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Research Article

f_q-Derivations of G-Algebra

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We introduce the notion of f_q-derivation as a new derivation of G-algebra. For an endomorphism map f of any G-algebra X, we show that at least one f_q-derivation of X exists. Moreover, for such a map, we show that a self-map d_q^f of X is f_q-derivation of X if X is an associative medial G-algebra. For a medial G-algebra X, d_q^f is f_q-derivation of X if d_q^f is an outside f_q-derivation of X. Finally, we show that if f is the identity endomorphism of X then the composition of two f_q-derivations of X is a f_q-derivation. Moreover, we give a condition to get a commutative composition.

1. Introduction

Derivation is an important area of research in the theory of algebraic structure in mathematics. The theory of derivations of algebraic structures came from the development of Galois theory and the theory of invariants. Many researches have been done on derivations on different algebras (see [1–4]).

Several authors [5–9] have studied derivations in BCI-algebra after the work done in 2004 by Jun and Xin where the notion of derivation in ring and near-ring theory was applied to BCI-algebra [4]. As in [5], for a self-map d, for any algebra X, d is a left-right derivation (briefly (l, r)-derivation) of X if it satisfies the identity $d(x * y) = (d(x) * y) \wedge (x * d(y))$ for all $x, y \in X$. If d satisfies the identity $d(x * y) = (x * d(y)) \wedge (d(x) * y)$ for all $x, y \in X$, then d is a right-left derivation (briefly (r, l)-derivation) of X. If d is both (l, r)- and (r, l)-derivation, then d is a derivation of X.

Recently, in 2013, a new derivation named f_q-derivation of BCI-algebras was introduced. That is, in general, for any self-map d_q^f of an algebra X, f_q-derivation of X is defined by $d_q^f(x) = f(x) * q$ for all x and $q \in X$. The map d_q^f is called an outside f_q-derivation of X if it satisfies $d_q^f(x * y) = (f(x) * d_q^f(y)) \wedge (d_q^f(x) * f(y))$, $\forall x, y \in X$. If the map d_q^f satisfies the identity $d_q^f(x * y) = (d_q^f(x) * f(y)) \wedge (f(x) * d_q^f(y))$, $\forall x, y \in X$, then the map d_q^f is called an inside f_q-derivation of X. If d_q^f is both outside and inside f_q-derivation of X, then it is a f_q-derivation of X ([10]).

The notion of G-algebra was introduced in [11]. The aim of the paper is to complete the studies on G-algebra; in particular, we aim to apply the notion of f_q-derivation on G-algebra and obtain some related properties. We start with definitions and propositions on G-algebra taken from [11]. Then, we redefine the notion of f_q-derivation in G-algebra and prove that every self-map d_q^f of an associative, medial G-algebra is f_q-derivation, where f is an endomorphism of X. We also show that every self-map d_q^f of an associative, medial G-algebra is f_q-derivation. Then, we show that if f is the identity endomorphism of X, then, for a medial G-algebra, d_q^f is a f_q-derivation of X if d_q^f is an outside f_q-derivation of X. Further, we show that if f is the identity endomorphism of X and $d_q^f, d_q'^f$ are both outside (resp., inside) f_q-derivations of X, then the composition is an outside (resp., inside) f_q-derivation of X and consequently f_q-derivation. We conclude the section with a condition given on two f_q-derivations of X to get a commutative composition.

Definition 1. A G-algebra is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the axioms:

(1) $x * x = 0$,

(2) $x * (x * y) = y$, for all x, y in X.

Proposition 2. *If $(X, *, 0)$ is a G-algebra, then the following conditions hold:*

(1) $x * 0 = x$,

(2) $0 * (0 * x) = x$, *for any* $x \in X$.

Proposition 3. *Let $(X, *, 0)$ be a G-algebra. Then, the following conditions hold for any $x, y \in X$:*

(1) $(x * (x * y)) * y = 0$,

(2) $x * y = 0 \Rightarrow x = y$,

(3) $0 * x = 0 * y \Rightarrow x = y$.

Definition 4. A G-algebra X satisfying $(x * y) * (z * u) = (x * z) * (y * u)$, for any x, y, z and $u \in X$, is called a medial G-algebra.

Lemma 5. *If X is a medial G-algebra, then, for any $x, y, z \in X$, the following axiom holds:*

$$(x * y) * z = (x * z) * y. \tag{1}$$

Theorem 6. *A G-algebra X is medial if and only if it satisfies the following conditions:*

(1) $y * x = 0 * (x * y)$ *for all* $x, y \in X$,

(2) $x * (y * z) = z * (y * x)$ *for all* $x, y, z \in X$.

2. Results

In this section we will introduce a new derivation of G-algebra motivated by [10, Definition 3.1]. We start by defining an endomorphism of G-algebra X.

Definition 7. Let X be a G-algebra and let f be a self-map of X. One says that f is an endomorphism if

$$f(x * y) = f(x) * f(y), \quad \forall x, y \in X. \tag{2}$$

Throughout the paper, d_q^f is a self-map of G-algebra X defined by $d_q^f(x) = f(x) * q$ for all $x \in X$, $q \in X$ and f is an endomorphism self-map of X unless otherwise mentioned.

For elements x and y of a G-algebra X, denote $x \wedge y$ by $y * (y * x)$. By considering that $x \wedge y = x$ in G-algebra, we redefine the notion of f_q-derivation in [10] to get the following definition.

Definition 8. A map d_q^f is called an outside f_q-derivation of X if

$$d_q^f(x * y) = f(x) * d_q^f(y), \quad \forall x, y \in X. \tag{3}$$

If the map d_q^f satisfies the following identity:

$$d_q^f(x * y) = d_q^f(x) * f(y), \quad \forall x, y \in X, \tag{4}$$

then the map is called an inside f_q-derivation of X. If d_q^f is both an outside and inside f_q-derivation of X, then d_q^f is a f_q-derivation of X.

Table 1

$*$	0	1	2
0	0	1	2
1	1	0	2
2	2	1	0

Table 2

x	0	0	0	1	1	1	2	2	2
y	0	1	2	0	1	2	0	1	2
$x * y$	0	1	2	1	0	2	2	1	0
$f(x * y)$	0	2	1	2	0	1	1	2	0
$d_0^f(x * y)$	0	2	1	2	0	1	1	2	0
$f(x)$	0	0	0	2	2	2	1	1	1
$d_0^f(x)$	0	0	0	2	2	2	1	1	1
$f(y)$	0	2	1	0	2	1	0	2	1
$d_0^f(y)$	0	2	1	0	2	1	0	2	1
$f(x) * d_0^f(y)$	0	2	1	2	0	1	1	2	0
$d_0^f(x) * f(y)$	0	2	1	2	0	1	1	2	0

Table 3

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Remark 9. If d_q^f is f_q-derivation, then

$$d_q^f(x * y) = f(x) * d_q^f(y) = d_q^f(x) * f(y). \tag{5}$$

Example 10. Consider the G-algebra given by Cayley table (Table 1).

Define an endomorphism:

$$f : X \longrightarrow X, \quad \text{such that } x \longmapsto \begin{cases} 0 & \text{if } x = 0, \\ 2 & \text{if } x = 1, \\ 1 & \text{if } x = 2. \end{cases} \tag{6}$$

If $q = 0$, then Table 2 shows that d_0^f is an outside and an inside f_q-derivation of X. Hence, d_0^f is a f_q-derivation of X.

If we take $q = 2$, then d_2^f is not an outside f_q-derivation of X or an inside f_q-derivation of X since $d_2^f(1 * 2) = 2$ while $f(1) * d_2^f(2) = 0$ and $d_2^f(1) * f(2) = 1$.

Example 11. Let $X = \{0, a, b, c\}$. Consider the G-algebra given by Cayley table (Table 3).

Define an endomorphism:

$$f : X \longrightarrow X, \quad \text{such that } x \longmapsto \begin{cases} 0 & \text{if } x = 0, \\ b & \text{if } x = a, \\ a & \text{if } x = b, \\ c & \text{if } x = c. \end{cases} \tag{7}$$

It can be shown by direct calculation that d_q^f is f_q-derivation of X for all $q \in X$.

Proposition 12. *For any G-algebra X, there exists at least one f_q-derivation of X, that is, the map d_0^f.*

Proof. Let $q = 0$; then $d_0^f(x * y) = (f(x * y)) * 0 = (f(x) * f(y)) * 0 = f(x) * f(y)$ and $f(x) * d_0^f(y) = f(x) * (f(y) * 0) = f(x) * f(y)$. We also have $d_0^f(x) * f(y) = (f(x) * 0) * f(y) = f(x) * f(y)$. Hence, d_0^f is f_q-derivation of X. □

Proposition 13. *If X is an associative G-algebra, then d_q^f is an outside f_q-derivation of X, for all $q \in X$.*

Proof. We have $d_q^f(x * y) = (f(x * y)) * q = (f(x) * f(y)) * q$ and $f(x) * d_q^f(y) = f(x) * (f(y) * q) = (f(x) * f(y)) * q$, as X is associative. Hence, d_q^f is an outside f_q-derivation of X. □

Proposition 14. *If X is a medial G-algebra, then d_q^f is an inside f_q-derivation of X, for all $q \in X$.*

Proof. Since $d_q^f(x * y) = f(x * y) * q = (f(x) * f(y)) * q$ and $d_q^f(x) * f(y) = (f(x) * q) * f(y) = (f(x) * f(y)) * q$, as X is medial, therefore, d_q^f is an inside f_q-derivation of X. □

The next theorem follows from Propositions 13 and 14.

Theorem 15. *Let X be an associative medial G-algebra; then d_q^f is a f_q-derivation of X, for all $q \in X$.*

Next we provide an alternative proof of Theorem 15.

Theorem 16. *Let X be an associative medial G-algebra. Then, d_q^f is both an outside f_q-derivation of X and an inside f_q-derivation of X for any $q \in X$.*

Proof. Let $x, y, q \in X$. Then, on one hand, we have

$$\begin{aligned} d_q^f(x * y) &= f(x * y) * q \\ &= ((f(x) * f(y)) * q) * 0 \\ &= ((f(x) * q) * f(y)) * 0 \\ &= ((f(x) * q) * f(y)) \\ &\quad * (((f(x) * q) * f(y)) * ((f(x) * q) * f(y))) \\ &= \left(d_q^f(x) * f(y)\right) \\ &\quad * \left(\left(d_q^f(x) * f(y)\right) * ((f(x) * f(y)) * q)\right) \\ &= \left(d_q^f(x) * f(y)\right) \\ &\quad * \left(\left(d_q^f(x) * f(y)\right) * (f(x) * (f(y) * q))\right) \\ &= \left(d_q^f(x) * f(y)\right) \\ &\quad * \left(\left(d_q^f(x) * f(y)\right) * \left(f(x) * d_q^f(y)\right)\right) \\ &= f(x) * d_q^f(y), \quad \text{as } y * (y * x) = x. \end{aligned} \tag{8}$$

By Definition 8, d_q^f is an outside f_q-derivation of X.

On the other hand,

$$\begin{aligned} d_q^f(x * y) &= f(x * y) * q = (f(x * y) * q) * 0 \\ &= (f(x * y) * q) \\ &\quad * ((f(x * y) * q) * (f(x * y) * q)) \\ &= ((f(x) * f(y)) * q) \\ &\quad * (((f(x) * f(y)) * q) * ((f(x) * f(y)) * q)) \\ &= (f(x) * (f(y) * q)) \\ &\quad * ((f(x) * (f(y) * q)) * ((f(x) * q) * f(y))) \\ &= \left(f(x) * d_q^f(y)\right) \\ &\quad * \left(\left(f(x) * d_q^f(y)\right) * \left(d_q^f(x) * f(y)\right)\right) \\ &= d_q^f(x) * f(y). \end{aligned} \tag{9}$$

Therefore, d_q^f is an inside f_q-derivation of X. □

Using Proposition 3(2), we get the following.

Proposition 17. *If d_q^f is an outside (resp., inside) f_q-derivation of X, then $d_q^f(0) = f(x) * d_q^f(x)$, $\forall x \in X$ (resp., $d_q^f(0) = d_q^f(x) * f(x)$, $\forall x \in X$).*

Proof. It is obvious. □

Theorem 18. *Let X be a medial G-algebra. If d_q^f is an outside f_q-derivation of X, then d_q^f is a f_q-derivation of X.*

Proof. From Proposition 14, we know that d_q^f is an inside f_q-derivation of X. Thus, d_q^f is a f_q-derivation of X. □

Definition 19. A map d_q^f is said to be regular if $d_q^f(0) = 0$.

Proposition 20. *Let d_q^f be a f_q-derivation of X. If either $f(x) * d_q^f(y) = 0$ or $d_q^f(x) * f(y) = 0$, then d_q^f is a regular derivation.*

Proof. Since d_q^f is a f_q-derivation, we have $d_q^f(x * y) = f(x) * d_q^f(y) = d_q^f(x) * f(y)$. Consider that $f(x) * d_q^f(y) = 0$; then $d_q^f(0) = d_q^f(x * x) = f(x) * d_q^f(x) = 0$. Similarly, if $d_q^f(x) * f(y) = 0$, we have $d_q^f(0) = d_q^f(x * x) = d_q^f(x) * f(x) = 0$. This proves that d_q^f is a regular derivation. □

Proposition 21. *Let d_q^f be a regular f_q-derivation of X; then $d_q^f(x) = f(x)$, $\forall x \in X$.*

Proof. Since d_q^f is a regular f_q-derivation of X, then $d_q^f(0) = 0$. So $d_q^f(0) = d_q^f(x * x) = d_q^f(x) * f(x) = 0$. Therefore, $d_q^f(x) = f(x)$ from Proposition 3(2). □

Definition 22. Let X be a G-algebra and let d_q^f, $d_q'^f$ be two self-maps of X. Define $d_q^f \circ d_q'^f : X \to X$ by

$$\left(d_q^f \circ d_q'^f\right)(x) = d_q^f\left(d_q'^f(x)\right) \quad \forall x \in X. \tag{10}$$

Proposition 23. *Let X be a G-algebra and let f be the identity endomorphism of X. If d_q^f, $d_q'^f$ are outside f_q-derivations of X, then $d_q^f \circ d_q'^f$ is also an outside f_q-derivation.*

Proof. Consider the element $x * y$. Then $(d_q^f \circ d_q'^f)(x * y) = d_q^f(d_q'^f(x * y))$. As $d_q'^f$ and d_q^f are outside f_q-derivations, we have $d_q^f(d_q'^f(x * y)) = d_q^f(f(x) * d_q'^f(y)) = f(x) * (d_q^f \circ d_q'^f)(y)$. Thus, $d_q^f \circ d_q'^f$ is an outside f_q-derivation. □

Similarly, we can prove the following proposition.

Proposition 24. *For a G-algebra X, let f be the identity endomorphism of X. If d_q^f, $d_q'^f$ are inside f_q-derivations of X, then $d_q^f \circ d_q'^f$ is also an inside f_q-derivation.*

Combining Proposition 23 and Proposition 24 we have the following theorem.

Theorem 25. *Let X be a G-algebra and let f be the identity endomorphism of X. If d_q^f, $d_q'^f$ are both outside (resp., inside) f_q-derivations of X, then the composition is an outside (resp., inside) f_q-derivation of X.*

Proposition 26. *Let X be a G-algebra and let d_q^f, $d_q'^f$ be f_q-derivations of X such that $d_q^f \circ f = f \circ d_q^f$ and $d_q'^f \circ f = f \circ d_q'^f$; then $d_q^f \circ d_q'^f = d_q'^f \circ d_q^f$.*

Proof. Consider $d_q'^f$ as an outside f_q-derivation of X and d_q^f as an inside f_q-derivation of X; then for all $x, y \in X$ we have

$$\begin{aligned}\left(d_q^f \circ d_q'^f\right)(x * y) &= d_q^f\left(d_q'^f(x * y)\right) \\ &= d_q^f\left(f(x) * d_q'^f(y)\right) \\ &= d_q^f(f(x)) * f\left(d_q'^f(y)\right) \\ &= \left(d_q^f \circ f\right)(x) * \left(f \circ d_q'^f\right)(y).\end{aligned} \tag{11}$$

On the other hand,

$$\begin{aligned}\left(d_q'^f \circ d_q^f\right)(x * y) &= d_q'^f\left(d_q^f(x * y)\right) \\ &= d_q'^f\left(d_q^f(x) * f(y)\right) \\ &= f\left(d_q^f(x)\right) * d_q'^f(f(y)) \\ &= \left(f \circ d_q^f\right)(x) * \left(d_q'^f \circ f\right)(y) \\ &= \left(d_q^f \circ f\right)(x) * \left(f \circ d_q'^f\right)(y).\end{aligned} \tag{12}$$

From (11) and (12), we can see that $(d_q^f \circ d_q'^f)(x * y) = (d_q'^f \circ d_q^f)(x * y)$. By putting $y = 0$, we get

$$\left(d_q^f \circ d_q'^f\right)(x) = \left(d_q'^f \circ d_q^f\right)(x). \tag{13}$$

Hence, $d_q^f \circ d_q'^f = d_q'^f \circ d_q^f$. □

3. Conclusion

In this paper, the notion of f_q-derivation of G-algebra is introduced and some related properties are investigated. The main results are Theorems 15 and 18 where we show that a self-map d_q^f of a G-algebra is f_q-derivation if G-algebra satisfies some properties. In Theorem 25, we show that in G-algebra the composition of two f_q-derivations of X is f_q-derivation if f is identity endomorphism of X. Moreover, we give in Proposition 26 a condition on two f_q-derivations of X to get a commutative composition.

Competing Interests

The author declares that there are no competing interests regarding the publication of this paper.

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