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Research Article

An Operational Matrix of Fractional Differentiation of the Second Kind of Chebyshev Polynomial for Solving Multiterm Variable Order Fractional Differential Equation

Jianping Liu, Xia Li, and Limeng Wu

Hebei Normal University of Science and Technology, Qinhuangdao, Hebei 066004, China

Correspondence should be addressed to Jianping Liu; liujianping0408@126.com

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The multiterm fractional differential equation has a wide application in engineering problems. Therefore, we propose a method to solve multiterm variable order fractional differential equation based on the second kind of Chebyshev Polynomial. The main idea of this method is that we derive a kind of operational matrix of variable order fractional derivative for the second kind of Chebyshev Polynomial. With the operational matrices, the equation is transformed into the products of several dependent matrices, which can also be viewed as an algebraic system by making use of the collocation points. By solving the algebraic system, the numerical solution of original equation is acquired. Numerical examples show that only a small number of the second kinds of Chebyshev Polynomials are needed to obtain a satisfactory result, which demonstrates the validity of this method.

1. Introduction

The concept of fractional order derivative goes back to the 17th century [1, 2]. It is only a few decades ago that it was realized that the arbitrary order derivative provides an excellent framework for modeling the real-world problems in a variety of disciplines from physics, chemistry, biology, and engineering, such as viscoelasticity and damping, diffusion and wave propagation, and chaos [3–6].

Orthogonal functions have received noticeable consideration for solving fractional differential equation (FDE). By using orthogonal functions, the FDE can be reduced to solve an algebraic system, and then original problems are simplified. Ahmadian et al. [7] proposed a computational method based on Jacobi Polynomials for solving fuzzy linear FDE on interval $[0, 1]$. Kazem et al. [8] constructed a general formulation for the fractional order Legendre functions. Yüzbaşı [9] gave the numerical solutions of fractional Riccati type differential equations by means of the Bernstein Polynomials. Kazem [10] constructed a general formulation for the Jacobi operational matrix for fractional integral equations.

Tau method and collocation method are widely used tools for the solution of FDE. Operational approach of the tau method was employed for solving fractional problems [11]. A numerical approach was provided for the FDE based on a spectral tau method [12]. An efficient method based on the shifted Chebyshev-tau idea was presented for solving the space fractional diffusion equations [13]. Tau method is very effective for constant coefficient nonlinear problems, but the method is not generally adopted for nonlinear FDE. In practice, since collocation method has the advantages of less computation and easy implementation, it is more widely applied for solving variable coefficient nonlinear problems. The collocation method was used for solving the nonlinear fractional integrodifferential equations [14]. The third kind of Chebyshev wavelets collocation method was introduced for solving the time fractional convection diffusion equations with variable coefficients [15].

From the literatures above, we conclude that many authors employed tau and collocation method for solving different kinds of FDE based on different kinds of orthogonal functions or their variants. However, for the aforementioned

FDE, the derivative order is a fixed constant, which does not change spatially and temporally; variable order multiterm FDE is not mentioned and solved. Therefore, our main motivation is to give a numerical technology for solving variable order linear and nonlinear multiterm FDE based on the second kind of Chebyshev Polynomial. With further development of science research, it is found that variable order fractional calculus can provide an effective mathematical framework for the complex dynamical problems. The modeling and application of variable order differential equation has been a front subject. In addition, the FDE is a special case of variable order ones, so it can also be solved by our proposed technology.

Variable order derivative is proposed by Samko and Ross [16] in 1993, and then Lorenzo and Hartley [17, 18] studied variable order calculus in theory more deeply. Coimbra and Diaz [19, 20] used variable order derivative to research nonlinear dynamics and control problems of viscoelasticity oscillator. Pedro et al. [21] researched diffusive-convective effects on the oscillatory flow past a sphere by variable order modeling. The development of numerical algorithms to solve variable order FDE is necessary.

Since the kernel of the variable order operators is very complex for having a variable exponent, it is difficult to gain the solution of variable order differential equation. Only a few authors studied numerical methods of variable order fractional differential equations. Coimbra [19] employed a consistent approximation with first-order accurate for solving variable order differential equations. Sun et al. [22] proposed a second-order Runge-Kutta method to numerically integrate the variable order differential equation. Lin et al. [23] studied the stability and the convergence of an explicit finite-difference approximation for the variable order fractional diffusion equation with a nonlinear source term. Chen et al. [24, 25] paid their attention to Bernstein Polynomials to solve variable order linear cable equation and variable order time fractional diffusion equation. A numerical method based on the Legendre Polynomials is presented for a class of variable order FDE [26]. Chen et al. [27] introduced the numerical solution for a class of nonlinear variable order FDE with Legendre wavelets.

To the best of our knowledge, it is not seen that operational matrix of variable order derivative based on the second kind of Chebyshev Polynomial is used to solve multiterm variable order FDE. In addition, for most literatures, they solved variable order FDE defined on the interval $[0,1]$. Accordingly, based on the second kind of Chebyshev Polynomial, we propose a new efficient technique for solving multiterm variable order FDE defined on the interval $[0,R]$.

The multiterm variable order FDE is given as follows:

$$\begin{aligned} &D^{\alpha(t)} f(t) \\ &\quad = F\left(t, f(t), D^{\beta_1(t)} f(t), D^{\beta_2(t)} f(t), \ldots, D^{\beta_k(t)} f(t)\right), \quad (1) \\ &\qquad 0 < t < R, \end{aligned}$$

where $D^{\alpha(t)}f(t)$ and $D^{\beta_i(t)}f(t)$ are fractional derivative in Caputo sense. When $\alpha(t)$ and $\beta_i(t)$, $i = 1,2,\ldots,k$ are all constants, (1) becomes (2); namely,

$$\begin{aligned} &D^{\alpha} f(t) \\ &\quad = F\left(t, f(t), D^{\beta_1} f(t), D^{\beta_2} f(t), \ldots, D^{\beta_k} f(t)\right), \quad (2) \\ &\qquad 0 < t < R. \end{aligned}$$

Thus, (2) is a special case of (1). Our proposed method can solve both (1) and (2). They often appear in oscillatory equations, such as vibration equation, fractional Van Der Pol equation, the Rayleigh equation with fractional damping, and fractional Riccati differential equation.

The basic idea of this method is that we derive differential operational matrices based on the second kind of Chebyshev Polynomial. With the operational matrices, the equation is transformed into the products of several dependent matrices, which can also be viewed as an algebraic system by making use of the collocation points. By solving the algebraic system, the numerical solution is acquired. Since the second kinds of Chebyshev Polynomials are orthogonal to each other, the operational matrices based on Chebyshev Polynomials greatly reduce the size of computational work while accurately providing the series solution. From some numerical examples, we can see that our results are in good agreement with the analytical solution, which demonstrates the validity of this method. Therefore, it has the potential to utilize wider applicability.

The paper is organized as follows. In Section 2, some necessary definitions and properties of the variable order fractional derivatives are introduced. The basic definitions of the second kind of Chebyshev Polynomial and function approximation are given in Sections 3 and 4, respectively. In Section 5, a kind of operational matrix of the second kind of Chebyshev Polynomial is derived, and then we applied the operational matrices to solve the equation as given at beginning. In Section 6, we present some numerical examples to demonstrate the efficiency of the method. We end the paper with a few concluding remarks in Section 7.

2. Basic Definition of Caputo Variable Order Fractional Derivatives

Definition 1. Caputo variable fractional derivative with order $\alpha(t)$ is defined by

$$\begin{aligned} D^{\alpha(t)} u(t) = {}& \frac{1}{\Gamma(1-\alpha(t))} \int_{0+}^{t} (t-\tau)^{-\alpha(t)} u'(\tau)\, d\tau \\ &+ \frac{(u(0+) - u(0-))\, t^{-\alpha(t)}}{\Gamma(1-\alpha(t))}. \end{aligned} \quad (3)$$

If we assume the starting time in a perfect situation, we can get Definition 2 as follows.

Definition 2. Consider

$$D^{\alpha(t)}u(t) = \frac{1}{\Gamma(1-\alpha(t))}\int_0^t (t-\tau)^{-\alpha(t)} u'(\tau)\, d\tau \quad (0 < \alpha(t) < 1). \tag{4}$$

By Definition 2, we can get the following formula [25]:

$$D_t^{\alpha(t)}(t^n) = \begin{cases} \dfrac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))} t^{n-\alpha(t)}, & n = 1, 2, \ldots, \\ 0, & n = 0. \end{cases} \tag{5}$$

3. Shifted Second Kind of Chebyshev Polynomial

The second kind of Chebyshev Polynomial defined on the interval $I = [-1, 1]$ is orthogonal based on the weight function $\omega(x) = \sqrt{1-x^2}$. They satisfy the following formulas:

$$\begin{gathered} U_0(x) = 1, \\ U_1(x) = 2x, \\ U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \\ n = 1, 2, \ldots, \\ \int_{-1}^{1} \sqrt{1-x^2}U_n(x)U_m(x)\, dx = \begin{cases} 0, & m \neq n, \\ \dfrac{\pi}{2}, & m = n. \end{cases} \end{gathered} \tag{6}$$

When $t \in [0, R]$, let $x = 2t/R - 1$; we can get shifted second kind of Chebyshev Polynomial $\widetilde{U}_n(t) = U_n(2t/R - 1)$, whose weight function is $\omega(t) = \sqrt{tR - t^2}$ with $t \in [0, R]$. They satisfy the following formulas:

$$\begin{gathered} \widetilde{U}_0(t) = 1, \\ \widetilde{U}_1(t) = 2\left(\frac{2t}{R} - 1\right) = \frac{4t}{R} - 2, \\ \widetilde{U}_{n+1}(t) = 2\left(\frac{2t}{R} - 1\right)\widetilde{U}_n(t) - \widetilde{U}_{n-1}(t), \\ n = 1, 2, 3, \ldots, \\ \int_0^R \sqrt{tR - t^2}\widetilde{U}_n(t)\widetilde{U}_m(t)\, dt = \begin{cases} 0, & m \neq n, \\ \dfrac{\pi}{8}R^2, & m = n. \end{cases} \end{gathered} \tag{7}$$

The shifted second kind of Chebyshev Polynomial $\widetilde{U}_n(t)$ can also be expressed as

$$\widetilde{U}_n(t) = \begin{cases} 1, & n = 0, \\ \displaystyle\sum_{k=0}^{[n/2]} (-1)^k \frac{(n-k)!}{k!(n-2k)!}\left(\frac{4t}{R} - 2\right)^{n-2k}, & n \geq 1, \end{cases} \tag{8}$$

where $[n/2]$ denotes the maximum integer which is no more than $n/2$.

Let

$$\Psi(t) = \left[\widetilde{U}_0(t), \widetilde{U}_1(t), \ldots, \widetilde{U}_n(t)\right]^T, \quad T(t) = \left[1, t, \ldots, t^n\right]^T; \tag{9}$$

then

$$\Psi(t) = AT(t). \tag{10}$$

Let

$$A = BC. \tag{11}$$

If n is an even number, then

$$B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & (-1)^0 \dfrac{(1-0)!}{0!(1-0)!}\left(\dfrac{4}{R}\right)^{1-0} & 0 & \cdots & 0 \\ (-1)^1 \dfrac{(2-1)!}{1!(2-2)!}\left(\dfrac{4}{R}\right)^{2-2} & 0 & (-1)^0 \dfrac{(2-0)!}{0!(2-0)!}\left(\dfrac{4}{R}\right)^{2-0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^{n/2} \dfrac{(n-n/2)!}{(n/2)!(n-2\cdot n/2)!}\left(\dfrac{4}{R}\right)^{n-2\cdot n/2} & \cdots & (-1)^{n/2-1} \dfrac{(n-n/2+1)!}{(n/2-1)![n-2(n/2-1)]!}\left(\dfrac{4}{R}\right)^{n-2(n/2-1)} & \cdots & (-1)^0 \dfrac{(n-0)!}{0!(n-0)!}\left(\dfrac{4}{R}\right)^{n-0} \end{bmatrix}. \tag{12}$$

If n is an odd number, then

$$B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & (-1)^0 \dfrac{(1-0)!}{0!\,(1-0)!}\left(\dfrac{4}{R}\right)^{1-0} & 0 & \cdots & 0 \\ (-1)^1 \dfrac{(2-1)!}{1!\,(2-2)!}\left(\dfrac{4}{R}\right)^{2-2} & 0 & (-1)^0 \dfrac{(2-0)!}{0!\,(2-0)!}\left(\dfrac{4}{R}\right)^{2-0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & (-1)^{(n-1)/2} \dfrac{(n-(n-1)/2)!}{((n-1)/2)!\,(n-2\cdot(n-1)/2)!}\left(\dfrac{4}{R}\right)^{n-2\cdot(n-1)/2} & 0 & \cdots & (-1)^0 \dfrac{(n-0)!}{0!\,(n-0)!}\left(\dfrac{4}{R}\right)^{n-0} \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \binom{1}{0}\left(-\dfrac{R}{2}\right)^{1-0} & \binom{1}{1}\left(-\dfrac{R}{2}\right)^{1-1} & 0 & \cdots & 0 \\ \binom{2}{0}\left(-\dfrac{R}{2}\right)^{2-0} & \binom{2}{1}\left(-\dfrac{R}{2}\right)^{2-1} & \binom{2}{2}\left(-\dfrac{R}{2}\right)^{2-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{n}{0}\left(-\dfrac{R}{2}\right)^{n-0} & \binom{n}{1}\left(-\dfrac{R}{2}\right)^{n-1} & \binom{n}{2}\left(-\dfrac{R}{2}\right)^{n-2} & \vdots & \binom{n}{n}\left(-\dfrac{R}{2}\right)^{n-n} \end{bmatrix}. \tag{13}$$

Therefore, we can easily gain

$$T_n(t) = A^{-1}\Psi(t). \tag{14}$$

4. Function Approximation

Theorem 3. *Assume a function $f(t) \in [0,R]$ be n times continuously differentiable. Let $u_n(t) = \sum_{i=0}^{n} \lambda_i \widetilde{U}_i(t) = \Lambda^T \Psi_n(t)$ be the best square approximation function of $f(t)$, where $\Lambda = [\lambda_0, \lambda_1, \ldots, \lambda_n]^T$ and $\Psi_n(t) = [\widetilde{U}_0(t), \widetilde{U}_1(t), \ldots, \widetilde{U}_n(t)]^T$; then*

$$\left\| f(t) - u_n(t) \right\| \le \frac{MS^{n+1}R}{(n+1)!}\sqrt{\frac{\pi}{8}}, \tag{15}$$

where $M = \max_{t\in[0,R]} f^{(n+1)}(t)$ and $S = \max\{R - t_0, t_0\}$.

Proof. We consider the Taylor Polynomial:

$$\begin{aligned} f(t) &= f(t_0) + f'(t_0)(t - t_0) + \cdots \\ &\quad + f^{(n)}(t_0)\frac{(t-t_0)^n}{n!} + f^{(n+1)}(\eta)\frac{(t-t_0)^{n+1}}{(n+1)!}, \\ &\quad t_0 \in [0,R], \end{aligned} \tag{16}$$

where η is between t and t_0.

Let

$$\begin{aligned} p_n(t) &= f(t_0) + f'(t_0)(t - t_0) + \cdots \\ &\quad + f^{(n)}(t_0)\frac{(t-t_0)^n}{n!}; \end{aligned} \tag{17}$$

then

$$\left| f(x) - p_n(x) \right| = \left| f^{(n+1)}(\eta)\frac{(t-t_0)^{n+1}}{(n+1)!} \right|. \tag{18}$$

Since $u_n(t) = \sum_{i=0}^{n} \lambda_i \widetilde{U}_i(t) = \Lambda^T \Psi_n(t)$ is the best square approximation function of $f(t)$, we can gain

$$\begin{aligned} \left\| f(t) - u_n(t) \right\|^2 &\le \left\| f(t) - p_n(t) \right\|^2 \\ &= \int_0^R \omega(t)\left[f(t) - p_n(t)\right]^2 dt \\ &= \int_0^R \omega(t)\left[f^{(n+1)}(\eta)\frac{(t-t_0)^{n+1}}{(n+1)!}\right]^2 dt \\ &\le \frac{M^2}{[(n+1)!]^2}\int_0^R (t-t_0)^{2n+2}\,\omega(t)\,dt \\ &= \frac{M^2}{[(n+1)!]^2}\int_0^R (t-t_0)^{2n+2}\sqrt{tR - t^2}\,dt. \end{aligned} \tag{19}$$

Let $S = \max\{R - t_0, t_0\}$; therefore

$$\begin{aligned} \left\| f(t) - u_n(t) \right\|^2 &\le \frac{M^2 S^{2n+2}}{[(n+1)!]^2}\int_0^R \sqrt{tR - t^2}\,dt \\ &= \frac{M^2 S^{2n+2}}{[(n+1)!]^2}\frac{\pi R^2}{8}. \end{aligned} \tag{20}$$

And by taking the square roots, Theorem 3 can be proved. □

5. Operational Matrices of $D^{\alpha(t)}\Psi_n(t)$ and $D^{\beta_i(t)}\Psi_n(t)$ $i = 1, 2, \ldots, k$ Based on Shifted Second Kind of Chebyshev Polynomial

Consider

$$D^{\alpha(t)}\Psi_n(t) = D^{\alpha(t)}\left[AT_n(t)\right] = AD^{\alpha(t)}\left[1 \;\; t \;\; \cdots \;\; t^n\right]^T. \tag{21}$$

According to (5), we can get

$$\begin{aligned} D^{\alpha(t)}\Psi_n(t) &= A\left[0 \quad \frac{\Gamma(2)}{\Gamma(2-\alpha(t))}t^{1-\alpha(t)} \quad \cdots \quad \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))}t^{n-\alpha(t)}\right]^T \\ &= A\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha(t))}t^{-\alpha(t)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))}t^{-\alpha(t)} \end{bmatrix}\begin{bmatrix} 1 \\ t \\ \vdots \\ t^n \end{bmatrix} \\ &= AMA^{-1}\Psi_n(t), \end{aligned} \tag{22}$$

where

$$M = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha(t))}t^{-\alpha(t)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))}t^{-\alpha(t)} \end{bmatrix}. \tag{23}$$

AMA^{-1} is called the operational matrix of $D^{\alpha(t)}\Psi_n(t)$. Therefore,

$$\begin{aligned} D^{\alpha(t)} f(t) &\approx D^{\alpha(t)}\left(\Lambda^T\Psi_n(t)\right) = \Lambda^T D^{\alpha(t)}\Psi_n(t) \\ &= \Lambda^T AMA^{-1}\Psi_n(t). \end{aligned} \tag{24}$$

Similarly, we can get

$$D^{\beta_i(t)}\Psi_n(t) = AN_iA^{-1}\Psi_n(t), \quad i = 1, 2, \ldots, k, \tag{25}$$

where

$$N_i = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\beta_i(t))}t^{-\beta_i(t)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-\beta_i(t))}t^{-\beta_i(t)} \end{bmatrix}. \tag{26}$$

AN_iA^{-1} is called the operational matrix of $D^{\beta_i(t)}\Psi_n(t)$. Thus,

$$\begin{aligned} D^{\beta_i(t)} f(t) &\approx D^{\beta_i(t)}\left(\Lambda^T\Psi_n(t)\right) = \Lambda^T D^{\beta_i(t)}\Psi_n(t) \\ &= \Lambda^T AN_iA^{-1}\Psi_n(t). \end{aligned} \tag{27}$$

The original equation (1) is transformed into the form as follows:

$$\begin{aligned} \Lambda^T AMA^{-1}\Psi_n(t) &= F\left[t, \Lambda^T\Psi_n(t), \Lambda^T AN_1A^{-1}\Psi_n(t),\right. \\ &\quad \left.\ldots, \Lambda^T AN_kA^{-1}\Psi_n(t)\right], \quad t \in [0, R]. \end{aligned} \tag{28}$$

In this paper, we use collocation method to solve the coefficient $\Lambda = [\lambda_0, \lambda_1, \ldots, \lambda_n]^T$. By taking the collocation points, (28) will become an algebraic system. We can gain the solution $\Lambda = [\lambda_0, \lambda_1, \ldots, \lambda_n]^T$ by Newton method. Finally, the numerical solution $u_n(t) = \Lambda^T\Psi_n(t)$ is gained.

6. Numerical Examples and Results Analysis

In this section, we verify the efficiency of proposed method to support the above theoretical discussion. For this purpose, we consider linear and nonlinear multiterm variable order FDE and corresponding multiterm FDE. For multiterm variable order FDE, we compare our approach with the analytical solution. For multiterm FDE, we compare our computational results with the analytical solution and solutions in [28] by using other methods. The results indicate that our method is a powerful tool for solving multiterm variable order FDE and multiterm FDE. Numerical examples show that only a small number of the second kinds of Chebyshev Polynomials are needed to obtain a satisfactory result. Furthermore, our method has higher precision than [28]. In this section, the notation

$$\begin{aligned} \varepsilon &= \max_{i=0,1,\ldots,n}\left|f(t_i) - u_n(t_i)\right|, \\ t_i &= R\frac{(2i+1)}{2(n+1)}, \quad i = 0, 1, \ldots, n, \end{aligned} \tag{29}$$

is used to show the accuracy of our proposed method.

Example 1. (a) Consider the linear FDE with variable order as follows:

$$\begin{aligned} &aD^{\alpha(t)} f(t) + b(t) D^{\beta_1(t)} f(t) + c(t) D^{\beta_2(t)} f(t) \\ &\quad + e(t) D^{\beta_3(t)} f(t) + k(t) f(t) = g(t), \\ &\qquad t \in [0, R], \\ &y(0) = 2, \\ &y'(0) = 0, \end{aligned} \tag{30}$$

where

$$\begin{aligned} f(t) = &-a\frac{t^{2-\alpha(t)}}{\Gamma(3-\alpha(t))} - b(t)\frac{t^{2-\beta_1(t)}}{\Gamma(3-\beta_1(t))} \\ &- c(t)\frac{t^{2-\beta_2(t)}}{\Gamma(3-\beta_2(t))} - e(t)\frac{t^{2-\beta_3(t)}}{\Gamma(3-\beta_3(t))} \\ &+ k(t)\left(2 - \frac{t^2}{2}\right). \end{aligned} \tag{31}$$

The analytical solution is $f(t) = 2 - t^2/2$. We use our proposed technology to solve it.

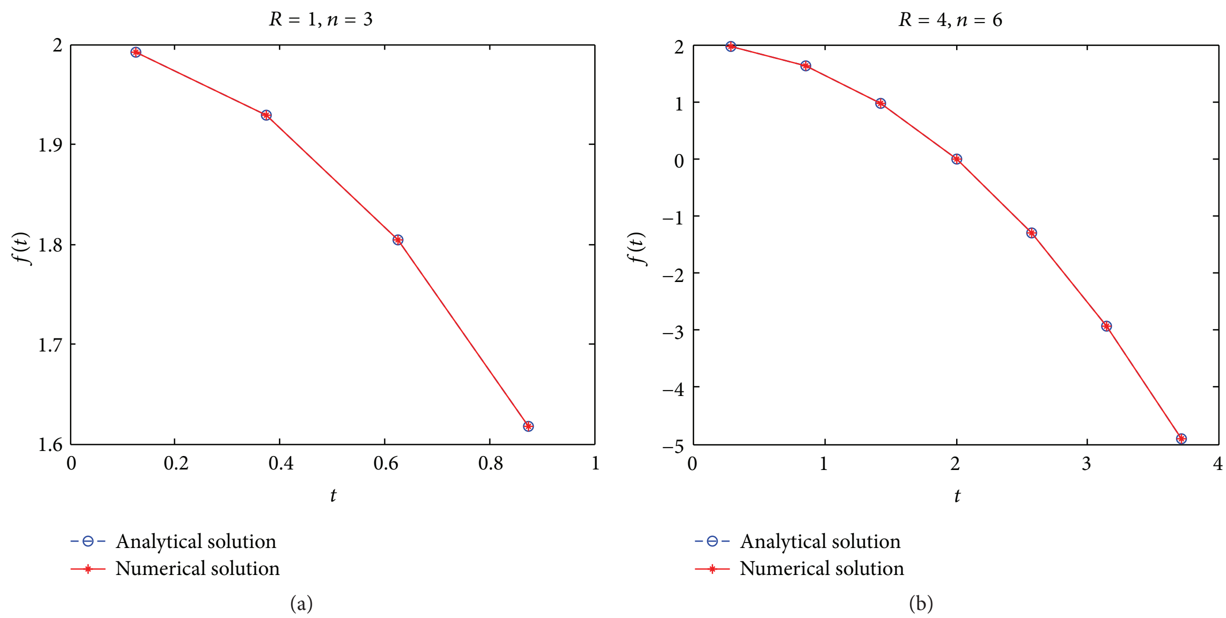

FIGURE 1: Analytical solution and numerical solution of Example 1(a) for different R.

Let $f(t) \approx u_n(t) = \Lambda^T \Psi_n(t)$, $\alpha(t) = 2t$, $\beta_1(t) = t/3$, $\beta_2(t) = t/4$, and $\beta_3(t) = t/5$; according to (28), we have

$$\begin{aligned} &a\Lambda^T AMA^{-1}\Psi_n(t) + b(t)\Lambda^T AN_1A^{-1}\Psi_n(t) \\ &\quad + c(t)\Lambda^T AN_2A^{-1}\Psi_n(t) \\ &\quad + e(t)\Lambda^T AN_3A^{-1}\Psi_n(t) + k(t)\Lambda^T\Psi_n(t) \\ &= g(t). \end{aligned} \tag{32}$$

Take the collocation points $t_i = R((2i+1)/2(n+1))$, $i = 0, 1, \ldots, n$, to process (32), and then get

$$\begin{aligned} &a\Lambda^T AMA^{-1}\Psi_n(t_i) + b(t_i)\Lambda^T AN_1A^{-1}\Psi_n(t_i) \\ &\quad + c(t_i)\Lambda^T AN_2A^{-1}\Psi_n(t_i) \\ &\quad + e(t_i)\Lambda^T AN_3A^{-1}\Psi_n(t_i) + k(t_i)\Lambda^T\Psi_n(t_i) \\ &= g(t_i), \quad i = 1, 2, \ldots, n. \end{aligned} \tag{33}$$

By solving the algebraic system (33), we can gain the vector $\Lambda = [\lambda_0, \lambda_1, \ldots, \lambda_n]^T$. Subsequently, numerical solution $u_n(t) = \Lambda^T\Psi_n(t)$ is obtained. Likely [28], we present numerical solution by our method for

$$\begin{aligned} a &= 1, \\ b(t) &= \sqrt{t}, \\ c(t) &= t^{1/3}, \\ e(t) &= t^{1/4}, \\ k(t) &= t^{1/5}. \end{aligned} \tag{34}$$

TABLE 1: Values of ε of Example 1(a) for different R.

R	$n = 3$	$n = 4$	$n = 5$	$n = 6$
$R = 1$	0	$2.2204e-16$	$2.2204e-16$	$3.3529e-14$
$R = 2$	0	$4.4409e-16$	$1.3323e-15$	$9.5812e-14$
$R = 4$	$2.2204e-16$	$3.5527e-15$	$3.1974e-14$	$6.1018e-13$

In Table 1, we list the values of ε at the collocation points. From Table 1, we could find that a small number of Chebyshev Polynomials are needed to reach perfect solution for different R. Figure 1 shows the analytical solution and numerical solution for different R at collocation points. We can conclude that the numerical solution is very close to the analytical solution. The same trend is observed for other values of $\alpha(t)$ and $\beta_i(t)$, $i = 1, 2, \ldots, k$. All the values of ε are small enough to meet the practical engineering application.

Let $\alpha(t) = 2$, $\beta_1(t) = 1.234$, $\beta_2(t) = 1$, $\beta_3(t) = 0.333$, and $R = 1$ as [28]; Example 1(a) becomes a multiterm order FDE, namely, Example 1(b). This problem has been solved in [28].

(b) See [28]:

$$\begin{aligned} &aD^2 f(t) + b(t)D^{\beta_1} f(t) + c(t)Df(t) \\ &\quad + e(t)D^{\beta_3} f(t) + k(t)f(t) = g(t), \quad t \in [0,1], \\ &y(0) = 2, \\ &y'(0) = 0, \end{aligned} \tag{35}$$

where

$$\begin{aligned} f(t) = -a - b(t)\frac{t^{2-\beta_1}}{\Gamma(3-\beta_1)} - c(t)t \\ - e(t)\frac{t^{2-\beta_3}}{\Gamma(3-\beta_3)} + k(t)\left(2 - \frac{t^2}{2}\right). \end{aligned} \tag{36}$$

Table 2: Computational results of Example 1(b) for $R = 1$.

t	Λ	ε
$n = 3$	$[1.8438, -0.1250, -0.0313, -0.0000]^T$	$4.4409e-16$
$n = 4$	$[1.8438, -0.1250, -0.0312, 0.0000, -0.0000]^T$	$1.4633e-13$
$n = 5$	$[1.8437, -0.1250, -0.0313, 0.0000, -0.0000, 0.0000]^T$	$3.2743e-12$
$n = 6$	$[1.8438, -0.1250, -0.0312, 0.0000, 0.0000, 0.0000, -0.0000]^T$	$1.0725e-13$

Table 3: Values of ε of Example 2(b) for $R = 2, 4$.

R	$n = 3$	$n = 4$	$n = 5$	$n = 6$
$R = 2$	$8.8818e-16$	$9.1038e-15$	$2.2959e-13$	$9.4480e-14$
$R = 4$	$8.8818e-16$	$1.0214e-14$	$7.3275e-15$	$3.8369e-13$

The analytical solution is $f(t) = 2 - t^2/2$. Example 1(b) is a special case of Example 1(a), so we still obtain the solution by our method as Example 1(a). The computational results are seen in Table 2. We list the vector $\Lambda = [\lambda_0, \lambda_1, \ldots, \lambda_n]^T$ and the values of ε at the collocation points.

As seen from Table 2, the vector $\Lambda = [\lambda_0, \lambda_1, \ldots, \lambda_n]^T$ obtained is mainly composed of three terms, namely, $\lambda_0, \lambda_1, \lambda_2$, which is in agreement with the analytical solution $f(t) = 2-t^2/2$. The values of ε are smaller than [28] with the same size of Chebyshev Polynomials (in [28], the value of ε is $6.88384e-5$ for $n = 5$). In addition, we extend the interval from $[0, 1]$ to $[0, 2]$ and $[0, 4]$. Similarly, we also get the perfect results as shown in Table 3, which is not solved in [28].

Example 2. (a) As the second example, the nonlinear multiterm variable order FDE

$$D^{\alpha(t)} f(t) + D^{\beta_1(t)} f(t) D^{\beta_2(t)} f(t) + f^2(t) = g(t), \quad t \in [0, R], \tag{37}$$

with

$$g(t) = t^6 + \frac{6}{\Gamma(4-\alpha(t))} t^{3-\alpha(t)} + \frac{36}{\Gamma(4-\beta_1(t))\Gamma(4-\beta_2(t))} t^{6-\beta_1(t)-\beta_2(t)}, \tag{38}$$

subject to the initial conditions $f(0) = f'(0) = f''(0) = 0$ is considered. The analytical solution is $f(t) = t^3$.

Let $f(t) = \Lambda^T \Psi(t)$; according to (28), we have

$$a\Lambda^T AMA^{-1}\Psi_n + \left(\Lambda^T AN_1 A^{-1}\Psi_n\right)\left(\Lambda^T AN_2 A^{-1}\Psi_n\right) + \left(\Lambda^T \Psi_n\right)^2 = g(t). \tag{39}$$

Let $\alpha(t) = t^2$, $\beta_1(t) = \sin t$, and $\beta_2(t) = t/4$; by taking the collocation points, the solution of Example 2(a) could be gained. The values of ε are displayed in Table 4 for different R. From the result analysis, our method could gain satisfactory solution. Figure 2 obviously shows that the numerical solution converges to the analytical solution.

Table 4: Values of ε of Example 2(a) with $\alpha(t) = t^2$, $\beta_1(t) = \sin t$, and $\beta_2(t) = t/4$.

R	$n = 3$	$n = 4$	$n = 5$	$n = 6$
$R = 1$	$5.1560e-15$	$2.2690e-14$	$5.4114e-14$	$1.0227e-13$
$R = 2$	$1.8874e-15$	$1.4660e-14$	$6.6386e-14$	$7.4174e-13$
$R = 4$	$4.3512e-15$	$3.1200e-14$	$5.1616e-08$	$1.0565e-11$

If $\alpha(t), \beta_1(t), \beta_2(t)$ are constants, Example 2(a) becomes a multiterm order FDE in [28]. This problem for $R = 1$ has also been solved in [28].

(b) See [28]:

$$D^{\alpha} f(t) + D^{\beta_1} f(t) D^{\beta_2} f(t) + f^2(t) = g(t), \quad 2 < \alpha < 3,\ 1 < \beta_1 < 2,\ 0 < \beta_2 < 1,\ t \in [0, 1], \tag{40}$$

with

$$g(t) = t^6 + \frac{6}{\Gamma(4-\alpha)} t^{3-\alpha} + \frac{36}{\Gamma(4-\beta_1)\Gamma(4-\beta_2)} t^{6-\beta_1-\beta_2}. \tag{41}$$

The same as [28], we let $\alpha = 2.5$, $\beta_1 = 1.5$, and $\beta_2 = 0.9$ and $\alpha = 2.75$, $\beta_1 = 1.75$, and $\beta_2 = 0.75$ for $R = 1$ and then use our method to solve them. The computational results are shown in Tables 5 and 6. As seen from Tables 5 and 6, the vector $\Lambda = [\lambda_0, \lambda_1, \ldots, \lambda_n]^T$ obtained is mainly composed of four terms, namely, $\lambda_0, \lambda_1, \lambda_2, \lambda_3$, which is in agreement with the analytical solution $f(t) = t^3$. It is evident that the numerical solution obtained converges to the analytical solution for $\alpha = 2.5$, $\beta_1 = 1.5$, and $\beta_2 = 0.9$ and $\alpha = 2.75$, $\beta_1 = 1.75$, and $\beta_2 = 0.75$. The values of ε are smaller than [28] with the same n size.

In addition, we extend the interval from $[0, 1]$ to $[0, 2]$ and $[0, 3]$. Similarly, we also get the perfect results in Tables 7 and 8, but the problems are not solved in [28].

At last, the proposed method is used to solve the multiterm initial value problem with nonsmooth solution.

Example 3. Let us consider the FDE as follows:

$$\left|t + \frac{1}{3}\right| D^{\alpha} y + \left|t - \frac{1}{3}\right| D^{\beta} y + y = \left|t^2 - \frac{1}{9}\right| \left\{\left(t^2 - \frac{1}{9}\right)^2 + \left(6t^3 - \frac{2t}{3}\right)\left|t - \frac{1}{3}\right| + \left(30t^2 - \frac{2}{3}\right)\left|t + \frac{1}{3}\right|\right\}, \tag{42}$$

$$1 < \alpha \le 2,\ 0 < \beta \le 1,\ y(0) = \frac{1}{729},\ y'(0) = 0,\ t \in [0, 3],$$

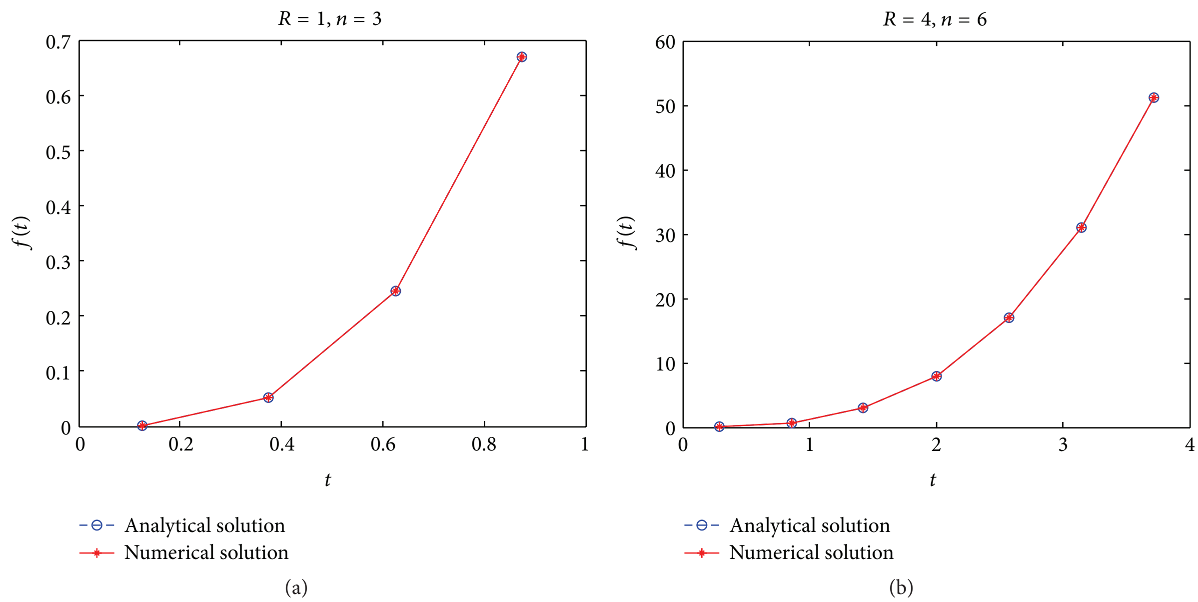

FIGURE 2: Analytical solution and numerical solution of Example 2(a) with $\alpha(t) = t^2$, $\beta_1(t) = \sin t$, and $\beta_2(t) = t/4$ for different R.

TABLE 5: Computational results of Example 2(b) for $R = 1$ with $\alpha = 2.5$, $\beta_1 = 1.5$, and $\beta_2 = 0.9$.

t	Λ	ε
$n = 3$	$[0.2188, 0.2187, 0.0937, 0.0156]^T$	$1.2628e-15$
$n = 4$	$[0.2188, 0.2187, 0.0938, 0.0156, 0.0000]^T$	$1.5910e-14$
$n = 5$	$[0.2188, 0.2187, 0.0937, 0.0156, 0.0000, -0.0000]^T$	$4.7362e-13$
$n = 6$	$[0.2188, 0.2187, 0.0937, 0.0156, -0.0000, -0.0000, 0.0000]^T$	$1.2801e-11$

TABLE 6: Computational results of Example 2(b) for $R = 1$ with $\alpha = 2.75$, $\beta_1 = 1.75$, and $\beta_2 = 0.75$.

t	Λ	ε
$n = 3$	$[0.2187, 0.2188, 0.0938, 0.0156]^T$	$1.3983e-15$
$n = 4$	$[0.2187, 0.2187, 0.0938, 0.0156, 0.0000]^T$	$7.6964e-14$
$n = 5$	$[0.2188, 0.2187, 0.0937, 0.0156, -0.0000, -0.0000]^T$	$1.4200e-12$
$n = 6$	$[0.2188, 0.2188, 0.0938, 0.0156, -0.0000, -0.0000, 0.0000]^T$	$1.8479e-11$

TABLE 7: Values of ε of Example 2(b) for $R = 2, 3$ with $\alpha = 2.5$, $\beta_1 = 1.5$, and $\beta_2 = 0.9$.

R	$n = 3$	$n = 4$	$n = 5$	$n = 6$
$R = 2$	$1.0474e-15$	$3.1058e-14$	$1.2829e-13$	$6.3259e-13$
$R = 3$	$9.8203e-15$	$6.3154e-14$	$4.4387e-13$	$4.1653e-12$

TABLE 8: Values of ε of Example 2(b) for $R = 2, 3$ with $\alpha = 2.75$, $\beta_1 = 1.75$, and $\beta_2 = 0.75$.

R	$n = 3$	$n = 4$	$n = 5$	$n = 6$
$R = 2$	$2.9790e-15$	$1.2483e-14$	$6.2233e-13$	$8.4933e-13$
$R = 3$	$1.1374e-14$	$1.0100e-14$	$1.5589e-14$	$1.3921e-11$

in which only for $\alpha = 2$ and $\beta = 1$, the analytical solution is known and given by $y = |(t^2 - 1/9)^3|$.

By applying the proposed method to solve the equation, we can obtain that the value of ε is $1.5853e-3$ for $n = 9$. The computational results are shown as Figures 3 and 4.

As seen from Figure 3, it is evident that the numerical solution obtained converges to the analytical solution. We also plot the absolute error between the analytical solution and numerical solution in Figure 4. It shows that the absolute error is small, which could meet the needs of general projects. In a word, the proposed method possesses simple form, satisfactory accuracy, and wide field of application.

7. Conclusion

In this paper, we present an operational matrix technology based on the second kind of Chebyshev Polynomial to solve multiterm FDE and multiterm variable order FDE. This technology reduces the original equation to a system of algebraic

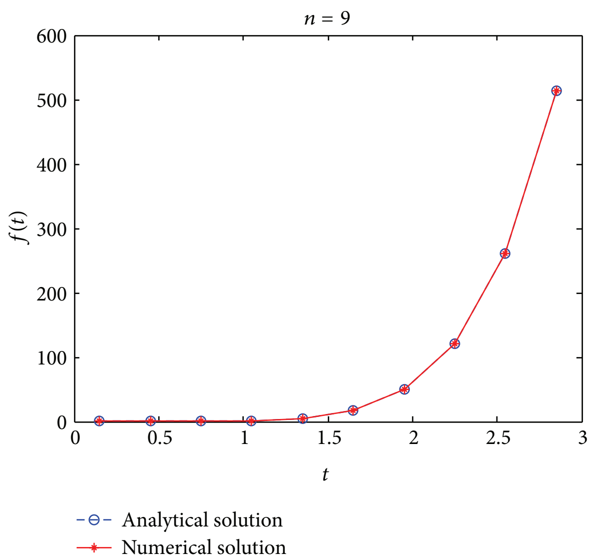

FIGURE 3: Analytical solution and numerical solution of Example 3.

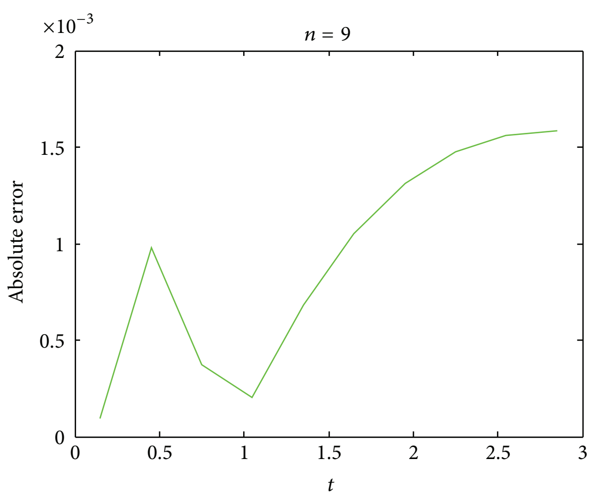

FIGURE 4: Absolute error of the proposed method of Example 3.

equations, which greatly simplifies the problem. In order to confirm the efficiency of the proposed techniques, several numerical examples are implemented, including linear and nonlinear terms. By comparing the numerical solution with the analytical solution and that of other methods in the literature, we demonstrate the high accuracy and efficiency of the proposed techniques.

In addition, the proposed method can be applied by developing for the other related fractional problem, such as variable fractional order integrodifferential equation, variable order time fractional diffusion equation, and variable fractional order linear cable equation. This is one possible area of our future work.

Competing Interests

The authors declare that they have no competing interests.

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