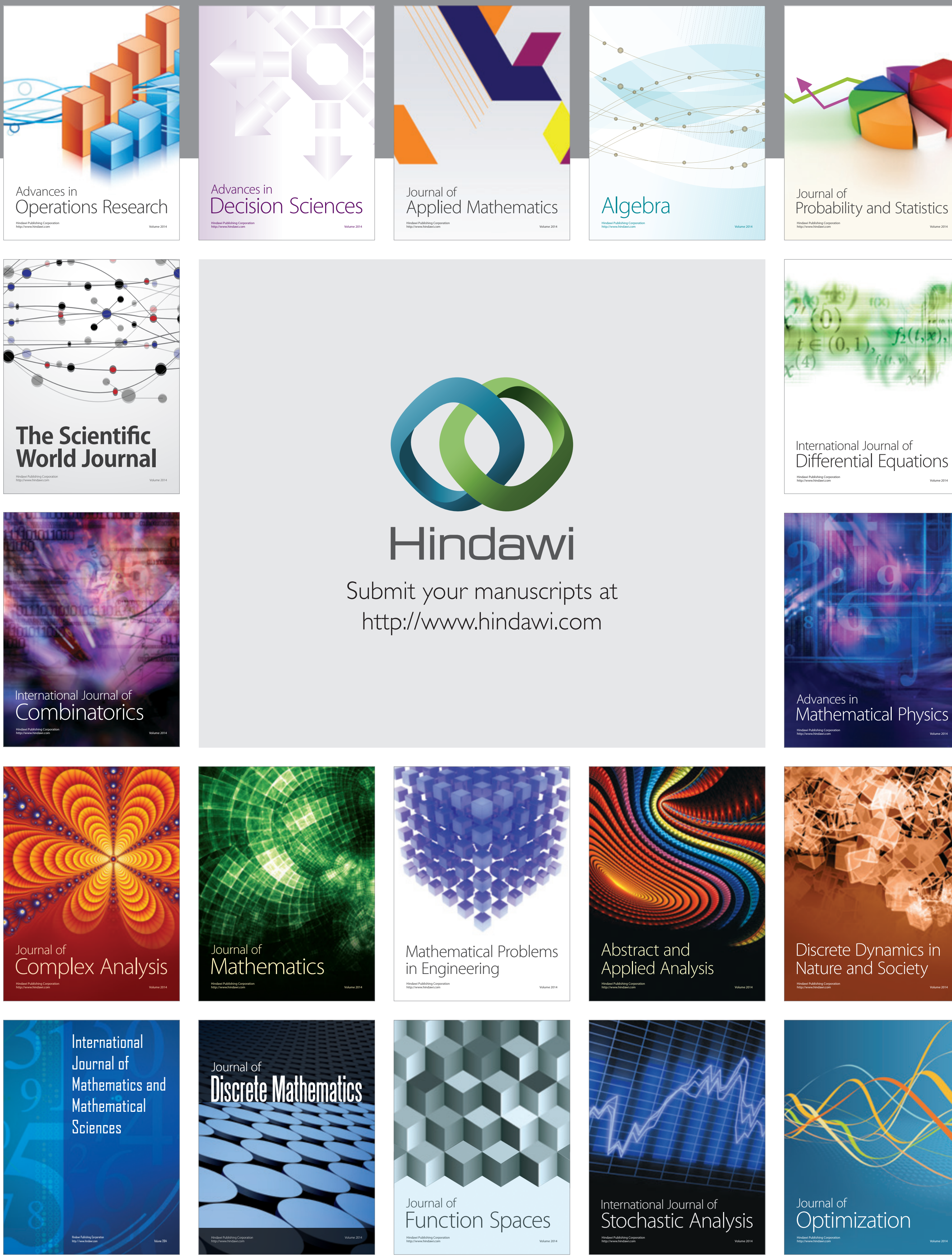
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