ON f-DERIVATIONS OF BCI-ALGEBRAS

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The notion of left-right (resp., right-left) f-derivation of a BCI-algebra is introduced, and some related properties are investigated. Using the idea of regular f-derivation, we give characterizations of a p-semisimple BCI-algerba.

1. Introduction and preliminaries

In the theory of rings and near-rings, the properties of derivations are an important topic to study, see [2, 3, 7, 10]. In [6], Jun and Xin applied the notions in rings and near-rings theory to BCI-algebras, and obtained some related properties. In this paper, the notion of left-right (resp., right-left) f-derivation of a BCI-algebra is introduced, and some related properties are investigated. Using the idea of regular f-derivation, we give characterizations of a p-semisimple BCI-algebra.

By a BCI-algebra we mean an algebra $(X;*,0)$ of type (2,0) satisfying the following conditions:

(I) $((x*y)*(x*z))*(z*y)=0$;
(II) $(x*(x*y))*y=0$;
(III) $x*x=0$;
(IV) $x*y=0$ and $y*x=0$ imply that $x=y$;

for all $x,y,z\in X$.

In any BCI-algebra X, one can define a partial order "$\leq$" by putting $x\leq y$ if and only if $x*y=0$.

A subset S of a BCI-algebra X is called *subalgebra* of X if $x*y\in S$ for all $x,y\in S$. A subset I of a BCI-algebra X is called an *ideal* of X if it satisfies (i) $0\in I$; (ii) $x*y\in I$ and $y\in I$ imply that $x\in I$ for all $x,y\in X$.

A mapping f of a BCI-algebra X into itself is called an *endomorphism* of X if $f(x*y)=f(x)*f(y)$ for all $x,y\in X$. Note that $f(0)=0$. Especially, f is *monic* if for any $x,y\in X$, $f(x)=f(y)$ implies that $x=y$.

A BCI-algebra X has the following properties:

(1) $x*0=x$;
(2) $(x*y)*z=(x*z)*y$;

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(3) $x \leq y$ implies that $x * z \leq y * z$ and $z * y \leq z * x$;
(4) $x * (x * (x * y)) = x * y$;
(5) $(x * z) * (y * z) \leq x * y$;
(6) $0 * (x * y) = (0 * x) * (0 * y)$;
(7) $x * 0 = 0$ implies that $x = 0$.

For a BCI-algebra X, denote by X_+ (resp., $G(X)$) the BCK-part (resp., the BCI-G part) of X, that is, $X_+ = \{x \in X \mid 0 \leq x\}$ (resp., $G(X) = \{x \in X \mid 0 * x = x\}$). Note that $G(X) \cap X_+ = \{0\}$. If $X_+ = \{0\}$, then X is called a *p-semisimple BCI-algebra*.

In a p-semisimple BCI-algebra X, the following hold:
(8) $(x * z) * (y * z) = x * y$;
(9) $0 * (0 * x) = x$;
(10) $x * (0 * y) = y * (0 * x)$;
(11) $x * y = 0$ implies that $x = y$;
(12) $x * a = x * b$ implies that $a = b$;
(13) $a * x = b * x$ implies that $a = b$;
(14) $a * (a * x) = x$.

Let X be a p-semisimple BCI-algebra. We define addition "+" as $x + y = x * (0 * y)$ for all $x, y \in X$. Then $(X, +)$ is an abelian group with identity 0 and $x - y = x * y$. Conversely, let $(X, +)$ be an abelian group with identity 0 and let $x * y = x - y$. Then X is a p-semisimple BCI-algebra and $x + y = x * (0 * y)$ for all $x, y \in X$ (see [5]).

For a BCI-algebra X, we denote $x \wedge y = y * (y * x)$, in particular, $0 * (0 * x) = a_x$, and $L_p(X) = \{a \in X \mid x * a = 0 \Rightarrow x = a \text{ for any } x \in X\}$. We call the elements of $L_p(X)$ *the p-atoms* of X. For any $a \in X$, let $V(a) = \{x \in X \mid a * x = 0\}$, which is called *the branch of X with respect to a*. It follows that $x * y \in V(a * b)$ whenever $x \in V(a)$ and $y \in V(b)$ for all $x, y \in X$ and $a, b \in L_p(X)$. Note that $L_p(X) = \{x \in X \mid a_x = x\}$, which is *the p-semisimple part* of X, and X is a p-semisimple BCI-algebra if and only if $L_p(X) = X$ (see [6]). Note also that $a_x \in L_p(X)$, that is, $0 * (0 * a_x) = a_x$, which implies that $a_x * y \in L_p(X)$ for all $y \in X$. It is clear that $G(X) \subseteq L_p(X)$, $x * (x * a) = a$, and $a * x \in L_p(X)$ for all $a \in L_p(X)$ and $x \in X$. For more details, refer to [1, 8, 11].

Definition 1.1 [9]. A BCI-algebra X is said to be *commutative* if $x = x \wedge y$ whenever $x \leq y$ for all $x, y \in X$.

Definition 1.2 [4]. A BCI-algebra X is said to be *branchwise commutative* if $x \wedge y = y \wedge x$ for all $x, y \in V(a)$ and all $a \in L_p(X)$.

LEMMA 1.3 [6]. *A BCI-algebra X is commutative if and only if it is branchwise commutative.*

Definition 1.4 [6]. Let X be a BCI-algebra. By a *left-right derivation (briefly, (l,r)-derivation)* of X, a self-map d of X satisfying the identity $d(x * y) = (d(x) * y) \wedge (x * d(y))$ for all $x, y \in X$ is meant. If d satisfies the identity $d(x * y) = (x * d(y)) \wedge (d(x) * y)$ for all $x, y \in X$, then it is said that d is a *right-left derivation (briefly, (r,l)-derivation)* of X. Moreover, if d is both an (r,l)- and an (l,r)-derivation, it is said that d is a *derivation*.

2. f-derivations

In what follows, let f be an endomorphism of X unless otherwise specified.

Definition 2.1. Let X be a BCI-algebra. By a *left-right f-derivation (briefly, (l,r)-f-derivation)* of X, a self-map d_f of X satisfying the identity $d_f(x*y) = (d_f(x)*f(y)) \wedge (f(x)*d_f(y))$ for all $x, y \in X$ is meant, where f is an endomorphism of X. If d_f satisfies the identity $d_f(x*y) = (f(x)*d_f(y)) \wedge (d_f(x)*f(y))$ for all $x, y \in X$, then it is said that d_f is a *right-left f-derivation (briefly, (r,l)-f-derivation)* of X. Moreover, if d_f is both an (r,l)- and an (l,r)-f-derivation, it is said that d_f is an *f-derivation*.

Example 2.2. Let $X = \{0,1,2,3,4,5\}$ be a BCI-algebra with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	2	2	2	2
1	1	0	2	2	2	2
2	2	2	0	0	0	0
3	3	2	1	0	0	0
4	4	2	1	1	0	1
5	5	2	1	1	1	0

Define a map $d_f : X \to X$ by

$$d_f(x) = \begin{cases} 2 & \text{if } x = 0,1, \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

and define an endomorphism f of X by

$$f(x) = \begin{cases} 0 & \text{if } x = 0,1, \\ 2 & \text{otherwise.} \end{cases} \tag{2.2}$$

Then it is easily checked that d_f is both derivation and f-derivation of X.

Example 2.3. Let X be a BCI-algebra as in Example 2.2. Define a map $d_f : X \to X$ by

$$d_f(x) = \begin{cases} 2 & \text{if } x = 0,1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Then it is easily checked that d_f is a derivation of X.

Define an endomorphism f of X by

$$f(x) = 0, \quad \forall x \in X. \tag{2.4}$$

Then d_f is not an f-derivation of X since

$$d_f(2*3) = d_f(0) = 2, \tag{2.5}$$

but

$$(d_f(2)*f(3)) \wedge (f(2)*d_f(3)) = (0*0) \wedge (0*0) = 0 \wedge 0 = 0, \tag{2.6}$$

and thus $d_f(2*3) \neq (d_f(2)*f(3)) \wedge (f(2)*d_f(3))$.

Remark 2.4. From Example 2.3, we know that there is a derivation of X which is not an f-derivation of X.

Example 2.5. Let $X = \{0,1,2,3,4,5\}$ be a BCI-algebra with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	3	2	3	2
1	1	0	5	4	3	2
2	2	2	0	3	0	3
3	3	3	2	0	2	0
4	4	2	1	5	0	3
5	5	3	4	1	2	0

Define a map $d_f : X \to X$ by

$$d_f(x) = \begin{cases} 0 & \text{if } x = 0,1, \\ 2 & \text{if } x = 2,4, \\ 3 & \text{if } x = 3,5, \end{cases} \tag{2.7}$$

and define an endomorphism f of X by

$$f(x) = \begin{cases} 0 & \text{if } x = 0,1, \\ 2 & \text{if } x = 2,4, \\ 3 & \text{if } x = 3,5. \end{cases} \tag{2.8}$$

Then it is easily checked that d_f is both derivation and f-derivation of X.

Example 2.6. Let X be a BCI-algebra as in Example 2.5. Define a map $d_f : X \to X$ by

$$d_f(x) = \begin{cases} 0 & \text{if } x = 0,1, \\ 2 & \text{if } x = 2,4, \\ 3 & \text{if } x = 3,5. \end{cases} \tag{2.9}$$

Then it is easily checked that d_f is a derivation of X.

Define an endomorphism f of X by

$$f(0) = 0, \quad f(1) = 1, \quad f(2) = 3, \quad f(3) = 2, \quad f(4) = 5, \quad f(5) = 4. \tag{2.10}$$

Then d_f is not an f-derivation of X since

$$d_f(2 * 3) = d_f(3) = 3, \tag{2.11}$$

but

$$(d_f(2) * f(3)) \wedge (f(2) * d_f(3)) = (2 * 2) \wedge (3 * 3) = 0 \wedge 0 = 0, \tag{2.12}$$

and thus $d_f(2 * 3) \neq (d_f(2) * f(3)) \wedge (f(2) * d_f(3))$.

Example 2.7. Let X be a BCI-algebra as in Example 2.5. Define a map $d_f : X \to X$ by

$$d_f(0) = 0, \quad d_f(1) = 1, \quad d_f(2) = 3, \quad d_f(3) = 2, \quad d_f(4) = 5, \quad d_f(5) = 4. \tag{2.13}$$

Then d_f is not a derivation of X since

$$d_f(2 * 3) = d_f(3) = 2, \tag{2.14}$$

but

$$(d_f(2) * 3) \wedge (2 * d_f(3)) = (3 * 3) \wedge (2 * 2) = 0 \wedge 0 = 0, \tag{2.15}$$

and thus $d_f(2 * 3) \neq (d_f(2) * 3) \wedge (2 * d_f(3))$.

Define an endomorphism f of X by

$$f(0) = 0, \quad f(1) = 1, \quad f(2) = 3, \quad f(3) = 2, \quad f(4) = 5, \quad f(5) = 4. \tag{2.16}$$

Then it is easily checked that d_f is an f-derivation of X.

Remark 2.8. From Example 2.7, we know that there is an f-derivation of X which is not a derivation of X.

For convenience, we denote $f_x = 0 * (0 * f(x))$ for all $x \in X$. Note that $f_x \in L_p(X)$.

Theorem 2.9. *Let d_f be a self-map of a BCI-algebra X defined by $d_f(x) = f_x$ for all $x \in X$. Then d_f is an (l,r)-f-derivation of X. Moreover, if X is commutative, then d_f is an (r,l)-f-derivation of X.*

Proof. Let $x, y \in X$.

Since

$$\begin{aligned} 0 * (0 * (f_x * f(y))) &= 0 * (0 * ((0 * (0 * f(x))) * f(y))) \\ &= 0 * (0 * ((0 * f(y)) * (0 * f(x)))) \\ &= 0 * (0 * (0 * f(y * x))) = 0 * f(y * x) \\ &= 0 * (f(y) * f(x)) = (0 * f(y)) * (0 * f(x)) \\ &= (0 * (0 * f(x))) * f(y) = f_x * f(y), \end{aligned} \tag{2.17}$$

we have $f_x * f(y) \in L_P(X)$, and thus

$$f_x * f(y) = (f(x) * f_y) * ((f(x) * f_y) * (f_x * f(y))). \tag{2.18}$$

It follows that

$$\begin{aligned} d_f(x * y) = f_{x*y} &= 0 * (0 * f(x * y)) = 0 * (0 * (f(x) * f(y))) \\ &= (0 * (0 * f(x))) * (0 * (0 * f(y))) = f_x * f_y \\ &= (0 * (0 * f_x)) * (0 * (0 * f(y))) = 0 * (0 * (f_x * f(y))) \\ &= f_x * f(y) = (f(x) * f_y) * ((f(x) * f_y) * (f_x * f(y))) \\ &= (f_x * f(y)) \wedge (f(x) * f_y) = (d_f(x) * f(y)) \wedge (f(x) * d_f(y)), \end{aligned} \tag{2.19}$$

and so d_f is an (l,r)-f-derivation of X. Now, assume that X is commutative. Using Lemma 1.3, it is sufficient to show that $d_f(x) * f(y)$ and $f(x) * d_f(y)$ belong to the same branch for all $x, y \in X$, we have

$$\begin{aligned} d_f(x) * f(y) &= f_x * f(y) = 0 * (0 * (f_x * f(y))) \\ &= (0 * (0 * f_x)) * (0 * (0 * f(y))) \\ &= f_x * f_y \in V(f_x * f_y), \end{aligned} \tag{2.20}$$

and so $f_x * f_y = (0 * (0 * f(x))) * (0 * (0 * f_y)) = 0 * (0 * (f(x) * f_y)) = 0 * (0 * (f(x) * d_f(y))) \leq f(x) * d_f(y)$, which implies that $f(x) * d_f(y) \in V(f_x * f_y)$. Hence, $d_f(x) * f(y)$ and $f(x) * d_f(y)$ belong to the same branch, and so

$$\begin{aligned} d_f(x * y) &= (d_f(x) * f(y)) \wedge (f(x) * d_f(y)) \\ &= (f(x) * d_f(y)) \wedge (d_f(x) * f(y)). \end{aligned} \tag{2.21}$$

This completes the proof. □

Proposition 2.10. *Let d_f be a self-map of a BCI-algebra X. Then the following hold.*

(i) *If d_f is an (l,r)-f-derivation of X, then $d_f(x) = d_f(x) \wedge f(x)$ for all $x \in X$.*

(ii) *If d_f is an (r,l)-f-derivation of X, then $d_f(x) = f(x) \wedge d_f(x)$ for all $x \in X$ if and only if $d_f(0) = 0$.*

Proof. (i) Let d_f be an (l,r)-f-derivation of X. Then,

$$\begin{aligned} d_f(x) = d_f(x * 0) &= (d_f(x) * f(0)) \wedge (f(x) * d_f(0)) \\ &= (d_f(x) * 0) \wedge (f(x) * d_f(0)) = d_f(x) \wedge (f(x) * d_f(0)) \\ &= (f(x) * d_f(0)) * ((f(x) * d_f(0)) * d_f(x)) \\ &= (f(x) * d_f(0)) * ((f(x) * d_f(x)) * d_f(0)) \\ &\leq f(x) * (f(x) * d_f(x)) = d_f(x) \wedge f(x). \end{aligned} \tag{2.22}$$

But $d_f(x) \wedge f(x) \leq d_f(x)$ is trivial and so (i) holds.

(ii) Let d_f be an (r,l)-f-derivation of X. If $d_f(x) = f(x) \wedge d_f(x)$ for all $x \in X$, then for $x = 0$, $d_f(0) = f(0) \wedge d_f(0) = 0 \wedge d_f(0) = d_f(0) * (d_f(0) * 0) = 0$.

Conversely, if $d_f(0) = 0$, then $d_f(x) = d_f(x * 0) = (f(x) * d_f(0)) \wedge (d_f(x) * f(0)) = (f(x) * 0) \wedge (d_f(x) * 0) = f(x) \wedge d_f(x)$, ending the proof. □

Proposition 2.11. *Let d_f be an (l,r)-f-derivation of a BCI-algebra X. Then,*

(i) *$d_f(0) \in L_p(X)$, that is, $d_f(0) = 0 * (0 * d_f(0))$;*

(ii) *$d_f(a) = d_f(0) * (0 * f(a)) = d_f(0) + f(a)$ for all $a \in L_p(X)$;*

(iii) *$d_f(a) \in L_p(X)$ for all $a \in L_p(X)$;*

(iv) *$d_f(a + b) = d_f(a) + d_f(b) - d_f(0)$ for all $a, b \in L_p(X)$.*

Proof. (i) The proof follows from Proposition 2.10(i).

(ii) Let $a \in L_p(X)$, then $a = 0 * (0 * a)$, and so $f(a) = 0 * (0 * f(a))$, that is, $f(a) \in L_p(X)$. Hence

$$\begin{aligned} d_f(a) &= d_f(0 * (0 * a)) \\ &= (d_f(0) * f(0 * a)) \wedge (f(0) * d_f(0 * a)) \\ &= (d_f(0) * f(0 * a)) \wedge (0 * d_f(0 * a)) \\ &= (0 * d_f(0 * a)) * ((0 * d_f(0 * a)) * (d_f(0) * f(0 * a))) \\ &= (0 * d_f(0 * a)) * ((0 * (d_f(0) * f(0 * a))) * d_f(0 * a)) \\ &= 0 * (0 * (d_f(0) * (f(0) * f(a)))) \\ &= 0 * (0 * (d_f(0) * (0 * f(a)))) \\ &= d_f(0) * (0 * f(a)) = d_f(0) + f(a). \end{aligned} \tag{2.23}$$

(iii) The proof follows directly from (ii).

(iv) Let $a, b \in L_p(X)$. Note that $a + b \in L_p(X)$, so from (ii), we note that

$$\begin{aligned} d_f(a+b) &= d_f(0) + f(a+b) \\ &= d_f(0) + f(a) + d_f(0) + f(b) - d_f(0) = d_f(a) + d_f(b) - d_f(0). \end{aligned} \tag{2.24}$$

□

PROPOSITION 2.12. *Let d_f be a (r,l)-f-derivation of a BCI-algebra X. Then,*

(i) *$d_f(a) \in G(X)$ for all $a \in G(X)$;*

(ii) *$d_f(a) \in L_p(X)$ for all $a \in G(X)$;*

(iii) *$d_f(a) = f(a) * d_f(0) = f(a) + d_f(0)$ for all $a \in L_p(X)$;*

(iv) *$d_f(a+b) = d_f(a) + d_f(b) - d_f(0)$ for all $a, b \in L_p(X)$.*

Proof. (i) For any $a \in G(X)$, we have $d_f(a) = d_f(0 * a) = (f(0) * d_f(a)) \wedge (d_f(0) * f(a)) = (d_f(0) * f(a)) * ((d_f(0) * f(a)) * (0 * d_f(a))) = 0 * d_f(a)$, and so $d_f(a) \in G(X)$.

(ii) For any $a \in L_p(X)$, we get

$$\begin{aligned} d_f(a) &= d_f(0 * (0 * a)) = (0 * d_f(0 * a)) \wedge (d_f(0) * f(0 * a)) \\ &= (d_f(0) * f(0 * a)) * ((d_f(0) * f(0 * a)) * (0 * d_f(0 * a))) \\ &= 0 * d_f(0 * a) \in L_p(X). \end{aligned} \tag{2.25}$$

(iii) For any $a \in L_p(X)$, we get

$$\begin{aligned} d_f(a) &= d_f(a * 0) = (f(a) * d_f(0)) \wedge (d_f(a) * f(0)) \\ &= d_f(a) * (d_f(a) * (f(a) * d_f(0))) = f(a) * d_f(0) \\ &= f(a) * (0 * d_f(0)) = f(a) + d_f(0). \end{aligned} \tag{2.26}$$

(iv) The proof follows from (iii). This completes the proof. □

Using Proposition 2.12, we know there is an (l,r)-f-derivation which is not an (r,l)-f-derivation as shown in the following example.

Example 2.13. Let $\mathbb{Z}$ be the set of all integers and "$-$" the minus operation on $\mathbb{Z}$. Then $(\mathbb{Z},-,0)$ is a BCI-algebra. Let $d_f : X \to X$ be defined by $d_f(x) = f(x) - 1$ for all $x \in \mathbb{Z}$. Then,

$$\begin{aligned}
(d_f(x) - f(y)) \wedge (f(x) - d_f(y)) &= (f(x) - 1 - f(y)) \wedge (f(x) - (f(y) - 1)) \\
&= (f(x-y) - 1) \wedge (f(x-y) + 1) \\
&= (f(x-y) + 1) - 2 = f(x-y) - 1 \\
&= d_f(x-y).
\end{aligned} \tag{2.27}$$

Hence, d_f is an (l,r)-f-derivation of X. But $d_f(0) = f(0) - 1 = -1 \neq 1 = f(0) - d_f(0) = 0 - d_f(0)$, that is, $d_f(0) \notin G(X)$. Therefore, d_f is not an (r,l)-f-derivation of X by Proposition 2.12(i).

3. Regular f-derivations

Definition 3.1. An f-derivation d_f of a BCI-algebra X is said to be *regular* if $d_f(0) = 0$.

Remark 3.2. We know that the f-derivations d_f in Examples 2.5 and 2.7 are regular.

Proposition 3.3. *Let X be a commutative BCI-algebra and let d_f be a regular (r,l)-f-derivation of X. Then the following hold.*

(i) *Both $f(x)$ and $d_f(x)$ belong to the same branch for all $x \in X$.*

(ii) *d_f is an (l,r)-f-derivation of X.*

Proof. (i) Let $x \in X$. Then,

$$\begin{aligned}
0 = d_f(0) &= d_f(a_x * x) \\
&= (f(a_x) * d_f(x)) \wedge (d(a_x) * f(x)) \\
&= (d(a_x) * f(x)) * ((d(a_x) * f(x)) * (f(a_x) * d_f(x))) \\
&= (d(a_x) * f(x)) * ((d(a_x) * f(x)) * (f_x * d_f(x))) \\
&= f_x * d_f(x) \quad \text{since } f_x * d_f(x) \in L_P(X),
\end{aligned} \tag{3.1}$$

and so $f_x \leq d_f(x)$. This shows that $d_f(x) \in V(f_x)$. Clearly, $f(x) \in V(f_x)$.

(ii) By (i), we have $f(x) * d_f(y) \in V(f_x * f_y)$ and $d_f(x) * f(y) \in V(f_x * f_y)$. Thus $d_f(x * y) = (f(x) * d_f(y)) \wedge (d_f(x) * f(y)) = (d_f(x) * f(y)) \wedge (f(x) * d_f(y))$, which implies that d_f is an (l,r)-f-derivation of X. □

Remark 3.4. The f-derivations d_f in Examples 2.5 and 2.7 are regular f-derivations but we know that the (l,r)-f-derivation d_f in Example 2.2 is not regular. In the following, we give some properties of regular f-derivations.

Definition 3.5. Let X be a BCI-algebra. Then define $\ker d_f = \{x \in X \mid d_f(x) = 0 \text{ for all } f\text{-derivations } d_f\}$.

Proposition 3.6. *Let d_f be an f-derivation of a BCI-algebra X. Then the following hold:*

(i) $d_f(x) \le f(x)$ *for all* $x \in X$;

(ii) $d_f(x) * f(y) \le f(x) * d_f(y)$ *for all* $x, y \in X$;

(iii) $d_f(x * y) = d_f(x) * f(y) \le d_f(x) * d_f(y)$ *for all* $x, y \in X$;

(iv) $\ker d_f$ *is a subalgebra of* X. *Especially, if* f *is monic, then* $\ker d_f \subseteq X_+$.

Proof. (i) The proof follows by Proposition 2.10(ii).

(ii) Since $d_f(x) \le f(x)$ for all $x \in X$, then $d_f(x) * f(y) \le f(x) * f(y) \le f(x) * d_f(y)$.

(iii) For any $x, y \in X$, we have

$$\begin{aligned} d_f(x * y) &= \big(f(x) * d_f(y)\big) \wedge \big(d_f(x) * f(y)\big) \\ &= \big(d_f(x) * f(y)\big) * \big(\big(d_f(x) * f(y)\big) * \big(f(x) * d_f(y)\big)\big) \\ &= \big(d_f(x) * f(y)\big) * 0 = d_f(x) * f(y) \le d_f(x) * d_f(y), \end{aligned} \tag{3.2}$$

which proves (iii).

(iv) Let $x, y \in \ker d_f$, then $d_f(x) = 0 = d_f(y)$, and so $d_f(x * y) \le d_f(x) * d_f(y) = 0 * 0 = 0$ by (iii), and thus $d_f(x * y) = 0$, that is, $x * y \in \ker d_f$. Hence, $\ker d_f$ is a subalgebra of X. Especially, if f is monic, and letting $x \in \ker d_f$, then $0 = d_f(x) \le f(x)$ by (i), and so $f(x) \in X_+$, that is, $0 * f(x) = 0$, and thus $f(0 * x) = f(x)$, which implies that $0 * x = x$, and so $x \in X_+$, that is, $\ker d_f \subseteq X_+$. □

Theorem 3.7. *Let f be monic of a commutative BCI-algebra X. Then X is p-semisimple if and only if* $\ker d_f = \{0\}$ *for every regular f-derivation d_f of X.*

Proof. Assume that X is p-semisimple BCI-algebra and let d_f be a regular f-derivation of X. Then $X_+ = \{0\}$, and so $\ker d_f = \{0\}$ by using Proposition 3.6(iv). Conversely, let $\ker d_f = \{0\}$ for every regular f-derivation d_f of X. Define a self-map d_f of X by $d_f^*(x) = f_x$ for all $x \in X$. Using Theorem 2.9, d_f^* is an f-derivation of X. Clearly, $d_f^*(0) = f_0 = 0 * (0 * f(0)) = 0$, and so d_f^* is a regular f-derivation of X. It follows from the hypothesis that $\ker d_f^* = \{0\}$. In addition, $d_f^*(x) = f_x = 0 * (0 * f(x)) = f(0 * (0 * x)) = f(0) = 0$ for all $x \in X_+$, and thus $x \in \ker d_f^*$, which shows that $X_+ \subseteq \ker d_f^*$. Hence, by Proposition 3.6(iv), $X_+ = \ker d_f^* = \{0\}$. Therefore X is p-semisimple. □

Definition 3.8. An ideal A of a BCI-algebra X is said to be an *f-ideal* if $f(A) \subseteq A$.

Definition 3.9. Let d_f be a self-map of a BCI-algebra X. An f-ideal A of X is said to be *d_f-invariant* if $d_f(A) \subseteq A$.

Theorem 3.10. *Let d_f be a regular (r,l)-f-derivation of a BCI-algebra X, then every f-ideal A of X is d_f-invariant.*

Proof. By Proposition 2.10(ii), we have $d_f(x) = f(x) \wedge d_f(x) \le f(x)$ for all $x \in X$. Let $y \in d_f(A)$. Then $y = d_f(x)$ for some $x \in A$. It follows that $y * f(x) = d_f(x) * f(x) = 0 \in A$. Since $x \in A$, then $f(x) \in f(A) \subseteq A$ as A is an f-ideal. It follows that $y \in A$ since A is an ideal of X. Hence $d_f(A) \subseteq A$, and thus A is d_f-invariant. □

Theorem 3.11. *Let d_f be an f-derivation of a BCI-algebra X. Then d_f is regular if and only if every f-ideal of X is d_f-invariant.*

Proof. Let d_f be a derivation of a BCI-algebra X and assume that every f-ideal of X is d_f-invariant. Then since the zero ideal $\{0\}$ is f-ideal and d_f-invariant, we have $d_f(\{0\}) \subseteq \{0\}$, which implies that $d_f(0) = 0$. Thus d_f is regular. Combining this and Theorem 3.10, we complete the proof. □

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