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Research Article

New Generalized Hyperbolic Functions to Find New Exact Solutions of the Nonlinear Partial Differential Equations

Yusuf Pandir and Halime Ulusoy

Department of Mathematics, Faculty of Science, Bozok University, 66100 Yozgat, Turkey

Correspondence should be addressed to Yusuf Pandir; yusufpandir@gmail.com

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We firstly give some new functions called generalized hyperbolic functions. By the using of the generalized hyperbolic functions, new kinds of transformations are defined to discover the exact approximate solutions of nonlinear partial differential equations. Based on the generalized hyperbolic function transformation of the generalized KdV equation and the coupled equal width wave equations (CEWE), we find new exact solutions of two equations and analyze the properties of them by taking different parameter values of the generalized hyperbolic functions. We think that these solutions are very important to explain some physical phenomena.

1. Introduction

To find exact solutions to nonlinear evolution equations, some new techniques have been introduced in recent years such as the Hirota method, the extended Jacobi elliptic function expansion method, the exponential function method, (G'/G)-expansion method, the simplest equation method, the trial equation method, Kudryashov's method, and solitary ansatz method [1–19]. Also some authors define new functions which are named generalized hyperbolic functions for constructing new solutions [20–22]. There are a lot of nonlinear evolution equations that are integrated using the various mathematical methods. Soliton solutions, compactons, singular solitons, and other solutions have been found by using these approaches. These types of solutions are very important and appear in various areas of applied mathematics.

In Section 2, we give the definition and properties of generalized hyperbolic functions. In Section 3, as applications, we obtain exact solutions of the generalized KdV equation and the coupled equal width wave equations.

2. The Definition and Properties of the Symmetrical Hyperbolic Fibonacci and Lucas Functions

In this section, we will define new functions which are named the symmetrical hyperbolic Fibonacci and Lucas functions for constructing new exact solutions of NPDEs and then study the properties of these functions.

Definition 1. Suppose that ξ is an independent variable; p, q, and k are all constants. The generalized hyperbolic sine function is

$$\sinh_a(\xi) = \frac{pa^{k\xi} - qa^{-k\xi}}{2}, \tag{1}$$

generalized hyperbolic cosine function is

$$\cosh_a(\xi) = \frac{pa^{k\xi} + qa^{-k\xi}}{2}, \tag{2}$$

generalized hyperbolic tangent function is

$$\tanh_a(\xi) = \frac{pa^{k\xi} - qa^{-k\xi}}{pa^{k\xi} + qa^{-k\xi}}, \tag{3}$$

generalized hyperbolic cotangent function is

$$\coth_a(\xi) = \frac{pa^{k\xi} + qa^{-k\xi}}{pa^{k\xi} - qa^{-k\xi}}, \tag{4}$$

generalized hyperbolic secant function is

$$\operatorname{sech}_a(\xi) = \frac{2}{pa^{k\xi} + qa^{-k\xi}}, \tag{5}$$

generalized hyperbolic cosecant function is

$$\operatorname{cosech}_a(\xi) = \frac{2}{pa^{k\xi} - qa^{-k\xi}}, \tag{6}$$

and the above six kinds of functions are said to be generalized new hyperbolic functions. Thus we can prove the following theory of generalized hyperbolic functions on the basis of Definition 1.

Theorem 2. *The generalized hyperbolic functions satisfy the following relations:*

$$\begin{gathered}
\cosh_a^2(\xi) - \sinh_a^2(\xi) = pq,\\
1 - \tanh_a^2(\xi) = pq\operatorname{sech}_a^2(\xi),\\
1 - \coth_a^2(\xi) = -pq\operatorname{cosech}_a^2(\xi),\\
\operatorname{sech}_a(\xi) = \frac{1}{\cosh_a(\xi)},\\
\operatorname{cosech}_a(\xi) = \frac{1}{\sinh_a(\xi)},\\
\tanh_a(\xi) = \frac{\sinh_a(\xi)}{\cosh_a(\xi)},\\
\coth_a(\xi) = \frac{\cosh_a(\xi)}{\sinh_a(\xi)}.
\end{gathered} \tag{7}$$

The following just part of them are proved here for simplification.

Theorem 3. *The derivative formulae of generalized hyperbolic functions are as follows:*

$$\begin{gathered}
\frac{d(\sinh_a(\xi))}{d\xi} = k\ln a\cosh_a(\xi),\\
\frac{d(\cosh_a(\xi))}{d\xi} = k\ln a\sinh_a(\xi),\\
\frac{d(\tanh_a(\xi))}{d\xi} = kpq\ln a\operatorname{sech}_a^2(\xi),\\
\frac{d(\coth_a(\xi))}{d\xi} = -kpq\ln a\operatorname{cosech}_a^2(\xi),\\
\frac{d(\operatorname{sech}_a(\xi))}{d\xi} = -k\ln a\operatorname{sech}_a(\xi)\tanh_a(\xi),\\
\frac{d(\operatorname{cosech}_a(\xi))}{d\xi} = -k\ln a\operatorname{cosech}_a(\xi)\coth_a(\xi).
\end{gathered} \tag{8}$$

Proof of (8). According to (7) and (8), we can get

$$\begin{aligned}
\frac{d(\tanh_a(\xi))}{d\xi} &= \left(\frac{\sinh_a(\xi)}{\cosh_a(\xi)}\right)'\\
&= \frac{(\sinh_a(\xi))'\cosh_a(\xi) - (\cosh_a(\xi))'\sinh_a(\xi)}{\cosh_a^2(\xi)}\\
&= \frac{k\ln a\cosh_a^2(\xi) - k\ln a\sinh_a^2(\xi)}{\cosh_a^2(\xi)}\\
&= kpq\ln a\operatorname{sech}_a^2(\xi).
\end{aligned} \tag{9}$$

Similarly, we can prove other differential coefficient formulae in Theorem 3. □

Remark 4. We see that when $p = 1$, $q = 1$, $k = 1$, and $a = e$ in (1)–(6), new generalized hyperbolic functions $\sinh_a(\xi)$, $\cosh_a(\xi)$, $\tanh_a(\xi)$, $\coth_a(\xi)$, $\operatorname{sech}_a(\xi)$, and $\operatorname{cosech}_a(\xi)$ degenerate as hyperbolic functions $\sinh(\xi)$, $\cosh(\xi)$, $\tanh(\xi)$, $\coth(\xi)$, $\operatorname{sech}(\xi)$, and $\operatorname{cosech}(\xi)$, respectively. In addition, when $p = 0$ or $q = 0$ in (1)–(6), $\sinh_a(\xi)$, $\cosh_a(\xi)$, $\tanh_a(\xi)$, $\coth_a(\xi)$, $\operatorname{sech}_a(\xi)$, and $\operatorname{cosech}_a(\xi)$ degenerate as exponential function $(1/2)pa^{k(\xi)}$, $\pm(1/2)qa^{-k(\xi)}$, $2pa^{-k(\xi)}$, $\pm 2qa^{k(\xi)}$, and ± 1, respectively.

3. Applications

Example 5. Application to the generalized form of KdV equation is presented.

The generalized KdV equation that will be studied in this paper are given by [23]

$$(u^l)_t + \alpha u^n u_x + \beta(u^n u_{xx})_x + \gamma u(u^n)_{xxx} = 0, \tag{10}$$

where the first term represents the generalized evolution. The special case with $l = 1$ is the regular evolution term. The coefficients of α are the nonlinear terms while the coefficients of β and γ are the nonlinear dispersion terms. This equation with $l = 1$ has been already studied by [18], where, in addition to soliton solution, compactons and periodic solutions were also obtained. The hypothesis for solving this equation is

$$u(x,t) = \frac{A}{\cosh_a^s(\eta)}, \tag{11}$$

where

$$\eta = B(x - vt). \tag{12}$$

Here, in (11) and (12), A represents the amplitude of the soliton while B is the inverse width of the soliton and v is the velocity of the soliton. The exponent s is unknown at this point and will be evaluated during the course of the derivation of the solutions to (10). From (11), it is possible to obtain

$$\left(u^l\right)_t = \frac{klsvA^l B \ln a \tanh_a(\eta)}{\cosh_a^{sl}(\eta)},$$
$$u^n u_x = \frac{-ksA^{n+1} B \ln a \tanh_a(\eta)}{\cosh_a^{s(n+1)}(\eta)},$$
$$\left(u^n u_{xx}\right)_x = \frac{k^3 s A^{n+1} B^3 pq (\ln a)^3 (s+1)(sn+s+2) \tanh_a(\eta)}{\cosh_a^{s(n+1)+2}(\eta)} - \frac{k^3 s^3 (n+1) A^{n+1} B^3 (\ln a)^3 \tanh_a(\eta)}{\cosh_a^{s(n+1)}(\eta)},$$
$$u\left(u^n\right)_{xxx} = \frac{-k^3 s^3 n^3 A^{n+1} B^3 (\ln a)^3 \tanh_a(\eta)}{\cosh_a^{s(n+1)}(\eta)} + \frac{k^3 snpq A^{n+1} B^3 (\ln a)^3 (sn+1)(sn+2) \tanh_a(\eta)}{\cosh_a^{s(n+1)+2}(\eta)}. \tag{13}$$

These results will now be substituted in (10) to obtain the 1-soliton solution of the generalized KdV equation. Equation (10) by virtue of (13) reduces to

$$\begin{aligned}&\frac{klsvA^l B \ln a \tanh_a(\eta)}{\cosh_a^{sl}(\eta)} - \frac{\alpha ksA^{n+1} B \ln a \tanh_a(\eta)}{\cosh_a^{s(n+1)}(\eta)} \\ &+ \frac{\beta k^3 spq A^{n+1} B^3 (\ln a)^3 (s+1)(sn+s+2) \tanh_a(\eta)}{\cosh_a^{s(n+1)+2}(\eta)} \\ &- \frac{\beta k^3 s^3 (n+1) A^{n+1} B^3 (\ln a)^3 \tanh_a(\eta)}{\cosh_a^{s(n+1)}(\eta)} \\ &+ \frac{\gamma k^3 snpq A^{n+1} B^3 (\ln a)^3 (sn+1)(sn+2) \tanh_a(\eta)}{\cosh_a^{s(n+1)+2}(\eta)} \\ &- \frac{\gamma k^3 s^3 n^3 A^{n+1} B^3 (\ln a)^3 \tanh_a(\eta)}{\cosh_a^{s(n+1)}(\eta)} = 0.\end{aligned} \tag{14}$$

From (14), equating the exponents sl and $sn+s+2$ gives

$$sl = sn + s + 2, \tag{15}$$

that leads to

$$s = \frac{2}{l-n-1}. \tag{16}$$

Now from (14), the two linearly independent functions are $1/\cosh^{ns+s+j}$ for $j = 0, 2$. Thus setting their coefficients to zero gives

$$v = \frac{\alpha pq\left(\beta l(l+1-n) + n\gamma(l-1)(l+n-1)\right)}{2lA^{l-n-1}\left(\beta(n+1) + n^3\gamma\right)}, \tag{17}$$

$$B = \frac{\sqrt{-\alpha}(n+1-l)}{2k \ln a \sqrt{\beta(n+1) + n^3\gamma}}. \tag{18}$$

Thus the 1-soliton solution of the generalized KdV equation with generalized evolution is given by

$$u(x,t) = \frac{A}{\cosh_a^{2/(l-n-1)}[B(x-vt)]}. \tag{19}$$

This shows that the restriction on the nonlinear exponents of (10) must be

$$l > n + 1. \tag{20}$$

Remark 6. If we take the corresponding values for some parameters, the solution (19) can be reduced to the solution (17) mentioned in [23].

Example 7. Application to the coupled equal width wave equations is presented.

We consider the coupled equal width wave equations [24]

$$\begin{aligned}u_t + uu_x - u_{xxt} + rr_x &= 0,\\ r_t + rr_x - r_{xxt} &= 0.\end{aligned} \tag{21}$$

The hypothesis for solving these equations is

$$\begin{aligned}r(x,t) &= \frac{A_1}{\cosh_a^{s_1}(\eta)},\\ u(x,t) &= \frac{A_2}{\cosh_a^{s_2}(\eta)},\end{aligned} \tag{22}$$

where

$$\eta = B(x - vt). \tag{23}$$

Here in (22)–(23) A_1 and A_2 are the amplitudes of the r-soliton and u-soliton, respectively, while v_1 and v_2 are the velocity of the soliton and B_1 and B_2 are the inverse widths of the solitons. The exponents s_1 and s_2 are unknown at this point and their values will fall out in the process of deriving the solution of this equation. From (22), it is possible to obtain

$$u_t = \frac{ks_2 v A_2 B \ln a \tanh_a(\eta)}{\cosh_a^{s_2}(\eta)}, \quad r_t = \frac{ks_1 v A_1 B \ln a \tanh_a(\eta)}{\cosh_a^{s_1}(\eta)}, \tag{24}$$

$$u_x = \frac{-ks_2 A_2 B \ln a \tanh_a(\eta)}{\cosh_a^{s_2}(\eta)}, \quad r_x = \frac{-ks_1 A_1 B \ln a \tanh_a(\eta)}{\cosh_a^{s_1}(\eta)}, \tag{25}$$

$$u_{xxt} = \frac{k^3 s_2^3 v A_2 B^3 (\ln a)^3 \tanh_a(\eta)}{\cosh_a^{s_2}(\eta)} - \frac{k^3 s_2 pq (s_2+1) v A_2 B^3 (\ln a)^3 (s_2+2) \tanh_a(\eta)}{\cosh_a^{s_2+2}(\eta)}, \tag{26}$$

$$r_{xxt} = \frac{k^3 s_1^3 v A_1 B^3 (\ln a)^3 \tanh_a(\eta)}{\cosh_a^{s_1}(\eta)} - \frac{k^3 s_1 pq (s_1+1) v A_1 B^3 (\ln a)^3 (s_1+2) \tanh_a(\eta)}{\cosh_a^{s_1+2}(\eta)}. \tag{27}$$

Substituting these into (21) yields to

$$\frac{ks_2 v A_2 B \ln a \tanh_a(\eta)}{\cosh_a^{s_2}(\eta)} - \frac{ks_2 A_2^2 B \ln a \tanh_a(\eta)}{\cosh_a^{2s_2}(\eta)} - \frac{k^3 s_2^3 v A_2 B^3 (\ln a)^3 \tanh_a(\eta)}{\cosh_a^{s_2}(\eta)} + \frac{k^3 s_2 (s_2+1)(s_2+2) pq v A_2 B^3 (\ln a)^3 \tanh_a(\eta)}{\cosh_a^{s_2+2}(\eta)} - \frac{ks_1 A_1^2 B \ln a \tanh_a(\eta)}{\cosh_a^{2s_1}(\eta)} = 0, \tag{28}$$

$$\frac{ks_1 v A_1 B \ln a \tanh_a(\eta)}{\cosh_a^{s_1}(\eta)} - \frac{ks_1 A_1^2 B \ln a \tanh_a(\eta)}{\cosh_a^{2s_1}(\eta)} - \frac{k^3 s_1^3 v A_1 B^3 (\ln a)^3 \tanh_a(\eta)}{\cosh_a^{s_1}(\eta)} + \frac{k^3 s_1 (s_1+1)(s_1+2) v pq A_1 B^3 (\ln a)^3 \tanh_a(\eta)}{\cosh_a^{s_1+2}(\eta)} = 0. \tag{29}$$

Now from (28) and (29), equating the exponents $2s_1$ and s_1+2 and also $2s_2$ and s_2+2 gives, respectively,

$$2s_1 = s_1 + 2, \qquad 2s_2 = s_2 + 2, \tag{30}$$

that leads to

$$s_1 = s_2 = 2. \tag{31}$$

So from (28) and (29), the four linearly independent functions are $1/\cosh^{s_1+j}$ and $1/\cosh^{s_2+j}$ for $j = 0, 2$. Therefore, setting their respective coefficients to zero we obtain A_2, B, and v as follow.

Case 1. One has

$$A_2 = \frac{A_1}{2}\left(1 \pm i\sqrt{3}\right), \qquad B = -\frac{1}{2k\ln a}, \qquad v = \frac{A_1}{3pq}, \tag{32}$$

where A_1, k, p, and q are arbitrary constants. Substituting (32) into (22), we obtain new exact solution to (21),

$$r(x,t) = \frac{A_1}{\cosh_a^2(\eta)}, \qquad u(x,t) = \frac{A_2}{\cosh_a^2(\eta)}, \tag{33}$$

where $\eta = -(1/2k\ln a)(x - (A_1/3pq)t)$.

Case 2. One has

$$A_2 = \frac{A_1}{2}\left(1 \pm i\sqrt{3}\right), \qquad B = \frac{1}{2k\ln a}, \qquad v = \frac{A_1}{3pq}, \tag{34}$$

where A_1, k, p, and q are arbitrary constants. Substituting (34) into (22), we obtain new exact solution to (21),

$$r(x,t) = \frac{A_1}{\cosh_a^2(\eta)}, \qquad u(x,t) = \frac{A_2}{\cosh_a^2(\eta)}, \tag{35}$$

where $\eta = (1/2k\ln a)(x - (A_1/3pq)t)$.

Remark 8. If we search the corresponding values for some parameters and make some transformations, in particular the solution (35) can be reduced to the solution (32) mentioned in [24].

4. Conclusions and Remarks

We consider generalized hyperbolic functions and new kinds of generalized hyperbolic function transformation to construct new exact solutions of nonlinear partial differential equations. This paper obtains 1-soliton solution to the generalized KdV equation and the coupled equal width wave equations. Our methods also can be applied to construct new exact solutions of other nonlinear partial differential equations.

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