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ITERATIVE SOLUTION OF NEGATIVE EXPONENT EMDEN-FOWLER PROBLEMS

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ABSTRACT. A monotone iterative technique for proving existence of a positive solution pair (λ,y) of $y''(x) +\lambda\, a(x)y^{\mu}(x) = 0$, $0 < x < 1$, $y(0+) = y(1-) = 0$, $\mu \leq -1$ is shown. The method is computationally effective and an example is given.

KEY WORDS AND PHRASES. Monotone iterations, singular Emden-Fowler problems.
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1. INTRODUCTION.

In this paper a monotone iterative technique for finding a positive solution pair (λ,y) of

$$\begin{aligned} &y''(x) + \lambda\, a(x)\, y^{\mu}(x) = 0, \ 0 < x < 1 \\ &y(0+) = y(1-) = 0, \qquad (1.1) \\ &\mu \leq -1, \end{aligned}$$

is shown. We require $a(x)$ to be non-negative and continuous on $[0,1]$ with $x^{\mu-1}\, a(x)$ and $(1-x)^{\mu-1}\, a(x)$ being bounded on $(0,1)$. Under these conditions, a solution exists and $y'(0+)$ and $y'(1-)$ are finite. This was shown by Taliaferro [1]; however the methods in that paper were not constructive in nature. Development of an iterative sequence which is proved to converge to a positive solution is our aim in this paper. In other work the authors [2] have obtained results for $\mu > -1$. Negative exponent Emden-Fowler problems have some significant applications in permeable catalysis and fluid mechanics (Luning and Perry [3], [4]).

2. THE MONOTONE ITERATION.

The iterative sequence is defined as follows: Let $u_0(x) = x$,

$$K(x,\xi) = \begin{matrix} (1-x)\xi & 0 \leq \xi < x \\ (1-\xi)x & x \leq \xi \leq 1, \end{matrix}$$

and

$$u_n(x) = \lambda_n \int_0^1 K(x,\xi)\ a(\xi)\ u_{n-1}^{\mu}(\xi)\ d\xi \qquad n = 1,2,\ldots \tag{2.1}$$

$$\lambda_n = \frac{1}{\int_0^1 (1-\xi)\ a(\xi)\ u_{n-1}^{\mu}(\xi)\ d\xi}.$$

The conditions on $a(x)$ guarantee the existence of $u_n(x)$, $n = 1,2,3,\ldots$, on $0 < x < 1$ and that $u_n'(1-)$ is finite for all n. Moreover, since $K(x,\xi)$ is the Green's function for $u'' = 0$, $u(0) = u(1) = 0$, the sequences $\{u_n(x)\}$, $\{\lambda_n\}$ satisfy

$$\begin{aligned} &u_n''(x) + \lambda_n\ a(x)u_{n-1}^{\mu}(x) = 0 \\ &u_n(0+) = u_n(1-) = 0 \\ &u_n'\ (0+) = 1 \end{aligned} \tag{2.2}$$

The sequences $\{u_n(x)\}$, $\{\lambda_n\}$ are actually alternating monotone.

LEMMA. For $n \geq 1$, $u_{2n-1}(x) < u_{2n+1}(x) < u_{2n}(x) < u_{2n-2}(x)$, $0 < x < 2$, $\lambda_{2n} < \lambda_{2n+2} < \lambda_{2n+1} < \lambda_{2n-1}$. Moreover, for any $0 < M < 1$ there is at most one value of $x \in (0,1)$ such that $Mu_i^{-\mu}(x) - u_j^{-\mu}(x) = 0$, where the pair (i,j) can be $(2n-2,2n-1)$, $(2n,2n-1)$, $(2n-2,2n)$, or $(2n+1,2n-1)$.

Before proving the lemma, we note that the result implies that $\{u_n'(1-)\}_{n=1}^{\infty}$ is a bounded sequence.

PROOF OF THE LEMMA. First we compare u_0 and u_1. We have $(u_0 - u_1)''(x) = \lambda_1 a(x)\ u_0^{\mu}\ (x) > 0$, $0 < x < 1$, $(u_0-u_1)'(0+) = (u_0-u_1)(0) = 0$, so that $u_0 > u_1$, $0 < x < 1$. Let $f_{0,1}(x) = M^{-\mu}\ u_0(x) - u_1(x)$ where $0 < M < 1$. Now $f_{0,1}''(x) = -u_1''(x) = \lambda_1\ a(x)\ u_0^{\mu}(x) > 0$, $0 < x < 1$, $f_{0,1}(0) = 0$, $f_{0,1}'(0) = M^{-1/\mu}-1 < 0$. Thus $f_{0,1}(x)$ can cross the x axis at most once interior to $(0,1)$. Since $u_0 > u_1$, the definition of λ_k leads to $\lambda_2 < \lambda_1$.

In a completely similar way it is shown that $u_0 > u_2$, $\lambda_3 < \lambda_1$, and $f_{0,2}(x) = M^{-1/\mu}u_0(x)-u_2(x)$ has at most one zero interior to $(0,1)$. For the induction hypothesis we assume $(u_{2n}-u_{2n+1})(x) > 0$, $0 < x < 1$, $(u_{2n}-u_{2n+2})(x) > 0$, $0 < x < 1$, there is at most one $x \in (0,1)$ such that $(M^{-1/\mu}u_{2n}-u_{2n+1})(x) = 0$, and there is at most one $x \in (0,1)$ such that $(M^{-1/\mu}u_{2n}-u_{2n+2})(x) = 0$. This implies that $\lambda_{2n+2} < \lambda_{2n+1}$ and $\lambda_{2n+3} < \lambda_{2n+1}$. To show that $(u_{2n+2}-u_{2n+1})(x) > 0$, $0 < x < 1$, we proceed as follows

$$(u_{2n+2}-u_{2n+1})''(x) = \lambda_{2n+1}a(x)u_{2n}^{\mu}(x)-\lambda_{2n+2}a(x)u_{2n+1}^{\mu}(x)$$

$$= -\lambda_{2n+1}a(x)u_{2n}^{\mu}(x)u_{2n+1}^{\mu}(x)$$

$$\times \left[\frac{\lambda_{2n+2}}{\lambda_{2n+1}} u_{2n}^{-\mu}(x)-u_{2n+1}^{-\mu}(x) \right] .$$

Since $\lambda_{2n+2}/\lambda_{2n+1} < 1$ we conclude that $(u_{2n+2}-u_{2n+1})(x) > 0$, $0 < x < 1$. Similarly it is shown that there is at most one $x \in (0,1)$ such that $(M^{-1/\mu}u_{2n+2}-u_{2n+1})(x) = 0$. Therefore $\lambda_{2n+3} > \lambda_{2n+2}$. One then proceeds in the same manner to consider $(u_{2n+3}-u_{2n+1})(x)$, $(M^{-1/\mu}u_{2n+3}-u_{2n+1})(x)$, $(u_{2n+2}-u_{2n+3})(x)$, $(M^{-1/\mu}u_{2n+2}-u_{2n+3})(x)$, and $(u_{2n+2}-u_{2n+4})(x)$, $(M^{-1/\mu}u_{2n+2}-u_{2n+4})(x)$ along with the definition of λ_n to order the λ's. This completes the induction proof of the lemma.

Because $0 < \lambda_{2n} < \lambda_{2n+2} < \lambda_{2n+1} < \lambda_{2n-1}$, there exists $0 < \check{\lambda} < \hat{\lambda}$ such that $\lim_{n \to \infty} \lambda_{2n} = \check{\lambda}$, $\lim_{n \to \infty} \lambda_{2n+1} = \hat{\lambda}$. Since $0 < u_{2n+1}(x) < u_{2n}(x) < u_{2n-2}(x) < u_0(x)$, $0 < x < 1$, the sequences $\{u_{2n-1}\}$ and $\{u_{2n}\}$ are uniformly bounded monotone sequences.

We can also conclude that the sequences are equicontinuous because $\{|u_n'(x)|\}$ is uniformly bounded. By Ascoli's lemma there exist functions $\check{u}, \hat{u} \in C[0,1]$ such that $0 < \check{u}(x) \leq \hat{u}(x)$, $0 < x < 1$, and $\lim_{n \to \infty} u_{2n-1} = \check{u}$ and $\lim_{n \to \infty} u_{2n} = \hat{u}$ uniformly on $[0,1]$. Using the dominated convergence theorem and (2.1) we have

$$\begin{aligned} \hat{u}(x) &= \check{\lambda} \int_0^1 K(x,\xi)a(\xi)\check{u}^{\mu}(\xi)\, d\xi, \\ \check{u}(x) &= \hat{\lambda} \int_0^1 K(x,\xi)a(\xi)\hat{u}^{\mu}(\xi)\, d\xi \end{aligned} \tag{2.3}$$

or, equivalently,

$$\hat{u}''(x) + \check{\lambda}\, a(x)\check{u}^{\mu}(x) = 0, \quad 0 < x < 1$$

$$\check{u}''(x) + \hat{\lambda}\, a(x)\hat{u}^{\mu}(x) = 0, \qquad 0 < x < 1, \tag{2.4}$$

$$\hat{u}(0) = \check{u}(0) = \hat{u}(1) = \check{u}(1) = 0.$$

Now we have two possibilities. Either $\hat{u}(x) = \check{u}(x)$ on $(0,1)$ or $\check{u}(x) < \hat{u}(x)$ on some subinterval of $(0,1)$. If $\hat{u} = \check{u}$ then $\hat{\lambda} = \check{\lambda}$ and the pair $(\lambda,y) = (\hat{\lambda},\hat{u})$ is a solution to the problem (1.1). Otherwise we proceed as follows. Since $\check{u} \leq \hat{u}$ we have

$$-\hat{u}'' = \check{\lambda} a(x)\check{u}^{\mu} \geq \check{\lambda} a(x)\hat{u}^{\mu},$$

$$\hat{u}(0) = \hat{u}(1) = 0$$

Now let the constant $C \in (0,1)$ be chosen so that $C^{1-\mu}\hat{\lambda} = \check{\lambda}$. Then we have

$$\begin{aligned} -(C\check{u}'') &= C\,\hat{\lambda}\, a(x)\hat{u}^{\mu} \leq C\,\hat{\lambda}\, a(x)\, \check{u}^{\mu} \\ &= C^{1-\mu}\hat{\lambda} a(x)\,(C\check{u})^{\mu} \\ &= \check{\lambda}\, a(x)\,(C\check{u})^{\mu} \end{aligned}$$

$$C\check{u}(0) = C\check{u}(1) = 0.$$

In this way we have a supersolution $\phi = \hat{u}$ and a subsolution $\psi = C\check{u}$ for the problem

$$-y''(x) = \check{\lambda}\, a(x)\, y^{\mu}(x), \quad 0 < x < 1,$$

$$y(0) = y(1) = 0 \tag{2.5}$$

with $\phi \geq \psi$. Since $x^{\mu-1}a(x)$ and $(1-x)^{\mu-1}\, a(x)$ are bounded, the function $f(x,u) = \lambda a(x)u^{\mu}$ has $\partial f/\partial u$ bounded, $0 < x < 1$, $\psi(x) \leq u \leq \phi(x)$. Therefore there exists $\omega > 0$ such that

$$f(x,\xi) - f(x,\eta) > -\omega(\xi-\eta)$$

for all $x \in [0,1]$ and ξ, $\eta \in [\psi,\phi]$ with $\xi > \eta$ (where one sided limits are used at 0 and 1).

The foregoing conditions are sufficient for the iteration schemes (see Amann [5], page 648).

$$-v''_{k+1}(x) + \omega v_{k+1}(x) = \check{\lambda}\, a(x)\, v_k^{\mu}(x) + \omega v_k(x)$$
$$v_{k+1}(0) = v_{k+1}(0) = v_{k+1}(1) = 0 \qquad (2.6)$$
$$v_0 = \phi \quad \text{or} \quad v_0 = \psi$$

to converge to a minimal (if $v_0 = \psi$) or a maximal solution (if $v_0 = \phi$) of the problem (2.5), the solution lying in $[\psi,\phi]$.

Thus we have shown that a positive solution pair (λ,y) exists for problem (1.1) and is obtainable by monotone iteration. Moreover the eigenvalue $\lambda = \check{\lambda}$ is determined by the iteration scheme in (2.1). If the function y is not determined by the iteration scheme (2.1) (that is if $\hat{y} \neq \check{y}$)y is found by the application of the second iterative scheme, (2.6). To implement the method numerically, special care must be taken in all calculations near the endpoints.

3. EXAMPLE.

Calculations were performed for the problem

$$y''(x) + \lambda\, x^3(1-x)^3\, y^{-2}(x) = 0$$
$$y(0) = y(1) = 0.$$

The iterative method was applied using the integral formulation, using Simpson's rule away from the endpoints. The sequences were alternating monotone and we found $\lambda_{10} \doteq 7.1737$, $\lambda_{11} \doteq 7.2213$. Because $\max|u_{11}-u_{10}| < .0003$ we used $\lambda = (\lambda_{10} + \lambda_{11})/2$ and $y = (u_{10} + u_{11})/2$; that is, we considered that the iterative sequences had "pinched together". The calculated solution was

$$\lambda \doteq 7.1975$$

x	y(x)
0	0
.1	.0628
.2	.1181
.3	.2002
.4	.2242
.5	.2292

where y(x) is symmetric about $x = 1/2$. Uniqueness of solution follows from a result of Taliaferro [1].

4. NOTE.

Our conditions on a(x) are more restrictive than those for a nonconstructive existence result. For a(x) continuous on (0,1) a solution exists if and only if $\int_0^1 t(1-t)a(t)dt < \infty$ [1].

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