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Research Article

On Maximal and Minimal Fuzzy Sets in I-Topological Spaces

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The notion of maximal fuzzy open sets is introduced. Some basic properties and relationships regarding this notion and other notions of I-topology are given. Moreover, some deep results concerning the known minimal fuzzy open sets concept are given.

1. Introduction

In this paper, the unit interval $[0,1]$ will be denoted by I. Let X be a nonempty set. A member of I^X is called a fuzzy subset of X [1]. Throughout this paper, for $A, B \in I^X$ we write $A \leq B$ if and only if $A(x) \leq B(x)$ for all $x \in X$. By $A = B$, we mean that $A \leq B$ and $B \leq A$, that is, $A(x) = B(x)$ for all $x \in X$. Also we write $A < B$ if and only if $A \leq B$ and $A \neq B$. If $\{A_j : j \in J\}$ is a collection of fuzzy sets in X, then $(\bigvee A_j)(x) = \sup\{A_j(x) : j \in J\}$, $x \in X$, and $(\bigwedge A_j)(x) = \inf\{A_j(x) : j \in J\}$, $x \in X$. If $r \in [0,1]$, then r_X denotes the fuzzy set given by $r_X(x) = r$ for all $x \in X$. The complement A^c of a fuzzy set A in X is given by $A^c(x) = 1 - A(x)$, $x \in X$. If $U \subseteq X$, then $\mathcal{X}_U$ denotes the characteristic function of U.

In this paper, we will follow [2] for the definitions of I-topology, the product I-topology, the direct and the inverse images of a fuzzy set under maps and their notations, fuzzy continuity, and fuzzy openness. A fuzzy set p defined by

$$p(x) = \begin{cases} t & \text{if } x = x_p, \\ 0 & \text{if } x \neq x_p, \end{cases} \tag{1.1}$$

where $0 < t \leq 1$, is called a fuzzy point in X, $x_p \in X$ is called the support of p and $p(x_p) = t$ the value (level) of p [2]. Two fuzzy points p and q in X are said to be distinct if and only if their supports are distinct, that is, $x_p \neq x_q$. $A \in I^X$ is called a crisp subset of X if A is a characteristic function of some ordinary subset of X [2]. $A \in I^X$ is called a fuzzy crisp point in X if it is a characteristic function of singleton [2].

In this paper, we follow [3] for the definition of "belonging to". Namely, a fuzzy point p in X is said to belong to a fuzzy set A in X (notation: $p \in A$) if and only if $p(x_p) \leq A(x_p)$.

Let (X, τ) be a topological space and let U be a subset of X. U is called semiopen [4] if $U \subseteq \mathrm{Cl}(\mathrm{Int}(U))$ and U is called preopen [5] if $U \subseteq \mathrm{Int}(\mathrm{Cl}(U))$. A nonempty open subset U of X is called a minimal open set if the only nonempty open set which is contained in U is U. $\min(X, \tau)$ will denote the family of all minimal open sets in X. (X, τ) is said to be homogeneous if for any two points $x_1, x_2 \in X$, there exists an autohomeomorphism on (X, τ) takes x_1 to x_2. In 1997, Fora and Al-Bsoul [6] used minimal open sets to characterize and count finite homogeneous spaces. In 2001, Nakaoka and Nobuyuki [7] characterized minimal open sets and proved that any subset of a minimal open set is pre-open. A non-empty pre-open subset U of X is called a minimal pre-open set [8] if the only non-empty pre-open set which is contained in U is U. The author in [8] characterized minimal pre-open sets as pre-open singletons. A non-empty semi-open subset U of X is called a minimal semi-open set [9] if the only non-empty semi-open set which is contained in U is U. The authors in [9] proved that the set of minimal open sets and the set of minimal semi-open sets in a space are equal. An I-topological space $(X, \Im)$ is said to be fuzzy homogeneous [10] if for any two points $x_1, x_2 \in X$, there exists a fuzzy autohomeomorphism on $(X, \Im)$ takes x_1 to x_2. The authors in [11] extended the concept of minimal open sets to include I-topological spaces as follows; a fuzzy open set A of an I-topological space $(X, \Im)$ is called minimal fuzzy open set [11] if A is nonzero and there is no nonzero proper fuzzy open subset of A. $\min(X, \Im)$ will denote the family of all minimal fuzzy open sets in X. The authors in [11] obtained many results concerning minimal fuzzy open sets, and they proved that homogeneity in I-topological spaces forces the shape of minimal fuzzy open sets. Then the author in [12] generalized minimal fuzzy open sets by two methods. A proper non-empty open subset U of a topological space (X, τ) is called a maximal open set [13] if any open set which contains U is X or U. $\max(X, \tau)$ will denote the family of all maximal open sets in X. The authors in [13] obtained fundamental properties of maximal open sets such as decomposition theorem for a maximal open set and established basic properties of intersections of maximal open sets, such as the law of radical closure. By a dual concepts of minimal open sets and maximal open sets, the authors in [14] introduced the concepts of minimal closed sets and maximal closed sets and obtained results easily by dualizing the known results regarding minimal open sets and maximal open sets. This paper proposes mainly maximal fuzzy open sets in I-topological spaces. It also gives some deep results concerning the known minimal fuzzy open sets concept.

Throughout this paper, for any set X, $|X|$ will denote the cardinality of X. If A is a fuzzy set in X, then the support of A is denoted by $S(A)$ and defined by $S(\lambda) = \lambda^{-1}(0,1]$, and the set $\{x \in X : A(x) = 1\}$ will be denoted by $1(A)$.

2. Maximal Fuzzy Open Sets

Definition 2.1. Let $(X, \Im)$ be an I-topological space. A nonzero fuzzy open subset A of X is said to be maximal fuzzy open set if $A \neq 1_X$ and for any fuzzy open set B in X with $A \leq B$, $B = A$ or $B = 1_X$.

Throughout this paper, the set of all maximal fuzzy open subsets of the I-topological space $(X, \Im)$ will be denoted by $\max(X, \Im)$.

Theorem 2.2. *Let $(X, \Im)$ be an I-topological space and $A \in \max(X, \Im)$ with $1(A) = \emptyset$. Then for every $B \in \Im - \{1_X\}$ one has $B \leq A$.*

Proof. Suppose to the contrary that there exist $B \in \Im - \{1_X\}$ and $x_\circ \in X$ such that $B(x_\circ) > A(x_\circ)$. Since $(A \vee B)(x_\circ) = B(x_\circ) \neq A(x_\circ)$ and $A \in \max(X, \Im)$, $A \vee B = 1_X$. Therefore, for each $x \in X$, $A(x) < 1 = B(x)$ and hence $B = 1_X$, which is a contradiction. □

Corollary 2.3. *Let $(X, \Im)$ be an I-topological space. If $A \in \max(X, \Im)$ with $1(A) = \emptyset$, then $\max(X, \Im) = \{A\}$.*

Corollary 2.4. *Let $(X, \Im)$ be an I-topological space. If $|\max(X, \Im)| > 1$, then for every $A \in \max(X, \Im)$, $1(A) \neq \emptyset$.*

Example 2.5. Let $X = \mathbb{R}$ and τ be the usual topology on $\mathbb{R}$, then $\max(X, \tau) = \{\mathbb{R} - \{x\} : x \in \mathbb{R}\}$. Let $\Im = \{\mathcal{X}_U : U \in \tau\}$. then $\max(X, \Im) = \{\mathcal{X}_U : U \in \max(X, \tau)\}$ is uncountable. Therefore, the condition "$1(A) = \emptyset$" in Corollary 2.3 cannot be dropped.

Proposition 2.6 (see [13]). *Let (X, τ) be a topological space and let $U \in \max(X, \tau)$, then* $\mathrm{Cl}(U) = X$ *or* $\mathrm{Cl}(U) = U$.

The following example shows that the exact fuzzy version of Proposition 2.6 is not true in general.

Example 2.7. Let $X = \{x_1, x_2, x_3\}$ and let A, B be fuzzy sets in X defined as follows:

$$\begin{aligned} &A(x_1) = 0.3, \qquad A(x_2) = 0.3, \qquad A(x_3) = 1, \\ &B(x_1) = 0.3, \qquad B(x_2) = 0.3, \qquad B(x_3) = 0. \end{aligned} \tag{2.1}$$

Let $\Im = \{0_X, 1_X, A, B\}$, then $\max(X, \Im) = \{A\}$ and $\mathrm{Cl}(A) = \{(x_1, 0.7), (x_2, 0.7), (x_3, 1)\}$ but $\mathrm{Cl}(A)$ is neither A nor 1_X.

The following lemma will be used in the following main result.

Lemma 2.8. *Let $(X, \Im)$ be an I-topological space and $A \in \max(X, \Im)$. If $B \in \Im - \{0_X\}$ such that $A \wedge B = 0_X$, then $A = \mathcal{X}_{S(A)}$ and $B = A^c$.*

Proof. Choose $x_\circ \in S(B)$. Since $A \wedge B = 0_X$, $A(x_\circ) = 0$. Thus, $(A \vee B)(x_\circ) = B(x_\circ) \neq A(x_\circ)$. Since $A \in \max(X, \Im)$, $A \vee B = 1_X$. Since $A \wedge B = 0_X$, it follows that $A = \mathcal{X}_{S(A)}$ and $B = A^c$. □

The following result is a partial fuzzy version of Proposition 2.6.

Theorem 2.9. *Let $(X, \Im)$ be an I-topological space. If $A = \mathcal{X}_{S(A)} \in \max(X, \Im)$, then either* $\mathrm{Cl}(A) = A$ *or* $\mathrm{Cl}(A) = 1_X$.

Proof. Suppose that $\mathrm{Cl}(A) \neq 1_X$, then there exists $x_\circ \in X$ such that $(\mathrm{Cl}(A))(x_\circ) < 1$. Let $B = (\mathrm{Cl}(A))^c$, then $B \in \Im, B \neq 0_X$, and $A \wedge B = 0_X$. Hence by Lemma 2.8, it follows that $B = A^c$. Therefore, $\mathrm{Cl}(A) = A$. □

Proposition 2.10 (see [13]). *Let (X,τ) be a topological space. If $U \in \max(X,\tau)$ and S is a non empty subset of $X - U$, then $\mathrm{Cl}(S) = X - U$.*

The following result is the exact fuzzy version of Proposition 2.10.

Theorem 2.11. *Let $(X,\Im)$ be an I-topological space. If $A \in \max(X,\Im)$ and B is a non zero fuzzy set in X with $B \le A^c$, then $\mathrm{Cl}(B) = A^c$.*

Proof. Suppose to the contrary that $\mathrm{Cl}(B) \ne A^c$. Since $B \le A^c$ and A^c is a fuzzy closed set in X, then $\mathrm{Cl}(B) \le A^c$. Therefore, there exists $x_\circ \in X$ such that $(\mathrm{Cl}(B))(x_\circ) < A^c(x_\circ)$. Since $A(x_\circ) < (1_X - \mathrm{Cl}(B))(x_\circ)$ and $A \in \max(X,\Im)$, then $A \vee (1_X - \mathrm{Cl}(B)) = 1_X$. We are going to show that $\mathrm{Cl}(B) = 0_X$. Let $x \in X = 1(A) \cup (X - 1(A))$. If $x \in 1(A)$, then $A^c(x) = 0$ and hence $(\mathrm{Cl}(B))(x) \le A^c(x) = 0$. If $x \in (X - 1(A))$, then $A(x) < 1$. Since $A \vee (1_X - \mathrm{Cl}(B)) = 1_X$, we get $(1_X - \mathrm{Cl}(B))(x) = 1$ and hence $(\mathrm{Cl}(B))(x) = 0$. Hence, we complete the proof that $\mathrm{Cl}(B) = 0_X$. Therefore, $B = 0_X$, which is a contradiction. □

Recall that a fuzzy subset A of an I-topological space $(X,\Im)$ is called fuzzy preopen if $A \le \mathrm{int}(\mathrm{Cl}(A))$.

From now on the set $PO(X,\Im)$ will denote the set of all fuzzy preopen subsets of the I-topological space $(X,\Im)$.

Corollary 2.12. *Let $(X,\Im)$ be an I-topological space and $A \in \max(X,\Im)$, then $\{B \in I^X : B \le \mathrm{int}(A^c)\} \subseteq PO(X,\Im)$.*

Proof. Let $B \in I^X$ such that $B \le \mathrm{int}(A^c)$. If $B = 0_X$, then $B \in PO(X,\Im)$. If $B \ne 0_X$, then by Theorem 2.11, $\mathrm{Cl}(B) = A^c$ and hence $B \le \mathrm{int}(A^c) = \mathrm{int}(\mathrm{Cl}(B))$, that is, $B \in PO(X,\Im)$. □

Corollary 2.13. *Let $(X,\Im)$ be an I-topological space and $A \in \max(X,\Im)$ with A is a clopen fuzzy set, then $\{B \in I^X : B \le A^c\} \subseteq PO(X,\Im)$.*

Proof. Let $B \in I^X$ such that $B \le A^c$. If $B = 0_X$, then $B \in PO(X,\Im)$. If $B \ne 0_X$, then by Theorem 2.11, $\mathrm{Cl}(B) = A^c$ and hence $B \le A^c = \mathrm{int}(A^c) = \mathrm{int}(\mathrm{Cl}(B))$, that is, $B \in PO(X,\Im)$. □

Proposition 2.14 (see [13]). *Let (X,τ) be a topological space. If $U \in \max(X,\tau)$ and M is a subset of X with $U \subsetneq M$, then $\mathrm{Cl}(M) = X$.*

Corollary 2.15. *Let (X,τ) be a topological space. If $U \in \max(X,\tau)$ and M is a closed subset of X with $U \subsetneq M$, then $M = X$.*

The following example shows that the exact fuzzy version of each of Proposition 2.14 and Corollary 2.15 is invalid in general.

Example 2.16. Let X be a non empty set with the I-topology $\Im = \{0_X, 1_X, (0.3)_X\}$, then $\max(X,\Im) = \{(0.3)_X\}$ and $(0.3)_X < (0.4)_X$, but $\mathrm{Cl}((0.4)_X) = (0.7)_X$. This shows that the exact fuzzy version of Proposition 2.14 is invalid in general. On the other hand, since $(0.7)_X$ is a fuzzy closed subset of $(X,\Im)$ and $0_X < (0.7)_X < 1_X$, we get that the exact fuzzy version of Corollary 2.15 is invalid in general.

The following result is a partial fuzzy version of Corollary 2.15.

Theorem 2.17. *Let* $(X, \mathfrak{I})$ *be an I-topological space and* $A \in \max(X, \mathfrak{I})$. *If* B *is a fuzzy closed subset of* X *with* $A < B$, *then for every* $x \in X$ *with* $A(x) < B(x)$, *one has* $B(x) > 0.5$.

Proof. Suppose to the contrary that there exists a fuzzy closed subset B of X and $x_\circ \in X$ such that $A(x_\circ) < B(x_\circ)$ and $B(x_\circ) \leq 0.5$. Since B is a fuzzy closed subset of X, $B^c \in \mathfrak{I}$ and so either $B^c \vee A = A$ or $B^c \vee A = 1_X$. If $B^c \vee A = A$, then $1 - B(x_\circ) \leq A(x_\circ)$ and so $B(x_\circ) > 0.5$ which is a contradiction because $B(x_\circ) \leq 0.5$. If $B^c \vee A = 1_X$, then $B(x_\circ) = 0$ or $A(x_\circ) = 1$ which is also a contradiction because $A(x_\circ) < B(x_\circ)$. □

For crisp maximal fuzzy sets, the exact fuzzy version of Corollary 2.15 is valid as the following result shows.

Theorem 2.18. *Let* $(X, \mathfrak{I})$ *be an I-topological space. If* $A = \mathcal{X}_S \in \max(X, \mathfrak{I})$ *and* B *is a fuzzy closed subset of* X *with* $A < B$, *then* $B = 1_X$.

Proof. Suppose to the contrary that there exists $x_\circ \in X$ such that $B(x_\circ) < 1$. Since $A < B$, then $x_\circ \in X - S$. Therefore, $(B^c \vee A)(x_\circ) = B^c(x_\circ) \neq 0 = A(x_\circ)$ and so $B^c \vee A \neq A$. Hence, we must have $B^c \vee A = 1_X$. Choose $x_1 \in X$ such that $A(x_1) < B(x_1)$. Then $B^c(x_1) = 1$ and so $B(x_1) = 0$, but $0 = A(x_1) < B(x_1)$, which is a contradiction. □

Corollary 2.19. *Let* $(X, \mathfrak{I})$ *be an I-topological space. If* $A = \mathcal{X}_S \in \max(X, \mathfrak{I})$ *and* B *is a fuzzy subset of* X *with* $A < B$, *then* $\mathrm{Cl}(B) = 1_X$.

Corollary 2.20. *Let* $(X, \mathfrak{I})$ *be an I-topological space. If* $A = \mathcal{X}_S \in \max(X, \mathfrak{I})$ *and* B *is a fuzzy subset of* X *with* $A \leq B$, *then* B *is a fuzzy preopen set.*

Proof. If $B = A$, then B is fuzzy open and so it is a fuzzy preopen set. On the other hand, if $A < B$, then by Corollary 2.19, it follows that $B \leq 1_X = \mathrm{int}(1_X) = \mathrm{int}(\mathrm{Cl}(B))$. □

The following ordinary topological spaces result follows easily.

Proposition 2.21. *Let* (X, τ) *be an ordinary topological space, then* $\{U, U^c\} \subseteq \max(X, \tau)$ *for some* $U \subseteq X$ *if and only if* $\tau = \{\emptyset, X, U, U^c\}$.

The exact fuzzy version of Proposition 2.21 is true as the following result shows.

Theorem 2.22. *Let* $(X, \mathfrak{I})$ *be an I-topological space and let* $A \in I^X$ *with* $A \neq A^c$, *then* $\{A, A^c\} \subseteq \max(X, \mathfrak{I})$ *if and only if* $A = \mathcal{X}_{S(A)}$ *and* $\mathfrak{I} = \{0_X, 1_X, A, A^c\}$.

Proof. $\Longrightarrow$) Suppose that $\{A, A^c\} \subseteq \max(X, \mathfrak{I})$. Choose $x_\circ \in X$ such that $A(x_\circ) \neq A^c(x_\circ)$. Then $(A \vee A^c)(x_\circ) \neq A(x_\circ)$ or $(A \vee A^c)(x_\circ) \neq A^c(x_\circ)$. Since $\{A, A^c\} \subseteq \max(X, \mathfrak{I})$, then $A \vee A^c = 1_X$. Thus for every $x \in X$ with $A(x) < 1$, we must have $A^c(x) = 1$ and so $A(x) = 0$. Therefore, $A = \mathcal{X}_{S(A)}$. Let $B \in \mathfrak{I} - \{0_X\}$. If $B \leq A$, then $A^c \wedge B = 0_X$ and, by Lemma 2.8, $B = (A^c)^c = A$. If $B \nleq A$, then $A \vee B = 1_X$ and so $B = A^c$.

$\Longleftarrow$) Clear. □

3. Images and Products of Maximal Sets

From now on π_x and π_y will denote the projections on X and Y, respectively, τ_{prod} will denote the product topology of τ_1 and τ_2, and $\mathfrak{I}_{\mathrm{prod}}$ will denote the product I-topology of $\mathfrak{I}_1$ and $\mathfrak{I}_2$.

We start by the following result.

Proposition 3.1. *Let* (X, τ_1) *and* (Y, τ_2) *be two ordinary topological spaces. If* $f : (X, \tau_1) \longrightarrow (Y, \tau_2)$ *is continuous, open, and surjective, then for every* $U \in \max(X, \tau_1)$ *with* $f(U) \neq Y$ *one has* $f(U) \in \max(Y, \tau_2)$.

Proof. Suppose that $H \in \tau_2$ with $f(U) \subsetneq H \subset Y$. Then $f^{-1}(H) \in \tau_1$ with $U \subsetneq f^{-1}(H)$. Since $U \in \max(X, \tau_1)$, then $f^{-1}(H) = U$ or $f^{-1}(H) = X$. If $f^{-1}(H) = X$, then $f(f^{-1}(H)) = f(X) = Y$. Thus $f^{-1}(H) = U$ and so $H = f(f^{-1}(H)) = f(U)$. □

The following result is the exact fuzzy version of Proposition 3.1.

Theorem 3.2. *Let* $(X, \Im_1)$ *and* $(Y, \Im_2)$ *be two I-topological spaces and let* $f:(X, \Im_1) \rightarrow (Y, \Im_2)$ *be fuzzy continuous, fuzzy open, and surjective function. If* $A \in \max(X, \Im_1)$, *then either* $f(A) = 1_Y$ *or* $f(A) \in \max(Y, \Im_2)$.

Proof. Suppose that $f(A) \neq 1_Y$. It is sufficient to show that $f(A) \in \max(Y, \Im_2)$. Since f is fuzzy open and A is a fuzzy open set, then $f(A)$ is a fuzzy open set. Since $A \neq 0_X$, there exists $x_\circ \in X$ such that $A(x_\circ) > 0$ and so $(f(A))(f(x_\circ)) = \sup\{A(x) : f(x) = f(x_\circ)\} \geq A(x_\circ) > 0$, hence $f(A) \neq 0_Y$. Suppose that $B \in \Im_2$ such that $f(A) < B$. We are going to show $B = 1_Y$ which completes the proof. Choose $y_\circ \in Y$ such that $(f(A))(y_\circ) < B(y_\circ)$. Since f is onto, there exists $x_1 \in X$ such that $f(x_1) = y_\circ$. Thus, $A(x_1) \leq (f(A))(y_\circ) < B(y_\circ)$. Since f is fuzzy continuous, $f^{-1}(B) \in \Im_1$. Thus, we have $f^{-1}(B) \vee A \in \Im_1$, $A \leq f^{-1}(B) \vee A$, and $(f^{-1}(B) \vee A)(x_1) = \max\{A(x_1), B(y_\circ)\} = B(y_\circ) > A(x_1)$. Since $A \in \max(X, \Im_1)$, then $f^{-1}(B) \vee A = 1_X$. To see that $B = 1_Y$, let $y \in Y$ and choose $x \in X$ such that $f(x) = y$, then $1 = \max\{A(x), B(y)\} \leq \max\{f(A)(y), B(y)\} = B(y)$ and so $B(y) = 1$. □

Corollary 3.3. *Let* $(X, \Im_1)$ *and* $(Y, \Im_2)$ *be two I-topological spaces and let* $f : (X, \Im_1) \rightarrow (Y, \Im_2)$ *be a fuzzy homeomorphism. If* $A \in \max(X, \Im_1)$, *then* $f(A) \in \max(Y, \Im_2)$.

Proof. Let $A \in \max(X, \Im_1)$. According to Theorem 3.2, it is sufficient to see that $f(A) \neq 1_Y$. Since $A \in \max(X, \Im_1)$, there exists $x_\circ \in X$ such that $A(x_\circ) < 1$. Thus, $f(A)(f(x_\circ)) = A(x_\circ) < 1$. □

Proposition 3.4. *Let* (X, τ_1) *and* (Y, τ_2) *be two ordinary topological spaces. If* $U \in \max(X, \tau_1)$ *and* $V \in \max(Y, \tau_2)$, *then* $U \times V \notin \max(X \times Y, \tau_{prod})$.

Proof. Follows because $U \times Y \in \tau_{\text{prod}}$, but $U \times V \subset U \times Y \subset X \times Y$. □

Proposition 3.5. *Let* (X, τ_1) *and* (Y, τ_2) *be two ordinary topological spaces and let* $G \in \max(X \times Y, \tau_{prod})$ *such that* $\pi_x(G) \times \pi_y(G) \neq X \times Y$, *then there exists* $U \in \max(X, \tau_1)$ *or* $V \in \max(Y, \tau_2)$ *such that* $G = U \times Y$ *or* $G = X \times V$.

Proof. Since $G \subseteq \pi_x(G) \times \pi_y(G)$, $\pi_x(G) \times \pi_y(G) \in \tau_{\text{prod}}$, $\pi_x(G) \times \pi_y(G) \neq X \times Y$, and $G \in \max(X \times Y, \tau_{\text{prod}})$, it follows that $G = \pi_x(G) \times \pi_y(G)$. If $\pi_x(G) \neq X$ and $\pi_y(G) \neq Y$, then by Proposition 3.1, $\pi_x(G) \in \max(X, \tau_1)$ and $\pi_y(G) \in \max(Y, \tau_2)$, which is impossible by Proposition 3.4. Therefore, either $\pi_x(G) = X$ or $\pi_y(G) = Y$. Hence, there exists $U \in \max(X, \tau_1)$ or $V \in \max(Y, \tau_2)$ such that $G = U \times Y$ or $G = X \times V$. □

The following example shows that the condition "$\pi_x(G) \times \pi_y(G) \neq X \times Y$" in Proposition 3.5 cannot be dropped.

Example 3.6. Let $X = \mathbb{R}$ with the usual topology τ_u and let $G = \mathbb{R}^2 - \{(0,0)\}$, then $G \in \max(X \times X, \tau_{\text{prod}})$, $\pi_x(G) \times \pi_y(G) = X \times X$, and there is no $U \in \max(X, \tau_u)$ such that $G = U \times X$ or $G = X \times U$.

The following is the main result of this section.

Theorem 3.7. *Let $(X, \Im_1)$ and $(Y, \Im_2)$ be two I-topological spaces. If $A \in \max(X, \Im_1)$ and $B \in \max(Y, \Im_2)$, then $A \times B \in \max(X \times Y, \Im_{prod})$ if and only if there exists $0 < c < 1$ such that $A = c_X$ and $B = c_Y$.*

Proof. Suppose that $A \times B \in \max(X \times X, \Im_{\text{prod}})$. Since $A \times 1_Y, 1_X \times B \in \Im_{\text{prod}}$, $A \times B \leq A \times 1_Y < 1_{X \times Y}$, $A \times B \leq 1_X \times B < 1_{X \times Y}$, then $A \times B = A \times 1_Y$ and $A \times B = 1_X \times B$. Therefore, for every $x \in X$ and $y \in Y$, $\min\{A(x), B(y)\} = \min\{A(x), 1\} = \min\{1, B(y)\}$ and hence $A(x) = B(y)$. Thus, there exists $0 < c < 1$ such that $A = c_X$ and $B = c_Y$. Conversely, suppose for some $0 < c < 1$ that $A = c_X$ and $B = c_Y$. Let $M \in \Im_{\text{prod}} - \{1_{X \times Y}\}$ such that $c_X \times c_Y = c_{X \times Y} \leq M$. Choose families $\{A_\alpha : \alpha \in \Lambda\} \subseteq \Im_1$ and $\{B_\alpha : \alpha \in \Lambda\} \subseteq \Im_2$ such that $M = \bigvee\{A_\alpha \times B_\alpha : \alpha \in \Lambda\}$. Since $M \neq 1_{X \times Y}$, then for every $\alpha \in \Lambda$, $A_\alpha \neq 1_X$ or $B_\alpha \neq 1_Y$. We are going to show that $A_\alpha \times B_\alpha \leq c_{X \times Y}$ for every $\alpha \in \Lambda$. Let $\alpha \in \Lambda$. □

Case 1. $A_\alpha \neq 1_X$ and $B_\alpha \neq 1_Y$, then $A_\alpha \leq c_X$ and $B_\alpha \leq c_Y$ and hence $A_\alpha \times B_\alpha \leq c_{X \times Y}$.

Case 2. $A_\alpha \neq 1_X$ and $B_\alpha = 1_Y$, then for every $(x, y) \in X \times Y$, $(A_\alpha \times B_\alpha)(x, y) = \min\{A_\alpha(x), 1\} = A_\alpha(x) \leq c_X(x) = c_{X \times Y}(x, y)$. Hence, $A_\alpha \times B_\alpha \leq c_{X \times Y}$.

Case 3. $A_\alpha = 1_X$ and $B_\alpha \neq 1_Y$. Similar to Case 2, we get $A_\alpha \times B_\alpha \leq c_{X \times Y}$.

Therefore, we get that $M = \bigvee\{A_\alpha \times B_\alpha : \alpha \in \Lambda\} \leq c_{X \times Y}$ and hence $M = c_{X \times Y}$. This ends the proof of this direction.

4. Maximal Sets and T_c Property

We start this section by the following nice characterization of T_1 ordinary topological spaces.

Proposition 4.1. *Let (X, τ) be an ordinary topological space with $|X| > 1$, then (X, τ) is T_1 if and only if $\max(X, \tau) = \{X - \{x\} : x \in X\}$.*

Proof. It is straightforward. □

Corollary 4.2. *If (X, τ) is a T_1 ordinary topological space with $|X| > 1$, then $\max(X, \tau)$ covers X.*

Definition 4.3 (see [15]). An I-topological space $(X, \Im)$ is said to be T_c if every fuzzy crisp point in X is fuzzy closed.

Definition 4.4. Let $(X, \Im)$ be an I-topological space and let $x \in X$. Denote the fuzzy set

$$\bigvee\{A \in \Im : A(x) < 1\} \tag{4.1}$$

by $x(\mathfrak{I})$.

Proposition 4.5. *Let* $(X, \mathfrak{I})$ *be a* T_c *I-topological space and* $x \in X$*, then for every* $y \in X - \{x\}$, $(x(\mathfrak{I}))(y) = 1$.

Proof. Since $(X, \mathfrak{I})$ is T_c, then $\mathcal{X}_{X-\{x\}} \in \mathfrak{I}$ with $(\mathcal{X}_{X-\{x\}})(x) = 0 < 1$. Thus, for every $y \in X-\{x\}$, it follows that $1 = (\mathcal{X}_{X-\{x\}})(y) \leq (x(\mathfrak{I}))(y) \leq 1$ and hence $(x(\mathfrak{I}))(y) = 1$. □

Theorem 4.6. *Let* $(X, \mathfrak{I})$ *be a* T_c *I-topological space and* $x \in X$*, then*

$$\max(X, \mathfrak{I}) = \{x(\mathfrak{I}) : x \in X,\ (x(\mathfrak{I}))(x) < 1\}. \tag{4.2}$$

Proof. Let $B \in \max(X, \mathfrak{I})$. Choose $x \in X$ such that $B(x) < 1$. If $(x(\mathfrak{I}))(x) = 1$, then there exists $A \in \mathfrak{I}$ such that $B(x) < A(x) < 1$ and hence we have $B \vee A \in \mathfrak{I}$ with $B < B \vee A < 1_X$ which is impossible. Since $B \vee x(\mathfrak{I}) \in \mathfrak{I}$ and $(B \vee x(\mathfrak{I}))(x) < 1$, then $B \vee x(\mathfrak{I}) = B$. Thus, $x(\mathfrak{I}) \leq B$. On the other hand, by the definition of $x(\mathfrak{I})$ and that $B(x) < 1$, we have $B \leq x(\mathfrak{I})$. Therefore, $B = x(\mathfrak{I})$.

Conversely, let $x \in X$ such that $(x(\mathfrak{I}))(x) < 1$ and $M \in \mathfrak{I} - \{1_X\}$ such that $x(\mathfrak{I}) \leq M$. By Proposition 4.5, it follows that $M(y) = 1$ for every $y \in X - \{x\}$. Since $M \in \mathfrak{I} - \{1_X\}$, it follows that $M(x) < 1$, and by the definition of $x(\mathfrak{I})$, we must have $M \leq x(\mathfrak{I})$. Therefore, $M = x(\mathfrak{I})$ and hence $x(\mathfrak{I}) \in \max(X, \mathfrak{I})$. □

Corollary 4.7. *Let* $(X, \mathfrak{I})$ *be a* T_c *I-topological space, then the following are equivalent:*

(a) $\max(X, \mathfrak{I}) = \emptyset$,

(b) $(x(\mathfrak{I}))(x) = 1$ *for every* $x \in X$,

(c) $x(\mathfrak{I}) = 1_X$ *for every* $x \in X$.

Corollary 4.8. *For every non empty set* X *and for the discrete I-topology* $\mathfrak{I}_{disc}$ *on* X, $\max(X, \mathfrak{I}_{disc}) = \emptyset$.

Corollary 4.8 shows that Corollary 4.2 is invalid in I-topological spaces.

5. Maximal Sets and Homogeneity

We start this section by the following ordinary topological spaces result.

Proposition 5.1. *If* (X, τ) *is a homogeneous ordinary topological space which contains a maximal open set, then* $\max(X, \tau)$ *covers* X.

Proof. Let $x \in X$. Choose $M \in \max(X, \tau)$ and $y \in M$. Since (X, τ) is homogeneous, then there exists a homeomorphism $h : (X, \tau) \longrightarrow (X, \tau)$ such that $h(y) = x$. By Proposition 3.1, it follows that $h(M) \in \max(X, \tau)$. Since $x = h(y) \in h(M)$, the proof is ended. □

The following example shows that the condition "homogeneous" in Proposition 5.1 cannot be dropped.

Example 5.2. Let $X = \mathbb{R}$ and $\tau = \{\emptyset, X, \mathbb{Q}\}$, then $\max(X, \tau) = \{\mathbb{Q}\}$ and so $\max(X, \tau)$ does not cover X.

Theorem 5.3. *Let* $(X, \Im)$ *be a fuzzy homogeneous topological space. If* $A \in \max(X, \Im)$ *such that* $1(A) = \emptyset$*, then* A *is a constant fuzzy set.*

Proof. Suppose to the contrary that there exists $x_1, x_2 \in X$ such that $A(x_1) < A(x_2)$. Since $(X, \Im)$ is fuzzy homogeneous, there exists a fuzzy homeomorphism $f : (X, \Im) \longrightarrow (X, \Im)$ such that $f(x_1) = x_2$. Note that $(A \vee f^{-1}(A))(x_1) = \max\{A(x_1), A(x_2)\} = A(x_2) < 1$. Then $A \vee f^{-1}(A) \in \Im$ and $A < A \vee f^{-1}(A) < 1_X$, which contradicts the fuzzy maximality of A. □

The following example shows that the condition "fuzzy homogeneous" in Theorem 5.3 cannot be dropped.

Example 5.4. Let $X = \{2, 3\}$ with the I-topology $\Im = \{0_X, 1_X, A\}$ where $A(2) = 0.3$ and $A(3) = 0.5$, then $\max(X, \Im) = \{A\}$ and $1(A) = \emptyset$ while A is not constant.

The following example shows that the condition "$1(A) = \emptyset$" in Theorem 5.3 cannot be dropped.

Example 5.5. Let $X = \{2, 3\}$ with the I-topology $\Im = \{0_X, 1_X, A, B\}$ where $A(2) = 0$, $A(3) = 1$, $B(2) = 1$, and $B(3) = 0$, then $(X, \Im)$ is fuzzy homogeneous and $A \in \max(X, \Im)$ while A is not constant.

Theorem 5.3 shows that Proposition 5.1 is invalid in I-topological spaces in general. However, the following main result is a partial fuzzy version of Theorem 5.3.

Theorem 5.6. *Let* $(X, \Im)$ *be a homogeneous I-topological space which consists a maximal fuzzy set, then* $\bigvee\{B : B \in \max(X, \Im)\} = 1_X$ *if and only if there exists* $A \in \max(X, \Im)$ *such that* $1(A) \neq \emptyset$.

Proof. $\Longrightarrow$) Suppose that $\bigvee\{B : B \in \max(X, \Im)\} = 1_X$ and suppose to the contrary that $1(B) = \emptyset$ for all $B \in \max(X, \Im)$. Choose $A \in \max(X, \Im)$. Then by Corollary 2.3, $\max(X, \Im) = \{A\}$. Applying Theorem 5.3, it follows that A is a constant fuzzy set and hence $\bigvee\{B : B \in \max(X, \Im)\} = A \neq 1_X$, which is a contradiction.

$\Longleftarrow$) Suppose $A \in \max(X, \Im)$ such that $1(A) \neq \emptyset$. Choose $x_\circ \in X$ such that $A(x_\circ) = 1$. To see that $\bigvee\{B : B \in \max(X, \Im)\} = 1_X$, let $x \in X$ such that $x \neq x_\circ$. Choose a fuzzy homeomorphism $f : (X, \Im) \longrightarrow (X, \Im)$ such that $f(x_\circ) = x$. By Corollary 3.3, it follows that $f(A) \in \max(X, \Im)$. Thus, $(\bigvee\{B : B \in \max(X, \Im)\})(x) \geq f(A)(x) = A(x_\circ) = 1$ and hence

$$\left(\bigvee\{B : B \in \max(X, \Im)\}\right)(x) = 1. \tag{5.1}$$

□

6. Minimal Fuzzy Open Sets

Definition 6.1. Let $(X, \Im)$ be an I-topological space. For each $x \in X$, we call the fuzzy set $\bigwedge\{A \in \Im : x \in S(A)\}$ the lower fuzzy set at x, and we denote it by A_x.

The following is an example of a zero lower fuzzy set.

Example 6.2. Let X be any non empty set together with any I-topology $\Im$ such that $\{c_x : 0 \leq c \leq 1\} \subseteq \Im$, then for every $x \in X$, $A_x = 0_X$.

The following lemma will be used in the next main result.

Lemma 6.3. *Let* $(X,\mathfrak{I})$ *be an I-topological space and let* $A \in \min(X,\mathfrak{I})$*. If* $x \in S(A)$*, then for every* $B \in \mathfrak{I}$ *with* $x \in S(B)$*, one has* $A \leq B$*.*

Proof. Since $x \in S(A) \cap S(B)$, $0_X \neq A \wedge B \leq A$. Since $A \in \min(X,\mathfrak{I})$, then $A \wedge B = A$ and hence $A \leq B$. □

Theorem 6.4. *Let* $(X,\mathfrak{I})$ *be an I-topological space and let* $A \in \min(X,\mathfrak{I})$*, then for every* $x \in S(A)$*,* $A = A_x$*.*

Proof. Let $x \in S(A)$. Then by Lemma 6.3, it follows that $A \leq B$ for all $B \in \mathfrak{I}$ with $x \in S(B)$ and hence $A \leq \bigwedge\{B \in \mathfrak{I} : x \in S(B)\} = A_x$. On the other hand, since $A \in \mathfrak{I}$ with $x \in S(A)$, then $A_x \leq A$. □

Corollary 6.5. *Let* $(X,\mathfrak{I})$ *be an I-topological space. If* $A \in \min(X,\mathfrak{I})$*, then* A *is a lower fuzzy set in* $(X,\mathfrak{I})$*.*

The following is an example of a non zero lower fuzzy set in an I-topological space that is not a minimal fuzzy set.

Example 6.6. Let $X = \mathbb{R}$ with the I-topology $\mathfrak{I} = \{\mathcal{X}_U : U \in \tau_u\}$, then $A_0 \leq \mathcal{X}_{(-1/n,1/n)}$ for each $n \in \mathbb{N}$ and so $A_0 \leq \bigwedge\{\mathcal{X}_{(-1/n,1/n)} : n \in \mathbb{N}\} = \mathcal{X}_{\{0\}}$. On the other hand, $A_0(0) = \inf\{A(0) : A \in \mathfrak{I}$ and $0 \in S(A)\} = 1$. Therefore, $A_0 = \mathcal{X}_{\{0\}} \notin \min(X,\mathfrak{I})$.

In ordinary topological spaces, we have the following result.

Proposition 6.7 (see [7]). *Let* (X,τ) *be a topological space. If* $U \in \min(X,\tau)$*, then every set* $V \subseteq U$ *is preopen.*

The following example shows that the exact fuzzy version of Proposition 6.7 is invalid in general.

Example 6.8. Let X be any nonempty set together with the I-topology $\mathfrak{I} = \{0_X, 1_X, (0.4)_X, (0.7)_X\}$, then $(0.4)_X \in \min(X,\mathfrak{I})$, $(0.2)_X \leq (0.4)_X$, but $\text{int}(\text{Cl}((0.2)_X)) = \text{int}((0.3)_X) = 0_X$, that is, $(0.2)_X$ is not a fuzzy preopen set.

Theorem 6.9. *Let* $(X,\mathfrak{I})$ *be an I-topological space and* $A \in \min(X,\mathfrak{I})$ *with* $A(x_\circ) > 0.5$ *for some* $x_\circ \in X$*, then for every fuzzy set* B *with* $S(B) \subseteq S(A)$ *and* $B(x_\circ) \geq 0.5$ *one has* $(\text{Cl}(B))(x) = 1$ *for all* $x \in S(B)$*.*

Proof. Suppose to the contrary that there exists a fuzzy set B such that $S(B) \subseteq S(A)$ and $B(x_\circ) \geq 0.5$ with $\text{Cl}(B)(x_1) < 1$ for some $x_1 \in S(B)$, since $S(B) \subseteq S(A)$, $A(x_1) > 0$. Thus, $(A \wedge (\text{Cl}(B))^c)(x_1) = \min\{A(x_1), 1 - (\text{Cl}(B))(x_1)\} > 0$ and hence $A \wedge (\text{Cl}(B))^c = A$. Therefore, $A \leq (\text{Cl}(B))^c$ and so $A(x_\circ) + (\text{Cl}(B))(x_\circ) \leq 1$, but $A(x_\circ) + (\text{Cl}(B))(x_\circ) > 0.5 + 0.5 = 1$, which is a contradiction. □

Corollary 6.10. *Let* $(X,\mathfrak{I})$ *be an I-topological space and* $A \in \min(X,\mathfrak{I})$ *with* $A(x_\circ) > 0.5$ *for some* $x_\circ \in X$*, then for every* $B \in I^X$ *with* $S(B) = S(A)$ *and* $B(x_\circ) \geq 0.5$*, one has* $A \leq \text{Cl}(B)$*.*

Proof. Let $B \in I^X$ with $S(B) = S(A)$ and $B(x_\circ) \geq 0.5$ and let $x \in X$. If $x \in S(B)$, then by Theorem 6.9, $(\mathrm{Cl}(B))(x) = 1$ and so $A(x) \leq (\mathrm{Cl}(B))(x)$. If $x \notin S(B)$, then $x \notin S(A)$ and so $0 = A(x) \leq (\mathrm{Cl}(B))(x)$. □

The following corollary is a partial fuzzy version of Proposition 6.7.

Corollary 6.11. *Let $(X, \mathfrak{I})$ be an I-topological space and $A \in \min(X, \mathfrak{I})$ with $A(x_\circ) > 0.5$ for some $x_\circ \in X$, then for every $B \in I^X$ such that $B(x_\circ) \geq 0.5$, $B \leq A$, and $S(B) = S(A)$, B is a fuzzy preopen set.*

Proof. Let $B \in I^X$ with $B(x_\circ) \geq 0.5$, $B \leq A$, and $S(B) = S(A)$, then by Corollary 6.10, $A \leq \mathrm{Cl}(B)$ and so $B \leq A \leq \mathrm{int}(\mathrm{Cl}(B))$. □

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References

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, pp. 338–353, 1965.
[2] C. K. Wong, "Fuzzy points and local properties of fuzzy topology," *Journal of Mathematical Analysis and Applications*, vol. 46, pp. 316–328, 1974.
[3] M. D. Weiss, "Fixed points, separation, and induced topologies for fuzzy sets," *Journal of Mathematical Analysis and Applications*, vol. 50, pp. 142–150, 1975.
[4] N. Levine, "Semi-open sets and semi-continuity in topological spaces," *The American Mathematical Monthly*, vol. 70, pp. 36–41, 1963.
[5] A. S. Mashhour, M. E. Abd El-Monsef, and S. N. El-Deeb, "On pre-continuous and weak pre-continuous mappings," *Proceedings of the Mathematical and Physical Society of Egypt*, vol. 53, pp. 47–53, 1982.
[6] A. Fora and A. Al-Bsoul, "Finite homogeneous spaces," *The Rocky Mountain Journal of Mathematics*, vol. 27, no. 4, pp. 1089–1094, 1997.
[7] F. Nakaoka and O. Nobuyuki, "Some applications of minimal open sets," *International Journal of Mathematics and Mathematical Sciences*, vol. 27, no. 8, pp. 471–476, 2001.
[8] S. Al Ghour, "Minimality and prehomogeneity," *Acta Mathematica Universitatis Comenianae. New Series*, vol. 72, no. 2, pp. 237–244, 2003.
[9] S. Al Ghour, K. Zoubi, and A. Fora, "Semihomogeneous topological spaces," *Publicationes Mathematicae Debrecen*, vol. 67, no. 1-2, pp. 131–143, 2005.
[10] A. Fora and S. Al Ghour, "Homogeneity in fuzzy spaces," *Questions and Answers in General Topology*, vol. 19, no. 2, pp. 159–164, 2001.
[11] S. Al Ghour and A. Fora, "Minimality and homogeneity in fuzzy spaces," *Journal of Fuzzy Mathematics*, vol. 12, no. 3, pp. 725–737, 2004.
[12] S. Al Ghour, "Some generalizations of minimal fuzzy open sets," *Acta Mathematica Universitatis Comenianae. New Series*, vol. 75, no. 1, pp. 107–117, 2006.
[13] F. Nakaoka and O. Nobuyuki, "Some properties of maximal open sets," *International Journal of Mathematics and Mathematical Sciences*, no. 21, pp. 1331–1340, 2003.
[14] F. Nakaoka and O. Nobuyuki, "Minimal closed set and maximal closed sets," *International Journal of Mathematics and Mathematical Sciences*, vol. 2006, Article ID 18647, 8 pages, 2006.
[15] A. A. Fora, "Separation axioms for fuzzy spaces," *Fuzzy Sets and Systems*, vol. 33, no. 1, pp. 59–75, 1989.