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Research Article

Positive Solutions for a Third Order Nonlinear Neutral Delay Difference Equation

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The existence, multiplicity, and properties of positive solutions for a third order nonlinear neutral delay difference equation are discussed. Six examples are given to illustrate the results presented in this paper.

1. Introduction and Preliminaries

Recently, some researchers used the Reccati transformation techniques, fixed point theorems, and iterative algorithms to study the oscillation, nonoscillation, asymptotic properties, and solvability for linear and nonlinear third order difference equations and systems; see, for example, [1–6] and the references therein. In particular, Saker [4], Andruch-Sobilo and Migda [1], and Grace and Hamedani [2] discussed the oscillation for the following third order difference equations:

$$\Delta^3 x(n) + p(n) x(n+1) = 0, \quad n \ge n_0,$$
$$\Delta^3 \left(x(n) - p(n) x(\sigma(n))\right) \pm q(n) x(\tau(n)) = 0, \quad n \ge n_0,$$
$$\Delta^3 (x(n) - x(n-\tau)) \pm q(n) |x(n-\sigma)|^3 \operatorname{sgn} x(n-\sigma) = 0, \quad n \ge 0. \tag{1}$$

Making use of the Schauder fixed point theorem, Banach fixed point theorem, and Mann iterative schemes, Yan and Liu [5] and Liu et al. [3], respectively, proved the existence of a bounded nonoscillatory solution for the third order difference equation:

$$\Delta^3 x(n) + f(n, x(n), x(n-\tau)) = 0, \quad n \ge n_0 \tag{2}$$

and the existence of positive solutions and convergence of the Mann iterative schemes for the following third order nonlinear neutral delay difference equation:

$$\Delta^3 (x(n) + b(n) x(n-\tau)) + \Delta h(n, x(h_1(n)), x(h_2(n)), \dots, x(h_k(n))) + f(n, x(f_1(n)), x(f_2(n)), \dots, x(f_k(n))) = c(n), \quad n \ge n_0. \tag{3}$$

However, the following third order nonlinear neutral delay difference equation:

$$\Delta^3 (x(n) + b(n) x(n-\tau) + c(n)) + \Delta^2 h(n, x(h_1(n)), \dots, x(h_k(n))) + \Delta g(n, x(g_1(n)), \dots, x(g_k(n))) + f(n, x(f_1(n)), \dots, x(f_k(n))) = d(n), \quad n \ge n_0, \tag{4}$$

where $\tau, k, n_0 \in \mathbb{N}$, $\{c(n)\}_{n\in\mathbb{N}_{n_0}}$, $\{b(n)\}_{n\in\mathbb{N}_{n_0}}$, and $\{d(n)\}_{n\in\mathbb{N}_{n_0}}$ are real sequences, $f, g, h \in C(\mathbb{N}\times\mathbb{R}^k, \mathbb{R})$ and $f_l, g_l, h_l : \mathbb{N}_{n_0} \to \mathbb{N}$ with

$$\lim_{n\to\infty} f_l(n) = \lim_{n\to\infty} g_l(n) = \lim_{n\to\infty} h_l(n) = +\infty, \quad l \in \{1, 2, \ldots, k\} \tag{5}$$

has not been studied. The purpose of this paper is to study solvability of (4). By utilizing the Krasnoselskii fixed point theorem, Schauder fixed point theorem and some new techniques, we establish the existence, multiplicity, and properties of positive solutions of (4). Six examples are constructed to illuminate our results.

Throughout this paper, we assume that Δ is the forward difference operator defined by $\Delta x(n) = x(n+1) - x(n)$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, and $\mathbb{N}_0$ and $\mathbb{N}$ denote the sets of nonnegative integers and positive integers, respectively,

$$\begin{gathered} \mathbb{N}_t = \{n : n \in \mathbb{N}_0 \text{ with } n \geq t\}, \quad t \in \mathbb{N}_0, \\ \alpha = \inf\left\{f_l(n), g_l(n), h_l(n) : 1 \leq l \leq k, n \in \mathbb{N}_{n_0}\right\}, \\ \beta = \min\{|n_0 - \tau|, \alpha\} \in \mathbb{N}. \end{gathered} \tag{6}$$

Let l_β^∞ denote the Banach space of all real sequences $x = \{x(n)\}_{n\in\mathbb{N}_\beta}$ with norm

$$\|x\| = \sup_{n\in\mathbb{N}_\beta} \left|\frac{x(n)}{n}\right| < +\infty \quad \text{for } x = \{x(n)\}_{n\in\mathbb{N}_\beta} \in l_\beta^\infty. \tag{7}$$

Let A, B, A_*, B^* and c^* be positive constants, $T \in \mathbb{N}$, $\{c(n)\}_{n\in\mathbb{N}_\beta}$, $\{A(n)\}_{n\in\mathbb{N}_\beta}$, and $\{B(n)\}_{n\in\mathbb{N}_\beta}$ be real sequences with

$$\begin{gathered} B(n) = B + \frac{|c(n)|}{n} > A(n) = A - \frac{|c(n)|}{n}, \quad n \in \mathbb{N}_\beta, \\ c(n) = c(n_0), \quad \beta \leq n \leq n_0 - 1, \\ c^* \geq \sup_{n\in\mathbb{N}_\beta} \frac{|c(n)|}{n}, \\ A_* = A - c^*, \qquad B^* = B + c^*. \end{gathered} \tag{8}$$

Put

$$\begin{aligned} &\Omega(A_*, B^*, T) \\ &= \Big\{x = \{x(n)\}_{n\in\mathbb{N}_\beta} \in l_\beta^\infty : A(T) \leq \frac{x(n)}{n} \leq B(T), \\ &\qquad \beta \leq n < T;\ A(n) \leq \frac{x(n)}{n} \leq B(n), n \geq T\Big\}. \end{aligned} \tag{9}$$

It is easy to see that $\Omega(A_*, B^*, T)$ is a bounded closed and convex subset of l_β^∞.

By a solution of (4), we mean a sequence $\{x(n)\}_{n\in\mathbb{N}_\beta} \in l_\beta^\infty$ with a positive integer $T \geq n_0 + \tau + \alpha$ such that (4) holds for all $n \geq T$.

The following lemmas play important roles in this paper.

Lemma 1 (see [6]). *A bounded and uniformly Cauchy subset $C \subseteq l_\beta^\infty$ is relatively compact.*

Lemma 2 (Krasnoselskii fixed point theorem). *Let X be a Banach space, D a bounded closed convex subset of X, and $S, G : D \to X$ mappings such that $Sx + Gy \in D$ for every pair $x, y \in D$. If S is a contraction and G is completely continuous, then the equation*

$$Sx + Gx = x \tag{10}$$

has a solution in D.

Lemma 3 (Schauder fixed point theorem). *Let D be a nonempty closed convex subset of a Banach space X and $T : D \to D$ a continuous mapping such that $T(D)$ is a relatively compact subset of X. Then T has at least one fixed point in D.*

Lemma 4. *Let $\tau, n \in \mathbb{N}$ and $\{q(k)\}_{k\in\mathbb{N}} \subset \mathbb{R}^+$. Then*

(i) $\sum_{i=1}^{\infty} \sum_{t=n+i\tau}^{\infty} q(t) \leq \sum_{t=n+\tau}^{\infty} (t/\tau) q(t)$;
(ii) $\sum_{i=1}^{\infty} \sum_{s=n+i\tau}^{\infty} \sum_{t=s}^{\infty} q(t) \leq \sum_{t=n+\tau}^{\infty} (t^2/\tau) q(t)$;
(iii) $\sum_{p=1}^{\infty} \sum_{i=n+p\tau}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} q(t) \leq \sum_{t=n+\tau}^{\infty} (t^3/\tau) q(t)$.

Proof. (i) Let $[t]$ denote the largest integer number not exceeding $t \in \mathbb{R}^+$. It is clear that

$$\begin{aligned} \sum_{i=1}^{\infty} \sum_{t=n+i\tau}^{\infty} q(t) &= \sum_{t=n+\tau}^{\infty} \left(1 + \left[\frac{t-n-\tau}{\tau}\right]\right) q(t) \\ &\leq \sum_{t=n+\tau}^{\infty} \frac{t}{\tau} q(t). \end{aligned} \tag{11}$$

(ii) It follows from (i) that

$$\begin{aligned} \sum_{i=1}^{\infty} \sum_{s=n+i\tau}^{\infty} \sum_{t=s}^{\infty} q(t) &= \sum_{i=1}^{\infty} \sum_{t=n+i\tau}^{\infty} (1 + t - n - i\tau) q(t) \\ &\leq \sum_{i=1}^{\infty} \sum_{t=n+i\tau}^{\infty} t q(t) \leq \sum_{t=n+\tau}^{\infty} \frac{t^2}{\tau} q(t). \end{aligned} \tag{12}$$

(iii) It follows from (ii) that

$$\begin{aligned} \sum_{p=1}^{\infty} \sum_{i=n+p\tau}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} q(t) &= \sum_{p=1}^{\infty} \sum_{i=n+p\tau}^{\infty} \sum_{t=i}^{\infty} (t - i + 1) q(t) \\ &\leq \sum_{p=1}^{\infty} \sum_{i=n+p\tau}^{\infty} \sum_{t=i}^{\infty} t q(t) \leq \sum_{t=n+\tau}^{\infty} \frac{t^3}{\tau} q(t). \end{aligned} \tag{13}$$

This completes the proof. □

2. Existence of Positive Solutions

Now we discuss the existence, multiplicity, and properties of positive solutions of (4) under various conditions on the sequence $\{b(n)\}_{n\in\mathbb{N}_\beta} \subseteq \mathbb{R}$.

Theorem 5. *Assume that there exist a constant b^* and three nonnegative sequences $\{F_n\}_{n\in\mathbb{N}_{n_0}}$, $\{G_n\}_{n\in\mathbb{N}_{n_0}}$, and $\{H_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying*

$$A + b^* B^* < B, \quad 0 \le b(n) \le b^* < 1, \; n \in \mathbb{N}_{n_0}, \tag{14}$$

$$\begin{gathered} |f(n, u_1, u_2, \ldots, u_k)| \le F_n, \\ |g(n, u_1, u_2, \ldots, u_k)| \le G_n, \\ |h(n, u_1, u_2, \ldots, u_k)| \le H_n, \\ (n, u_l) \in \mathbb{N}_{n_0} \times (\mathbb{R}^+ \setminus \{0\}), \quad 1 \le l \le k, \end{gathered} \tag{15}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=n}^{\infty} \max\left\{H_i, \sum_{s=i}^{\infty} G_s\right\} = 0, \tag{16}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=n}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} \max\{F_t, |d(t)|\} = 0. \tag{17}$$

Then

(i) *equation (4) possesses a positive solution $x = \{x(n)\}_{n\in\mathbb{N}_\beta} \in l_\beta^\infty$ with*

$$\lim_{n\to\infty} \frac{x(n) + b(n)x(n-\tau) + c(n)}{n} \in (A + b^* B^*, B), \tag{18}$$

$$A_* \le \liminf_{n\to\infty} \frac{x(n)}{n} \le \limsup_{n\to\infty} \frac{x(n)}{n} \le B^*; \tag{19}$$

(ii) *equation (4) possesses uncountably many positive solutions in l_β^∞.*

Proof. (i) Let $L \in (A + b^* B^*, B)$. It follows from (16) and (17) that there exists $T \ge n_0 + \tau + \alpha$ satisfying

$$\begin{aligned} &\frac{1}{T} \sum_{i=T}^{\infty} \left\{H_i + \sum_{s=i}^{\infty} G_s + \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} [F_t + |d(t)|]\right\} \\ &\quad < \min\{L - A - b^* B^*, B - L\}. \end{aligned} \tag{20}$$

Define two mappings S_L and $G_L : \Omega(A_*, B^*, T) \to l_\beta^\infty$ by

$$(S_L x)(n) = \begin{cases} nL - b(n)x(n-\tau) - c(n), & n \ge T, \\ \dfrac{n}{T}(S_L x)(T), & \beta \le n < T, \end{cases} \tag{21}$$

$$(G_L x)(n) = \begin{cases} \displaystyle\sum_{i=n}^{\infty} h(i, x(h_1(i)), \ldots, x(h_k(i))) \\ \displaystyle\quad - \sum_{i=n}^{\infty} \sum_{s=i}^{\infty} g(s, x(g_1(s)), \ldots, x(g_k(s))) \\ \displaystyle\quad + \sum_{i=n}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} [f(t, x(f_1(t)), \ldots, \\ \qquad x(f_k(t))) - d(t)], \\ \qquad n \ge T, \\ \dfrac{n}{T}(G_L x)(T), \quad \beta \le n < T, \end{cases} \tag{22}$$

for any $x = \{x(n)\}_{n\in\mathbb{N}_\beta} \in \Omega(A_*, B^*, T)$.

Now we show that

$$S_L x + G_L y \in \Omega(A_*, B^*, T), \quad x, y \in \Omega(A_*, B^*, T); \tag{23}$$

$$\|S_L x - S_L y\| \le b^* \|x - y\|, \quad x, y \in \Omega(A_*, B^*, T), \tag{24}$$

$$\|G_L y\| \le B, \quad y \in \Omega(A_*, B^*, T). \tag{25}$$

Using (14), (15), and (20)–(22), we get that for any $x = \{x(n)\}_{n\in\mathbb{N}_\beta}$, $y = \{y(n)\}_{n\in\mathbb{N}_\beta} \in \Omega(A_*, B^*, T)$

$$\begin{aligned} &\frac{(S_L x)(n) + (G_L y)(n)}{n} \\ &= L - \frac{b(n)}{n} x(n-\tau) - \frac{c(n)}{n} \\ &\quad + \frac{1}{n} \sum_{i=n}^{\infty} h(i, y(h_1(i)), \ldots, y(h_k(i))) \\ &\quad - \frac{1}{n} \sum_{i=n}^{\infty} \sum_{s=i}^{\infty} g(s, y(g_1(s)), \ldots, y(g_k(s))) \\ &\quad + \frac{1}{n} \sum_{i=n}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} [f(t, x(f_1(t)), \ldots, x(f_k(t))) - d(t)] \\ &\le L + \frac{|c(n)|}{n} \\ &\quad + \frac{1}{n} \sum_{i=n}^{\infty} |h(i, y(h_1(i)), \ldots, y(h_k(i)))| \\ &\quad + \frac{1}{n} \sum_{i=n}^{\infty} \sum_{s=i}^{\infty} |g(s, y(g_1(s)), \ldots, y(g_k(s)))| \end{aligned}$$

$$+\frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[\left|f\left(t,y\left(f_1(t)\right),\dots,y\left(f_k(t)\right)\right)\right|+|d(t)|\right]$$

$$\leq L+\frac{|c(n)|}{n}+\frac{1}{T}\sum_{i=T}^{\infty}H_i+\frac{1}{T}\sum_{i=T}^{\infty}\sum_{s=i}^{\infty}G_s$$

$$+\frac{1}{T}\sum_{i=T}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[F_t+|d(t)|\right]$$

$$< L+\frac{|c(n)|}{n}+\min\left\{L-A-b^*B^*,B-L\right\}$$

$$\leq B+\frac{|c(n)|}{n}=B(n),\quad n\geq T,$$

$$\frac{(S_Lx)(n)+(G_Ly)(n)}{n}=\frac{n}{T}\cdot\frac{(S_Lx)(T)+(G_Ly)(T)}{n}\leq B(T),\quad \beta\leq n<T,$$

$$\frac{(S_Lx)(n)+(G_Ly)(n)}{n}=L-\frac{b(n)}{n}x(n-\tau)-\frac{c(n)}{n}$$

$$+\frac{1}{n}\sum_{i=n}^{\infty}h\left(i,y\left(h_1(i)\right),\dots,y\left(h_k(i)\right)\right)$$

$$-\frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}g\left(s,y\left(g_1(s)\right),\dots,y\left(g_k(s)\right)\right)$$

$$+\frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[f\left(t,y\left(f_1(t)\right),\dots,y\left(f_k(t)\right)\right)-d(t)\right]$$

$$\geq L-\frac{n-\tau}{n}b(n)\frac{x(n-\tau)}{n-\tau}-\frac{|c(n)|}{n}$$

$$-\frac{1}{n}\sum_{i=n}^{\infty}\left|h\left(i,y\left(h_1(i)\right),\dots,y\left(h_k(i)\right)\right)\right|$$

$$-\frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\left|g\left(s,y\left(g_1(s)\right),\dots,y\left(g_k(s)\right)\right)\right|$$

$$-\frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[\left|f\left(t,y\left(f_1(t)\right),\dots,y\left(f_k(t)\right)\right)\right|+|d(t)|\right]$$

$$\geq L-b^*B^*-\frac{|c(n)|}{n}$$

$$-\frac{1}{T}\sum_{i=T}^{\infty}H_i-\frac{1}{T}\sum_{i=T}^{\infty}\sum_{s=i}^{\infty}G_s$$

$$-\frac{1}{T}\sum_{i=T}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[F_t+|d(t)|\right]$$

$$> L-b^*B^*-\frac{|c(n)|}{n}$$

$$-\min\left\{L-A-b^*B^*,B-L\right\}$$

$$\geq A-\frac{|c(n)|}{n}=A(n),\quad n\geq T,$$

$$\frac{(S_Lx)(n)+(G_Ly)(n)}{n}=\frac{n}{T}\cdot\frac{(S_Lx)(T)+(G_Ly)(T)}{n}\geq A(T),\quad \beta\leq n<T,$$

$$\left|\frac{(S_Lx)(n)-(S_Ly)(n)}{n}\right|=b(n)\frac{n-\tau}{n}\left|\frac{x(n-\tau)-y(n-\tau)}{n-\tau}\right|\leq b^*\|x-y\|,\quad n\geq T,$$

$$\left|\frac{(S_Lx)(n)-(S_Ly)(n)}{n}\right|=\frac{n}{T}\left|\frac{(S_Lx)(T)-(S_Ly)(T)}{n}\right|\leq b^*\|x-y\|,\quad \beta\leq n<T,$$

$$\left|\frac{(G_Ly)(n)}{n}\right|=\left|\frac{1}{n}\sum_{i=n}^{\infty}h\left(i,y\left(h_1(i)\right),\dots,y\left(h_k(i)\right)\right)\right.$$

$$-\frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}g\left(s,y\left(g_1(s)\right),\dots,y\left(g_k(s)\right)\right)$$

$$+\frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[f\left(t,y\left(f_1(t)\right),\dots,y\left(f_k(t)\right)\right)\right.$$

$$\left.\left.-d(t)\right]\right|$$

$$\leq\frac{1}{n}\sum_{i=n}^{\infty}\left|h\left(i,y\left(h_1(i)\right),\dots,y\left(h_k(i)\right)\right)\right|$$

$$+\frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\left|g\left(s,y\left(g_1(s)\right),\dots,y\left(g_k(s)\right)\right)\right|$$

$$+\frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[\left|f\left(t,y\left(f_1(t)\right),\dots,y\left(f_k(t)\right)\right)\right|\right.$$

$$\left.+|d(t)|\right]$$

$$\leq \frac{1}{T}\sum_{i=T}^{\infty} H_i + \frac{1}{T}\sum_{i=T}^{\infty}\sum_{s=i}^{\infty} G_s$$
$$+ \frac{1}{T}\sum_{i=T}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[F_t + |d(t)|\right]$$
$$< \min\{L - A - b^* B^*, B - L\} \leq B, \quad n \geq T,$$
$$\left|\frac{(G_L y)(n)}{n}\right| = \frac{n}{T}\left|\frac{(G_L y)(T)}{n}\right| \leq B, \quad \beta \leq n < T, \tag{26}$$

which yield the fact that (23)–(25) hold.

Next we show that G_L is completely continuous. Let $y^w = \{y^w(n)\}_{n\in\mathbb{N}_\beta}$ and $y = \{y(n)\}_{n\in\mathbb{N}_\beta} \in \Omega(A_*, B^*, T)$ with

$$\lim_{w\to\infty} y^w = y. \tag{27}$$

Using (16), (17), (27), and the continuity of f, g, and h, we know that for given $\varepsilon > 0$, there exist T_1, T_2, T_3 and $T_4 \in \mathbb{N}$ with $T_4 > T_3 > T_2 > T_1 > T$ satisfying

$$\frac{1}{T}\max\left\{\sum_{i=T_1+1}^{\infty} H_i + \sum_{i=T_1+1}^{\infty}\sum_{s=i}^{\infty} G_s + \sum_{i=T_1+1}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[F_t + |d(t)|\right],\right.$$
$$\sum_{i=T}^{T_1}\sum_{s=T_2+1}^{\infty} G_s + \sum_{i=T}^{T_1}\sum_{s=T_2+1}^{\infty}\sum_{t=s}^{\infty} F_t$$
$$\left.+\sum_{i=T}^{T_1}\sum_{s=i}^{T_2}\sum_{t=T_3+1}^{\infty} F_t\right\} < \frac{\varepsilon}{16}, \tag{28}$$

$$\frac{1}{T}\max\left\{\sum_{i=T}^{T_1}\left|h\left(i, y^w(h_1(i)), \ldots, y^w(h_k(i))\right) - h\left(i, y(h_1(i)), \ldots, y(h_k(i))\right)\right|,\right.$$
$$\sum_{i=T}^{T_1}\sum_{s=i}^{T_2}\left|g\left(s, y^w(g_1(s)), \ldots, y^w(g_k(s))\right) - g\left(s, y(g_1(s)), \ldots, y(g_k(s))\right)\right|,$$
$$\left.\sum_{i=T}^{T_1}\sum_{s=i}^{T_2}\sum_{t=s}^{T_3}\left|f\left(t, y^w(f_1(t)), \ldots, y^w(f_k(t))\right) - f\left(t, y(f_1(t)), \ldots, y(f_k(t))\right)\right|\right\}$$
$$< \frac{\varepsilon}{16}, \quad w \geq T_4. \tag{29}$$

Combining (15), (22), (28), and (29), we infer that

$$\|G_L y^w - G_L y\|$$
$$= \sup\left\{\frac{\left|(G_L y^w)(n) - (G_L y)(n)\right|}{n} : n \in \mathbb{N}_\beta\right\}$$
$$= \max\left\{\sup\left\{\frac{n}{T}\cdot\frac{\left|(G_L y^w)(T) - (G_L y)(T)\right|}{n} : \beta \leq n < T\right\},\right.$$
$$\left.\sup\left\{\frac{\left|(G_L y^w)(n) - (G_L y)(n)\right|}{n} : n \in \mathbb{N}_T\right\}\right\}$$
$$= \sup\left\{\frac{\left|(G_L y^w)(n) - (G_L y)(n)\right|}{n} : n \in \mathbb{N}_T\right\}$$
$$\leq \frac{1}{T}\sum_{i=T}^{\infty}\left|h\left(i, y^w(h_1(i)), \ldots, y^w(h_k(i))\right) - h\left(i, y(h_1(i)), \ldots, y(h_k(i))\right)\right|$$
$$+ \frac{1}{T}\sum_{i=T}^{\infty}\sum_{s=i}^{\infty}\left|g\left(s, y^w(g_1(s)), \ldots, y^w(g_k(s))\right) - g\left(s, y(g_1(s)), \ldots, y(g_k(s))\right)\right|$$
$$+ \frac{1}{T}\sum_{i=T}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left|f\left(t, y^w(f_1(t)), \ldots, y^w(f_k(t))\right) - f\left(t, y(f_1(t)), \ldots, y(f_k(t))\right)\right|$$
$$= \frac{1}{T}\sum_{i=T}^{T_1}\left|h\left(i, y^w(h_1(i)), \ldots, y^w(h_k(i))\right) - h\left(i, y(h_1(i)), \ldots, y(h_k(i))\right)\right|$$
$$+ \frac{1}{T}\sum_{i=T_1+1}^{\infty}\left|h\left(i, y^w(h_1(i)), \ldots, y^w(h_k(i))\right) - h\left(i, y(h_1(i)), \ldots, y(h_k(i))\right)\right|$$
$$+ \frac{1}{T}\sum_{i=T}^{T_1}\sum_{s=i}^{T_2}\left|g\left(s, y^w(g_1(s)), \ldots, y^w(g_k(s))\right) - g\left(s, y(g_1(s)), y(g_2(s)), \ldots, y(g_k(s))\right)\right|$$
$$+ \frac{1}{T}\sum_{i=T}^{T_1}\sum_{s=T_2+1}^{\infty}\left|g\left(s, y^w(g_1(s)), \ldots, y^w(g_k(s))\right) - g\left(s, y(g_1(s)), \ldots, y(g_k(s))\right)\right|$$

$$
\begin{aligned}
&+\frac{1}{T}\sum_{i=T_1+1}^{\infty}\sum_{s=i}^{\infty}\left|g\left(s,y^w\left(g_1(s)\right),\ldots,y^w\left(g_k(s)\right)\right)\right.\\
&\qquad\left.-g\left(s,y\left(g_1(s)\right),\ldots,y\left(g_k(s)\right)\right)\right|\\
&+\frac{1}{T}\sum_{i=T}^{T_1}\sum_{s=i}^{T_2}\sum_{t=s}^{T_3}\left|f\left(t,y^w\left(f_1(t)\right),\ldots,y^w\left(f_k(t)\right)\right)\right.\\
&\qquad\left.-f\left(t,y\left(f_1(t)\right),\ldots,y\left(f_k(t)\right)\right)\right|\\
&+\frac{1}{T}\sum_{i=T}^{T_1}\sum_{s=i}^{T_2}\sum_{t=T_3+1}^{\infty}\left|f\left(t,y^w\left(f_1(t)\right),\ldots,y^w\left(f_k(t)\right)\right)\right.\\
&\qquad\left.-f\left(t,y\left(f_1(t)\right),\ldots,y\left(f_k(t)\right)\right)\right|\\
&+\frac{1}{T}\sum_{i=T}^{T_1}\sum_{s=T_2+1}^{\infty}\sum_{t=s}^{\infty}\left|f\left(t,y^w\left(f_1(t)\right),\ldots,y^w\left(f_k(t)\right)\right)\right.\\
&\qquad\left.-f\left(t,y\left(f_1(t)\right),\ldots,y\left(f_k(t)\right)\right)\right|\\
&+\frac{1}{T}\sum_{i=T_1+1}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left|f\left(t,y^w\left(f_1(t)\right),\ldots,y^w\left(f_k(t)\right)\right)\right.\\
&\qquad\left.-f\left(t,y\left(f_1(t)\right),\ldots,y\left(f_k(t)\right)\right)\right|\\
&<\frac{\varepsilon}{16}+\frac{2}{T}\sum_{i=T_1+1}^{\infty}H_i+\frac{\varepsilon}{16}+\frac{2}{T}\sum_{i=T}^{T_1}\sum_{s=T_2+1}^{\infty}G_s\\
&+\frac{2}{T}\sum_{i=T_1+1}^{\infty}\sum_{s=i}^{\infty}G_s+\frac{\varepsilon}{16}+\frac{2}{T}\sum_{i=T}^{T_1}\sum_{s=i}^{T_2}\sum_{t=T_3+1}^{\infty}F_t\\
&+\frac{2}{T}\sum_{i=T}^{T_1}\sum_{s=T_2+1}^{\infty}\sum_{t=s}^{\infty}F_t\\
&+\frac{2}{T}\sum_{i=T_1+1}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}F_t<\varepsilon,\quad w\ge T_4,
\end{aligned}
\tag{30}
$$

which implies that G_L is continuous in $\Omega(A_*,B^*,T)$.

It follows from (15), (22), and (28) that for any $x=\{x(n)\}_{n\in\mathbb{N}_\beta}\in\Omega(A_*,B^*,T)$ and $t_1,t_2\ge T_4$

$$
\begin{aligned}
&\left|\frac{(G_Ly)(t_2)}{t_2}-\frac{(G_Ly)(t_1)}{t_1}\right|\\
&=\left|\frac{1}{t_2}\sum_{i=t_2}^{\infty}h\left(i,y\left(h_1(i)\right),\ldots,y\left(h_k(i)\right)\right)\right.\\
&\quad-\frac{1}{t_1}\sum_{i=t_1}^{\infty}h\left(i,y\left(h_1(i)\right),\ldots,y\left(h_k(i)\right)\right)\\
&\quad-\frac{1}{t_2}\sum_{i=t_2}^{\infty}\sum_{s=i}^{\infty}g\left(s,y\left(g_1(s)\right),\ldots,y\left(g_k(s)\right)\right)\\
&\quad+\frac{1}{t_1}\sum_{i=t_1}^{\infty}\sum_{s=i}^{\infty}g\left(s,y\left(g_1(s)\right),\ldots,y\left(g_k(s)\right)\right)\\
&\quad+\frac{1}{t_2}\sum_{i=t_2}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[f\left(t,y\left(f_1(t)\right),\ldots,y\left(f_k(t)\right)\right)-d(t)\right]\\
&\quad\left.-\frac{1}{t_1}\sum_{i=t_1}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[f\left(t,y\left(f_1(t)\right),\ldots,y\left(f_k(t)\right)\right)-d(t)\right]\right|\\
&\le\frac{2}{T_4}\left(\sum_{i=T_4}^{\infty}H_i+\sum_{i=T_4}^{\infty}\sum_{s=i}^{\infty}G_s+\sum_{i=T_4}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[F_t+|d(t)|\right]\right)\\
&<\varepsilon,
\end{aligned}
\tag{31}
$$

which means that $G_L(\Omega(A_*,B^*,T))$ is uniformly Cauchy, which together with (25) and Lemma 1 yields that $G_L(\Omega(A_*,B^*,T))$ is relatively compact. Hence G_L is completely continuous in $\Omega(A_*,B^*,T)$. Thus (14), (24), and Lemma 2 ensure that the mapping S_L+G_L has a fixed point $x=\{x(n)\}_{n\in\mathbb{N}_\beta}$, which together with (21) and (22) implies that

$$
\begin{aligned}
&x(n)\\
&=nL-b(n)x(n-\tau)-c(n)\\
&\quad+\sum_{i=n}^{\infty}h\left(i,x\left(h_1(i)\right),\ldots,x\left(h_k(i)\right)\right)\\
&\quad-\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}g\left(s,x\left(g_1(s)\right),\ldots,x\left(g_k(s)\right)\right)\\
&\quad+\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[f\left(t,x\left(f_1(t)\right),\ldots,x\left(f_k(t)\right)\right)-d(t)\right],\\
&\qquad n\ge T;
\end{aligned}
\tag{32}
$$

which gives that

$$
\begin{aligned}
&\Delta\left(x(n)+b(n)x(n-\tau)+c(n)\right)\\
&=L-h\left(n,x\left(h_1(n)\right),\ldots,x\left(h_k(n)\right)\right)\\
&\quad+\sum_{s=n}^{\infty}g\left(s,x\left(g_1(s)\right),\ldots,x\left(g_k(s)\right)\right)\\
&\quad-\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\left[f\left(t,x\left(f_1(t)\right),\ldots,x\left(f_k(t)\right)\right)-d(t)\right],\\
&\qquad n\ge T+\tau,
\end{aligned}
$$

$$\begin{aligned}
&\Delta^2 (x(n) + b(n) x(n-\tau) + c(n)) \\
&= -\Delta h(n, x(h_1(n)), \ldots, x(h_k(n))) \\
&\quad - g(n, x(g_1(n)), \ldots, x(g_k(n))) \\
&\quad + \sum_{t=n}^{\infty} [f(t, x(f_1(t)), \ldots, x(f_k(t))) - d(t)], \\
&\qquad n \ge T + \tau, \\
&\Delta^3 (x(n) + b(n) x(n-\tau) + c(n)) \\
&= -\Delta^2 h(n, x(h_1(n)), \ldots, x(h_k(n))) \\
&\quad - \Delta g(n, x(g_1(n)), \ldots, x(g_k(n))) \\
&\quad - [f(n, x(f_1(n)), \ldots, x(f_k(n))) - d(n)], \\
&\qquad n \ge T + \tau,
\end{aligned} \tag{33}$$

that is, $x = \{x(n)\}_{n\in\mathbb{N}_\beta} \in \Omega(A_*, B^*, T)$ is a positive solution of (4). It follows from (15)–(17), and (32) that

$$\begin{aligned}
&\left|\frac{x(n) + b(n) x(n-\tau) + c(n)}{n} - L\right| \\
&= \Bigg|\frac{1}{n}\sum_{i=n}^{\infty} h(i, x(h_1(i)), \ldots, x(h_k(i))) \\
&\quad - \frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty} g(s, x(g_1(s)), \ldots, x(g_k(s))) \\
&\quad + \frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty} [f(t, x(f_1(t)), \ldots, x(f_k(t))) - d(t)]\Bigg| \\
&\le \frac{1}{n}\sum_{i=n}^{\infty} H_i + \frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty} G_s + \frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty} F_t \longrightarrow 0 \\
&\qquad \text{as } n \longrightarrow \infty, \\
&A_* = A - c^* \le A - \frac{|c(n)|}{n} \\
&= A(n) \le \frac{x(n)}{n} \le B(n) = B + \frac{|c(n)|}{n} \\
&\le B + c^* = B^*, \quad n \ge T,
\end{aligned} \tag{34}$$

which yield that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{x(n) + b(n) x(n-\tau) + c(n)}{n} = L \in (A + b^* B^*, B), \\
&A_* \le \liminf_{n\to\infty} \frac{x(n)}{n} \le \limsup_{n\to\infty} \frac{x(n)}{n} \le B^*;
\end{aligned} \tag{35}$$

that is, (18) and (19) hold.

(ii) Let $L_1, L_2 \in (A+b^*B^*, B)$ and $L_1 \ne L_2$. As in the proof of (i), we infer that for each $l \in \{1, 2\}$, there exist a constant $T_l \ge n_0 + \tau + \alpha$ and two mappings S_{L_l} and $G_{L_l} : \Omega(A_*, B^*, T_l) \to l_\beta^\infty$ satisfying (20)–(22), where T, L, S_L and G_L are replaced by T_l, L_l, S_{L_l} and G_{L_l}, respectively, and $S_{L_l} + G_{L_l}$ possesses a fixed point $x_l = \{x_l(n)\}_{n\in\mathbb{N}_\beta}$, which is a positive solution of (4); that is,

$$\begin{aligned}
x_l(n) &= nL_l - b(n) x_l(n-\tau) - c(n) \\
&\quad + \sum_{i=n}^{\infty} h(i, x_l(h_1(i)), \ldots, x_l(h_k(i))) \\
&\quad - \sum_{i=n}^{\infty}\sum_{s=i}^{\infty} g(s, x_l(g_1(s)), \ldots, x_l(g_k(s))) \\
&\quad + \sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty} [f(t, x_l(f_1(t)), \ldots, x_l(f_k(t))) - d(t)], \\
&\qquad n \ge T_l.
\end{aligned} \tag{36}$$

Note that (16) and (17) imply that there exists $T_* > \max\{T_1, T_2\}$ satisfying

$$\frac{1}{T_*}\left(\sum_{i=T_*}^{\infty} H_i + \sum_{i=T_*}^{\infty}\sum_{s=i}^{\infty} G_s + \sum_{i=T_*}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty} F_t\right) < \frac{|L_1 - L_2|}{4}, \tag{37}$$

which together with (15) and (36) means that for any $n \ge T_*$

$$\begin{aligned}
&\left|\frac{x_1(n)}{n} - \frac{x_2(n)}{n}\right| \\
&= \Bigg|L_1 - L_2 - \frac{b(n)}{n}(x_1(n-\tau) - x_2(n-\tau)) \\
&\quad + \frac{1}{n}\sum_{i=n}^{\infty} [h(i, x_1(h_1(i)), \ldots, x_1(h_k(i))) \\
&\qquad - h(i, x_2(h_1(i)), \ldots, x_2(h_k(i)))] \\
&\quad - \frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty} [g(s, x_1(g_1(s)), \ldots, x_1(g_k(s))) \\
&\qquad - g(s, x_2(g_1(s)), \ldots, x_2(g_k(s)))] \\
&\quad + \frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty} [f(t, x_1(f_1(t)), \ldots, x_1(f_k(t))) \\
&\qquad - f(t, x_2(f_1(t)), \ldots, x_2(f_k(t)))]\Bigg|
\end{aligned}$$

$$\begin{aligned}
&\geq |L_1 - L_2| \\
&\quad - \frac{(n-\tau) b(n)}{n} \left| \frac{x_1(n-\tau) - x_2(n-\tau)}{n-\tau} \right| \\
&\quad - \frac{1}{T_*} \sum_{i=T_*}^{\infty} \left| h(i, x_1(h_1(i)), \dots, x_1(h_k(i))) \right. \\
&\qquad \left. - h(i, x_2(h_1(i)), \dots, x_2(h_k(i))) \right| \\
&\quad - \frac{1}{T_*} \sum_{i=T_*}^{\infty} \sum_{s=i}^{\infty} \left| g(s, x_1(g_1(s)), \dots, x_1(g_k(s))) \right. \\
&\qquad \left. - g(s, x_2(g_1(s)), \dots, x_2(g_k(s))) \right| \\
&\quad - \frac{1}{T_*} \sum_{i=T_*}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} \left| f(t, x_1(f_1(t)), \dots, x_1(f_k(t))) \right. \\
&\qquad \left. - f(t, x_2(f_1(t)), \dots, x_2(f_k(t))) \right| \\
&\geq |L_1 - L_2| - b^* \|x_1 - x_2\| \\
&\quad - \frac{2}{T_*} \left(\sum_{i=T_*}^{\infty} H_i + \sum_{i=T_*}^{\infty} \sum_{s=i}^{\infty} G_s + \sum_{i=T_*}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} F_t \right) \\
&> \frac{|L_1 - L_2|}{2} - b^* \|x_1 - x_2\|,
\end{aligned} \tag{38}$$

which implies that

$$\|x_1 - x_2\| > \frac{|L_1 - L_2|}{2(1+b^*)} > 0, \tag{39}$$

which yields that $x_1 \neq x_2$. Therefore (4) possesses uncountably many positive solutions in l_β^∞. This completes the proof. □

Theorem 6. *Assume that there exist constants b_* and b^* and three nonnegative sequences $\{F_n\}_{n\in\mathbb{N}_{n_0}}$, $\{G_n\}_{n\in\mathbb{N}_{n_0}}$, and $\{H_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying (15)–(17) and*

$$\begin{gathered}
b^* A + (B^* + c^*) \frac{b^*}{b_*} < b_* B + \frac{b_* A_*}{b^*} - c^*, \\
1 < b_* \leq b(n) \leq b^*, \quad n \in \mathbb{N}_{n_0}.
\end{gathered} \tag{40}$$

Then

(i) *equation (4) possesses a positive solution $x = \{x(n)\}_{n\in\mathbb{N}_\beta} \in l_\beta^\infty$ with (19) and*

$$\begin{aligned}
&\lim_{n\to\infty} \frac{x(n) + b(n) x(n-\tau) + c(n)}{n} \\
&\quad \in \left(b^* A + (B^* + c^*) \frac{b^*}{b_*}, b_* B + \frac{b_* A_*}{b^*} - c^* \right);
\end{aligned} \tag{41}$$

(ii) *equation (4) possesses uncountably many positive solutions in l_β^∞.*

Proof. (i) Put $L \in (b^* A + (B^* + c^*)(b^*/b_*), b_* B + b_* A_*/b^* - c^*)$. Observe that

$$\begin{aligned}
&\lim_{n\to\infty} \left[b^* A + \left(1 + \frac{\tau}{n}\right)(B^* + c^*) \frac{b^*}{b_*} \right] \\
&\quad = b^* A + (B^* + c^*) \frac{b^*}{b_*} < L < b_* B + \frac{b_* A_*}{b^*} - c^* \\
&\quad = \lim_{n\to\infty} \left[b_* B + \frac{b_* A_*}{b^*} - c^* \left(1 + \frac{\tau}{n}\right) \right],
\end{aligned} \tag{42}$$

which yields that there exists $N \in \mathbb{N}$ satisfying

$$\begin{aligned}
b^* A + (B^* + c^*) \frac{b^*}{b_*} &< b^* A + \left(1 + \frac{\tau}{N}\right)(B^* + c^*) \frac{b^*}{b_*} \\
&< L < b_* B + \frac{b_* A_*}{b^*} - c^* \left(1 + \frac{\tau}{N}\right) \\
&< b_* B + \frac{b_* A_*}{b^*} - c^*.
\end{aligned} \tag{43}$$

It follows from (16) and (17) that there exist $\theta \in (0,1)$ and $T > \max\{n_0 + \tau + \alpha, N\}$ satisfying

$$\theta = \frac{1}{b_*} \left(1 + \frac{\tau}{T}\right), \tag{44}$$

$$\begin{aligned}
&\frac{1}{T} \sum_{i=T+\tau}^{\infty} \left\{ H_i + \sum_{s=i}^{\infty} G_s + \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} [F_t + |d(t)|] \right\} \\
&\quad < \min \left\{ b_* B + \frac{b_* A_*}{b^*} - c^* \left(1 + \frac{\tau}{N}\right) - L, \right. \\
&\qquad \left. \frac{b_* L}{b^*} - b_* A - \left(1 + \frac{\tau}{N}\right)(B^* + c^*) \right\} \\
&\quad < \min \left\{ b_* B + \frac{b_* A_*}{b^*} - c^* \left(1 + \frac{\tau}{T}\right) - L, \right. \\
&\qquad \left. \frac{b_* L}{b^*} - b_* A - \left(1 + \frac{\tau}{T}\right)(B^* + c^*) \right\}.
\end{aligned} \tag{45}$$

Define two mappings S_L and $G_L : \Omega(A_*, B^*, T) \to l_\beta^\infty$ by

$$\begin{aligned}
&(S_L x)(n) \\
&= \begin{cases} \dfrac{nL}{b(n+\tau)} - \dfrac{x(n+\tau)}{b(n+\tau)} - \dfrac{c(n+\tau)}{b(n+\tau)}, & n \geq T, \\ \dfrac{n}{T} (S_L x)(T), & \beta \leq n < T, \end{cases}
\end{aligned} \tag{46}$$

$$(G_Lx)(n) = \begin{cases} \dfrac{1}{b(n+\tau)}\displaystyle\sum_{i=n+\tau}^{\infty} h(i, x(h_1(i)), \ldots, x(h_k(i))) \\ \quad -\dfrac{1}{b(n+\tau)}\displaystyle\sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty} g(s, x(g_1(s)), \ldots, x(g_k(s))) \\ \quad +\dfrac{1}{b(n+\tau)} \\ \quad \cdot \displaystyle\sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty} [f(t, x(f_1(t)), \ldots, x(f_k(t))) \\ \qquad -d(t)], \quad n \geq T, \\ \dfrac{n}{T}(G_Lx)(T), \qquad \beta \leq n < T \end{cases} \tag{47}$$

for any $x = \{x(n)\}_{n\in\mathbb{N}_\beta} \in \Omega(A_*, B^*, T)$.

Now we show that (23), (48), and (49) below hold

$$\|S_Lx - S_Ly\| \leq \theta\|x - y\|, \quad x, y \in \Omega(A_*, B^*, T); \tag{48}$$

$$\|G_Ly\| \leq B + \frac{A_*}{b^*}, \quad y \in \Omega(A_*, B^*, T). \tag{49}$$

Using (15) and (44)–(47), we get that for any $x = \{x(n)\}_{n\in\mathbb{N}_\beta}$, $y = \{y(n)\}_{n\in\mathbb{N}_\beta} \in \Omega(A_*, B^*, T)$

$$\begin{aligned} &\frac{(S_Lx)(n) + (G_Ly)(n)}{n} \\ &= \frac{L}{b(n+\tau)} - \frac{x(n+\tau)}{nb(n+\tau)} - \frac{c(n+\tau)}{nb(n+\tau)} \\ &\quad + \frac{1}{nb(n+\tau)}\sum_{i=n+\tau}^{\infty} h(i, y(h_1(i)), \ldots, y(h_k(i))) \\ &\quad - \frac{1}{nb(n+\tau)}\sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty} g(s, y(g_1(s)), \ldots, y(g_k(s))) \\ &\quad + \frac{1}{nb(n+\tau)} \\ &\quad \cdot \sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty} [f(t, y(f_1(t)), \ldots, y(f_k(t))) - d(t)] \\ &\leq \frac{L}{b_*} - \frac{x(n+\tau)}{(n+\tau)b(n+\tau)} + \frac{n+\tau}{nb(n+\tau)} \\ &\quad \cdot \frac{|c(n+\tau)|}{n+\tau} + \frac{1}{nb(n+\tau)} \\ &\quad \cdot \sum_{i=n+\tau}^{\infty} |h(i, y(h_1(i)), \ldots, y(h_k(i)))| + \frac{1}{nb(n+\tau)} \\ &\quad \cdot \sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty} |g(s, y(g_1(s)), \ldots, y(g_k(s)))| \\ &\quad + \frac{1}{nb(n+\tau)} \\ &\quad \cdot \sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty} [|f(t, y(f_1(t)), \ldots, y(f_k(t)))| + |d(t)|] \\ &\leq \frac{L}{b_*} - \frac{A_*}{b^*} + \frac{c^*}{b_*}\left(1 + \frac{\tau}{T}\right) + \frac{1}{Tb_*}\sum_{i=T+\tau}^{\infty} H_i \\ &\quad + \frac{1}{Tb_*}\sum_{i=T+\tau}^{\infty}\sum_{s=i}^{\infty} G_s + \frac{1}{Tb_*}\sum_{i=T+\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty} [F_t + |d(t)|] \\ &< \frac{L}{b_*} - \frac{A_*}{b^*} + \frac{c^*}{b_*}\left(1 + \frac{\tau}{T}\right) \\ &\quad + \frac{1}{b_*}\min\left\{b_*B + \frac{b_*A_*}{b^*} - c^*\left(1 + \frac{\tau}{T}\right) - L, \right. \\ &\qquad \left. \frac{b_*}{b^*}L - b_*A - \left(1 + \frac{\tau}{T}\right)(B^* + c^*)\right\} \\ &\leq B \leq B(n), \quad n \geq T, \end{aligned}$$

$$\begin{aligned} \frac{(S_Lx)(n) + (G_Ly)(n)}{n} &= \frac{n}{T} \cdot \frac{(S_Lx)(T) + (G_Ly)(T)}{n} \\ &\leq B(T), \quad \beta \leq n < T, \end{aligned}$$

$$\begin{aligned} &\frac{(S_Lx)(n) + (G_Ly)(n)}{n} \\ &= \frac{L}{b(n+\tau)} - \frac{x(n+\tau)}{nb(n+\tau)} - \frac{c(n+\tau)}{nb(n+\tau)} \\ &\quad + \frac{1}{nb(n+\tau)}\sum_{i=n+\tau}^{\infty} h(i, y(h_1(i)), \ldots, y(h_k(i))) \\ &\quad - \frac{1}{nb(n+\tau)}\sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty} g(s, y(g_1(s)), \ldots, y(g_k(s))) \\ &\quad + \frac{1}{nb(n+\tau)} \\ &\quad \cdot \sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty} [f(t, y(f_1(t)), \ldots, y(f_k(t))) - d(t)] \\ &\geq \frac{L}{b^*} - \frac{n+\tau}{nb(n+\tau)} \cdot \frac{x(n+\tau)}{n+\tau} - \frac{n+\tau}{nb(n+\tau)} \\ &\quad \cdot \frac{|c(n+\tau)|}{n+\tau} - \frac{1}{nb(n+\tau)} \end{aligned}$$

$$\cdot \sum_{i=n+\tau}^{\infty} |h(i, y(h_1(i)), \dots, y(h_k(i)))|$$

$$- \frac{1}{nb(n+\tau)} \sum_{i=n+\tau}^{\infty} \sum_{s=i}^{\infty} |g(s, y(g_1(s)), \dots, y(g_k(s)))|$$

$$- \frac{1}{nb(n+\tau)}$$

$$\cdot \sum_{i=n+\tau}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} [|f(t, y(f_1(t)), \dots, y(f_k(t)))| + |d(t)|]$$

$$\geq \frac{L}{b^*} - \frac{1}{b_*}\left(1+\frac{\tau}{T}\right) B^* - \frac{c^*}{b_*}\left(1+\frac{\tau}{T}\right)$$

$$- \frac{1}{Tb_*} \sum_{i=T+\tau}^{\infty} H_i - \frac{1}{Tb_*} \sum_{i=T+\tau}^{\infty} \sum_{s=i}^{\infty} G_s - \frac{1}{Tb_*}$$

$$\cdot \sum_{i=T+\tau}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} [F_t + |d(t)|]$$

$$> \frac{L}{b^*} - \frac{1}{b_*}\left(1+\frac{\tau}{T}\right)(B^* + c^*)$$

$$- \frac{1}{b_*} \min \left\{ b_* B + \frac{b_* A_*}{b^*} - c^*\left(1+\frac{\tau}{T}\right) - L, \right.$$

$$\left. \frac{b_*}{b^*} L - b_* A - \left(1+\frac{\tau}{T}\right)(B^* + c^*) \right\}$$

$$\geq A \geq A(n), \quad n \geq T,$$

$$\frac{(S_L x)(n) + (G_L y)(n)}{n} = \frac{n}{T} \cdot \frac{(S_L x)(T) + (G_L y)(T)}{n}$$

$$\geq A(T), \quad \beta \leq n < T,$$

$$\left| \frac{(S_L x)(n) - (S_L y)(n)}{n} \right| = \frac{n+\tau}{nb(n+\tau)} \left| \frac{x(n+\tau) - y(n+\tau)}{n+\tau} \right|$$

$$\leq \theta \|x - y\|, \quad n \geq T,$$

$$\left| \frac{(S_L x)(n) - (S_L y)(n)}{n} \right| = \frac{n}{T} \left| \frac{(S_L x)(T) - (S_L y)(T)}{n} \right|$$

$$\leq \theta \|x - y\|, \quad \beta \leq n < T,$$

$$\left| \frac{(G_L y)(n)}{n} \right|$$

$$= \left| \frac{1}{nb(n+\tau)} \sum_{i=n+\tau}^{\infty} h(i, y(h_1(i)), \dots, y(h_k(i))) \right.$$

$$- \frac{1}{nb(n+\tau)} \sum_{i=n+\tau}^{\infty} \sum_{s=i}^{\infty} g(s, y(g_1(s)), \dots, y(g_k(s)))$$

$$+ \frac{1}{nb(n+\tau)}$$

$$\left. \cdot \sum_{i=n+\tau}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} [f(t, y(f_1(t)), \dots, y(f_k(t))) - d(t)] \right|$$

$$\leq \frac{1}{nb(n+\tau)}$$

$$\cdot \sum_{i=n+\tau}^{\infty} |h(i, y(h_1(i)), \dots, y(h_k(i)))|$$

$$+ \frac{1}{nb(n+\tau)}$$

$$\cdot \sum_{i=n+\tau}^{\infty} \sum_{s=i}^{\infty} |g(s, y(g_1(s)), \dots, y(g_k(s)))|$$

$$+ \frac{1}{nb(n+\tau)}$$

$$\cdot \sum_{i=n+\tau}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} [|f(t, y(f_1(t)), \dots, y(f_k(t)))| + |d(t)|]$$

$$\leq \frac{1}{Tb_*} \sum_{i=T+\tau}^{\infty} H_i + \frac{1}{Tb_*} \sum_{i=T+\tau}^{\infty} \sum_{s=i}^{\infty} G_s$$

$$+ \frac{1}{Tb_*} \sum_{i=T+\tau}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} [F_t + |d(t)|]$$

$$< \frac{1}{b_*} \min \left\{ b_* B + \frac{b_* A_*}{b^*} - c^*\left(1+\frac{\tau}{T}\right) - L, \right.$$

$$\left. \frac{b_*}{b^*} L - b_* A - \left(1+\frac{\tau}{T}\right)(B^* + c^*) \right\}$$

$$\leq B + \frac{A_*}{b^*}, \quad n \geq T,$$

$$\left| \frac{(G_L y)(n)}{n} \right| = \frac{n}{T} \left| \frac{(G_L y)(T)}{n} \right| \leq B + \frac{A_*}{b^*}, \quad \beta \leq n < T, \tag{50}$$

which yield (23), (48), and (49).

Next we show that G_L is completely continuous. Let $y^w = \{y^w(n)\}_{n \in \mathbb{N}_\beta}$ and $y = \{y(n)\}_{n \in \mathbb{N}_\beta} \in \Omega(A_*, B^*, T)$ with (27). Using (16), (17), (27), (47), and the continuity of f, g, and h, we know that for given $\varepsilon > 0$, there exist T_1, T_2, T_3, and $T_4 \in \mathbb{N}$ with $T_4 > T_3 > T_2 > T_1 > T + \tau$ satisfying

$$\frac{1}{Tb_*} \max \left\{ \sum_{i=T_1+1}^{\infty} H_i + \sum_{i=T_1+1}^{\infty} \sum_{s=i}^{\infty} G_s \right.$$

$$+ \sum_{i=T_1+1}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} [F_t + |d(t)|], \sum_{i=T+\tau}^{T_1} \sum_{s=T_2+1}^{\infty} G_s$$

$$+\sum_{i=T+\tau}^{T_1}\sum_{s=T_2+1}^{\infty}\sum_{t=s}^{\infty}F_t+\sum_{i=T+\tau}^{T_1}\sum_{s=i}^{T_2}\sum_{t=T_3+1}^{\infty}F_t\Bigg\} < \frac{\varepsilon}{16}, \tag{51}$$

$$\begin{aligned}\frac{1}{Tb_*}\max\Bigg\{&\sum_{i=T+\tau}^{T_1}\left|h(i,y^w(h_1(i)),\dots,y^w(h_k(i)))-h(i,y(h_1(i)),\dots,y(h_k(i)))\right|,\\&\sum_{i=T+\tau}^{T_1}\sum_{s=i}^{T_2}\left|g(s,y^w(g_1(s)),\dots,y^w(g_k(s)))-g(s,y(g_1(s)),\dots,y(g_k(s)))\right|,\\&\sum_{i=T+\tau}^{T_1}\sum_{s=i}^{T_2}\sum_{t=s}^{T_3}\left|f(t,y^w(f_1(t)),\dots,y^w(f_k(t)))-f(t,y(f_1(t)),\dots,y(f_k(t)))\right|\Bigg\}\\&<\frac{\varepsilon}{16},\quad w\ge T_4.\end{aligned} \tag{52}$$

Combining (15), (47), (51), and (52), we infer that

$$\begin{aligned}&\|G_Ly^w-G_Ly\|\\&=\sup\left\{\frac{|(G_Ly^w)(n)-(G_Ly)(n)|}{n}:n\in\mathbb{N}_\beta\right\}\\&=\max\Bigg\{\sup\left\{\frac{n}{T}\cdot\frac{|(G_Ly^w)(T)-(G_Ly)(T)|}{n}:\beta\le n<T\right\},\\&\qquad\sup\left\{\frac{|(G_Ly^w)(n)-(G_Ly)(n)|}{n}:n\in\mathbb{N}_T\right\}\Bigg\}\\&\le\sup\Bigg\{\frac{1}{nb(n+\tau)}\cdot\sum_{i=n+\tau}^{\infty}\left|h(i,y^w(h_1(i)),\dots,y^w(h_k(i)))-h(i,y(h_1(i)),\dots,y(h_k(i)))\right|\\&\quad+\frac{1}{nb(n+\tau)}\cdot\sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty}\left|g(s,y^w(g_1(s)),\dots,y^w(g_k(s)))-g(s,y(g_1(s)),\dots,y(g_k(s)))\right|\\&\quad+\frac{1}{nb(n+\tau)}\cdot\sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left|f(t,y^w(f_1(t)),\dots,y^w(f_k(t)))-f(t,y(f_1(t)),\dots,y(f_k(t)))\right|:\\&\qquad n\in\mathbb{N}_T\Bigg\}\\&\le\frac{1}{Tb_*}\sum_{i=T+\tau}^{T_1}\left|h(i,y^w(h_1(i)),\dots,y^w(h_k(i)))-h(i,y(h_1(i)),\dots,y(h_k(i)))\right|\\&\quad+\frac{1}{Tb_*}\sum_{i=T_1+1}^{\infty}\left|h(i,y^w(h_1(i)),\dots,y^w(h_k(i)))-h(i,y(h_1(i)),\dots,y(h_k(i)))\right|\\&\quad+\frac{1}{Tb_*}\sum_{i=T+\tau}^{T_1}\sum_{s=i}^{T_2}\left|g(s,y^w(g_1(s)),\dots,y^w(g_k(s)))-g(s,y(g_1(s)),\dots,y(g_k(s)))\right|\\&\quad+\frac{1}{Tb_*}\sum_{i=T+\tau}^{T_1}\sum_{s=T_2+1}^{\infty}\left|g(s,y^w(g_1(s)),\dots,y^w(g_k(s)))-g(s,y(g_1(s)),\dots,y(g_k(s)))\right|\\&\quad+\frac{1}{Tb_*}\sum_{i=T_1+1}^{\infty}\sum_{s=i}^{\infty}\left|g(s,y^w(g_1(s)),\dots,y^w(g_k(s)))-g(s,y(g_1(s)),\dots,y(g_k(s)))\right|\\&\quad+\frac{1}{Tb_*}\sum_{i=T+\tau}^{T_1}\sum_{s=i}^{T_2}\sum_{t=s}^{T_3}\left|f(t,y^w(f_1(t)),\dots,y^w(f_k(t)))-f(t,y(f_1(t)),\dots,y(f_k(t)))\right|\\&\quad+\frac{1}{Tb_*}\sum_{i=T+\tau}^{T_1}\sum_{s=i}^{T_2}\sum_{t=T_3+1}^{\infty}\left|f(t,y^w(f_1(t)),\dots,y^w(f_k(t)))-f(t,y(f_1(t)),\dots,y(f_k(t)))\right|\\&\quad+\frac{1}{Tb_*}\sum_{i=T+\tau}^{T_1}\sum_{s=T_2+1}^{\infty}\sum_{t=s}^{\infty}\left|f(t,y^w(f_1(t)),\dots,y^w(f_k(t)))-f(t,y(f_1(t)),\dots,y(f_k(t)))\right|\end{aligned}$$

$$\begin{aligned}
&+\frac{1}{Tb_*}\sum_{i=T_1+1}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left|f\left(t,y^w\left(f_1(t)\right),\ldots,y^w\left(f_k(t)\right)\right)\right.\\
&\qquad\left.-f\left(t,y\left(f_1(t)\right),\ldots,y\left(f_k(t)\right)\right)\right|\\
&<\frac{\varepsilon}{16}+\frac{2}{Tb_*}\sum_{i=T_1+1}^{\infty}H_i+\frac{\varepsilon}{16}\\
&+\frac{2}{Tb_*}\sum_{i=T+\tau}^{T_1}\sum_{s=T_2+1}^{\infty}G_s+\frac{2}{Tb_*}\sum_{i=T_1+1}^{\infty}\sum_{s=i}^{\infty}G_s\\
&+\frac{\varepsilon}{16}+\frac{2}{Tb_*}\sum_{i=T+\tau}^{T_1}\sum_{s=i}^{T_2}\sum_{t=T_3+1}^{\infty}F_t\\
&+\frac{2}{Tb_*}\sum_{i=T+\tau}^{T_1}\sum_{s=T_2+1}^{\infty}\sum_{t=s}^{\infty}F_t+\frac{2}{Tb_*}\sum_{i=T_1+1}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}F_t\\
&<\varepsilon,\quad w\ge T_4,
\end{aligned}\tag{53}$$

which implies that G_L is continuous in $\Omega(A_*,B^*,T)$.

It follows from (47) and (51) that for any $x=\{x(n)\}_{n\in\mathbb{N}_\beta}\in\Omega(A_*,B^*,T)$ and $t_1,t_2\ge T_4$

$$\begin{aligned}
&\left|\frac{(G_Ly)(t_2)}{t_2}-\frac{(G_Ly)(t_1)}{t_1}\right|\\
&=\left|\frac{1}{t_2b(n+\tau)}\cdot\sum_{i=t_2+\tau}^{\infty}h\left(i,y\left(h_1(i)\right),\ldots,y\left(h_k(i)\right)\right)\right.\\
&-\frac{1}{t_1b(n+\tau)}\cdot\sum_{i=t_1+\tau}^{\infty}h\left(i,y\left(h_1(i)\right),\ldots,y\left(h_k(i)\right)\right)\\
&-\frac{1}{t_2b(n+\tau)}\cdot\sum_{i=t_2+\tau}^{\infty}\sum_{s=i}^{\infty}g\left(s,y\left(g_1(s)\right),\ldots,y\left(g_k(s)\right)\right)\\
&+\frac{1}{t_1b(n+\tau)}\cdot\sum_{i=t_1+\tau}^{\infty}\sum_{s=i}^{\infty}g\left(s,y\left(g_1(s)\right),\ldots,y\left(g_k(s)\right)\right)\\
&+\frac{1}{t_2b(n+\tau)}\cdot\sum_{i=t_2+\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[f\left(t,y\left(f_1(t)\right),\ldots,y\left(f_k(t)\right)\right)-d(t)\right]\\
&-\frac{1}{t_1b(n+\tau)}\left.\cdot\sum_{i=t_1+\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[f\left(t,y\left(f_1(t)\right),\ldots,y\left(f_k(t)\right)\right)-d(t)\right]\right|\\
&\le\frac{2}{T_4b_*}\cdot\left(\sum_{i=T_4+\tau}^{\infty}H_i+\sum_{i=T_4+\tau}^{\infty}\sum_{s=i}^{\infty}G_s+\sum_{i=T_4+\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[F_t+|d(t)|\right]\right)\\
&<\varepsilon,
\end{aligned}\tag{54}$$

which means that $G_L(\Omega(A_*,B^*,T))$ is uniformly Cauchy, which together with (49) and Lemma 1 yields that $G_L(\Omega(A_*,B^*,T))$ is relatively compact. That is, G_L is completely continuous in $\Omega(A_*,B^*,T)$. Thus (44), (48), and Lemma 2 ensure that the mapping S_L+G_L has a fixed point $x=\{x(n)\}_{n\in\mathbb{N}_\beta}\in\Omega(A_*,B^*,T)$, which together with (46) and (47) yields that

$$\begin{aligned}
x(n)&=\frac{nL}{b(n+\tau)}-\frac{x(n+\tau)}{b(n+\tau)}-\frac{c(n+\tau)}{b(n+\tau)}\\
&+\frac{1}{b(n+\tau)}\sum_{i=n+\tau}^{\infty}h\left(i,x\left(h_1(i)\right),\ldots,x\left(h_k(i)\right)\right)\\
&-\frac{1}{b(n+\tau)}\sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty}g\left(s,x\left(g_1(s)\right),\ldots,x\left(g_k(s)\right)\right)\\
&+\frac{1}{b(n+\tau)}\cdot\sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[f\left(t,x\left(f_1(t)\right),\ldots,x\left(f_k(t)\right)\right)-d(t)\right],\\
&\qquad n\ge T,
\end{aligned}\tag{55}$$

which means that

$$\begin{aligned}
&x(n+\tau)+b(n+\tau)x(n)+c(n+\tau)\\
&=nL+\sum_{i=n+\tau}^{\infty}h\left(i,x\left(h_1(i)\right),\ldots,x\left(h_k(i)\right)\right)\\
&-\sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty}g\left(s,x\left(g_1(s)\right),\ldots,x\left(g_k(s)\right)\right)\\
&+\sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[f\left(t,x\left(f_1(t)\right),\ldots,x\left(f_k(t)\right)\right)-d(t)\right],\\
&\qquad n\ge T.
\end{aligned}\tag{56}$$

It follows from (56) that

$$\begin{aligned}
&\Delta\left(x(n)+b(n)x(n-\tau)+c(n)\right)\\
&\quad= L-h\left(n,x\left(h_1(n)\right),\dots,x\left(h_k(n)\right)\right)\\
&\qquad+\sum_{s=n}^{\infty}g\left(s,x\left(g_1(s)\right),x\left(g_2(s)\right),\dots,x\left(g_k(s)\right)\right)\\
&\qquad-\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\left[f\left(t,x\left(f_1(t)\right),\dots,x\left(f_k(t)\right)\right)-d(t)\right],\\
&\qquad\qquad n\ge T+\tau;\\
&\Delta^3\left(x(n)+b(n)x(n-\tau)+c(n)\right)\\
&\quad=-\Delta^2h\left(n,x\left(h_1(n)\right),\dots,x\left(h_k(n)\right)\right)\\
&\qquad-\Delta g\left(n,x\left(g_1(n)\right),\dots,x\left(g_k(n)\right)\right)\\
&\qquad-\left[f\left(n,x\left(f_1(n)\right),\dots,x\left(f_k(n)\right)\right)-d(n)\right],\\
&\qquad\qquad n\ge T+\tau,
\end{aligned} \tag{57}$$

that is, $x=\{x(n)\}_{n\in\mathbb{N}_\beta}\in\Omega(A_*,B^*,T)$ is a positive solution of (4). By means of (15)–(17) and (56), we deduce that

$$\begin{aligned}
&\left|\frac{x(n)+b(n)x(n-\tau)+c(n)}{n}-L\right|\\
&\quad=\left|-\frac{\tau}{n}+\frac{1}{n}\sum_{i=n}^{\infty}h\left(i,x\left(h_1(i)\right),\dots,x\left(h_k(i)\right)\right)\right.\\
&\qquad-\frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}g\left(s,x\left(g_1(s)\right),\dots,x\left(g_k(s)\right)\right)\\
&\qquad+\frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[f\left(t,x\left(f_1(t)\right),\dots,x\left(f_k(t)\right)\right)\right.\\
&\qquad\qquad\left.\left.-d(t)\right]\right|\\
&\quad\le\frac{\tau}{n}+\frac{1}{n}\sum_{i=n}^{\infty}H_i+\frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}G_s\\
&\qquad+\frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}F_t\longrightarrow 0\quad\text{as } n\longrightarrow\infty,
\end{aligned} \tag{58}$$

which ensures that (41) holds. The proof of (19) is similar to that of Theorem 5 and is omitted.

(ii) Let $L_1,L_2\in(b^*A+(B^*+c^*)(b^*/b_*),b_*B+b_*A_*/b^*-c^*)$ and $L_1\ne L_2$. As in the proof of (i), we deduce that, for each $l\in\{1,2\}$, there exist $\theta_l\in(0,1)$, $T_l\ge n_0+\tau+\alpha$, and two mappings S_{L_l} and $G_{L_l}:\Omega(A_*,B^*,T_l)\to l_\beta^\infty$ satisfying (43)–(47), where θ,T,L,S_L, and G_L are replaced by θ_l,T_l,L_l,S_{L_l}, and G_{L_l}, respectively, and $S_{L_l}+G_{L_l}$ possesses a fixed point $x_l=\{x_l(n)\}_{n\in\mathbb{N}_\beta}\in\Omega(A_*,B^*,T_l)$, which is a positive solution of (4); that is,

$$\begin{aligned}
x_l(n)&=\frac{nL}{b(n+\tau)}-\frac{x_l(n+\tau)}{b(n+\tau)}-\frac{c(n+\tau)}{b(n+\tau)}+\frac{1}{b(n+\tau)}\\
&\quad\cdot\sum_{i=n+\tau}^{\infty}h\left(i,x_l\left(h_1(i)\right),\dots,x_l\left(h_k(i)\right)\right)-\frac{1}{b(n+\tau)}\\
&\quad\cdot\sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty}g\left(s,x_l\left(g_1(s)\right),\dots,x_l\left(g_k(s)\right)\right)+\frac{1}{b(n+\tau)}\\
&\quad\cdot\sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[f\left(t,x_l\left(f_1(t)\right),\dots,x_l\left(f_k(t)\right)\right)\right.\\
&\qquad\left.-d(t)\right],\quad n\ge T_l.
\end{aligned} \tag{59}$$

Note that (16) and (17) imply that there exists $T_*>\max\{T_1,T_2\}$ satisfying

$$\begin{aligned}
&\frac{1}{T_*b_*}\left(\sum_{i=T_*+\tau}^{\infty}H_i+\sum_{i=T_*+\tau}^{\infty}\sum_{s=i}^{\infty}G_s+\sum_{i=T_*+\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}F_t\right)\\
&\quad<\frac{\left|L_1-L_2\right|}{4b^*}.
\end{aligned} \tag{60}$$

In view of (15), (59), and (60), we infer that for any $n\ge T_*$

$$\begin{aligned}
&\left|\frac{x_1(n)}{n}-\frac{x_2(n)}{n}\right|\\
&\quad=\left|\frac{L_1-L_2}{b(n+\tau)}-\frac{x_1(n+\tau)-x_2(n+\tau)}{nb(n+\tau)}+\frac{1}{nb(n+\tau)}\right.\\
&\qquad\cdot\sum_{i=n+\tau}^{\infty}\left[h\left(i,x_1\left(h_1(i)\right),\dots,x_1\left(h_k(i)\right)\right)\right.\\
&\qquad\qquad\left.-h\left(i,x_2\left(h_1(i)\right),\dots,x_2\left(h_k(i)\right)\right)\right]\\
&\qquad-\frac{1}{nb(n+\tau)}
\end{aligned}$$

$$\cdot \sum_{i=n+\tau}^{\infty} \sum_{s=i}^{\infty} [g(s, x_1(g_1(s)), \ldots, x_1(g_k(s))) - g(s, x_2(g_1(s)), \ldots, x_2(g_k(s)))]$$
$$+ \frac{1}{nb(n+\tau)} \cdot \sum_{i=n+\tau}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} [f(t, x_1(f_1(t)), \ldots, x_1(f_k(t))) - f(t, x_2(f_1(t)), \ldots, x_2(f_k(t)))] \Bigg|$$
$$\geq \frac{|L_1 - L_2|}{b^*} - \frac{n+\tau}{nb(n+\tau)} \left| \frac{x_1(n+\tau) - x_2(n+\tau)}{n+\tau} \right|$$
$$- \frac{1}{T_* b_*} \sum_{i=T_*+\tau}^{\infty} |h(i, x_1(h_1(i)), \ldots, x_1(h_k(i))) - h(i, x_2(h_1(i)), \ldots, x_2(h_k(i)))|$$
$$- \frac{1}{T_* b_*} \sum_{i=T_*+\tau}^{\infty} \sum_{s=i}^{\infty} |g(s, x_1(g_1(s)), \ldots, x_1(g_k(s))) - g(s, x_2(g_1(s)), \ldots, x_2(g_k(s)))|$$
$$- \frac{1}{T_* b_*} \sum_{i=T_*+\tau}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} |f(t, x_1(f_1(t)), \ldots, x_1(f_k(t))) - f(t, x_2(f_1(t)), \ldots, x_2(f_k(t)))|$$
$$\geq \frac{|L_1 - L_2|}{b^*} - \frac{1}{b_*} \left(1 + \frac{\tau}{T_*}\right) \|x_1 - x_2\| - \frac{2}{T_* b_*} \cdot \left(\sum_{i=T_*+\tau}^{\infty} H_i + \sum_{i=T_*+\tau}^{\infty} \sum_{s=i}^{\infty} G_s + \sum_{i=T_*+\tau}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} F_t \right)$$
$$> \frac{|L_1 - L_2|}{2b^*} - \frac{1}{b_*} \left(1 + \frac{\tau}{T_*}\right) \|x_1 - x_2\|, \tag{61}$$

which implies that

$$\|x_1 - x_2\| > \frac{|L_1 - L_2|}{2b^* (1 + (1/b_*)(1 + \tau/T_*))} > 0, \tag{62}$$

which yields that $x_1 \neq x_2$. That is, (4) possesses uncountably many positive solutions in l_β^∞. This completes the proof. □

Theorem 7. *Assume that there exist constants b_* and b^* and three nonnegative sequences $\{F_n\}_{n\in\mathbb{N}_{n_0}}$, $\{G_n\}_{n\in\mathbb{N}_{n_0}}$, and $\{H_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying (15)–(17) and*

$$A < B + b_* B^*, \quad -1 < b_* \leq b(n) \leq b^* \leq 0, \ n \in \mathbb{N}_{n_0}. \tag{63}$$

Then

(i) *equation (4) possesses a positive solution $x = \{x(n)\}_{n\in\mathbb{N}_\beta} \in l_\beta^\infty$ with (19) and*

$$\lim_{n\to\infty} \frac{x(n) + b(n)x(n-\tau) + c(n)}{n} \in (A, B + b_* B^*); \tag{64}$$

(ii) *equation (4) possesses uncountably many positive solutions in l_β^∞.*

Proof. (i) Let $L \in (A, B + b_* B^*)$. It follows from (16) and (17) that there exists $T \geq n_0 + \tau + \alpha$ satisfying

$$\frac{1}{T} \sum_{i=T}^{\infty} \left\{ H_i + \sum_{s=i}^{\infty} G_s + \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} [F_t + |d(t)|] \right\} < \min\{L - A, B + b_* B^* - L\}. \tag{65}$$

Define two mappings S_L and $G_L : \Omega(A_*, B^*, T) \to l_\beta^\infty$ by (21) and (22).

Now we show that (23), (66) below hold:

$$\|S_L x - S_L y\| \leq |b_*| \|x - y\|, \quad x, y \in \Omega(A_*, B^*, T);$$
$$\|G_L y\| \leq B, \quad y \in \Omega(A_*, B^*, T). \tag{66}$$

Using (15), (21), (22), (63), and (65), we get that for any $x = \{x(n)\}_{n\in\mathbb{N}_\beta}$, $y = \{y(n)\}_{n\in\mathbb{N}_\beta} \in \Omega(A_*, B^*, T)$,

$$\frac{(S_L x)(n) + (G_L y)(n)}{n}$$
$$= L - \frac{b(n)}{n} x(n-\tau) - \frac{c(n)}{n} + \frac{1}{n} \sum_{i=n}^{\infty} h(i, y(h_1(i)), \ldots, y(h_k(i)))$$
$$- \frac{1}{n} \sum_{i=n}^{\infty} \sum_{s=i}^{\infty} g(s, y(g_1(s)), \ldots, y(g_k(s)))$$
$$+ \frac{1}{n} \sum_{i=n}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} [f(t, y(f_1(t)), \ldots, y(f_k(t))) - d(t)]$$

$$\leq L - \frac{(n-\tau)\, b(n)}{n} \cdot \frac{x(n-\tau)}{n-\tau} + \frac{|c(n)|}{n}$$

$$+ \frac{1}{n}\sum_{i=n}^{\infty} \left| h\left(i, y\left(h_1(i)\right), \dots, y\left(h_k(i)\right)\right) \right|$$

$$+ \frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty} \left| g\left(s, y\left(g_1(s)\right), \dots, y\left(g_k(s)\right)\right) \right|$$

$$+ \frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty} \left[\left| f\left(t, y\left(f_1(t)\right), \dots, y\left(f_k(t)\right)\right) \right| + |d(t)| \right]$$

$$\leq L - b_* B^* + \frac{|c(n)|}{n} + \frac{1}{T}\sum_{i=T}^{\infty} H_i + \frac{1}{T}\sum_{i=T}^{\infty}\sum_{s=i}^{\infty} G_s$$

$$+ \frac{1}{T}\sum_{i=T}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty} \left[F_t + |d(t)| \right] < L - b_* B^*$$

$$+ \frac{|c(n)|}{n} + \min\left\{L - A, B + b_* B^* - L\right\} \leq B + \frac{|c(n)|}{n}$$

$$= B(n), \quad n \geq T,$$

$$\frac{(S_L x)(n) + (G_L y)(n)}{n} = \frac{n}{T} \cdot \frac{(S_L x)(T) + (G_L y)(T)}{n}$$

$$\leq B(T), \quad \beta \leq n < T,$$

$$\frac{(S_L x)(n) + (G_L y)(n)}{n}$$

$$= L - \frac{b(n)}{n} x(n-\tau) - \frac{c(n)}{n}$$

$$+ \frac{1}{n}\sum_{i=n}^{\infty} h\left(i, y\left(h_1(i)\right), \dots, y\left(h_k(i)\right)\right)$$

$$- \frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty} g\left(s, y\left(g_1(s)\right), \dots, y\left(g_k(s)\right)\right)$$

$$+ \frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty} \left[f\left(t, y\left(f_1(t)\right), \dots, y\left(f_k(t)\right)\right) - d(t) \right]$$

$$\geq L - \frac{|c(n)|}{n} - \frac{1}{n}\sum_{i=n}^{\infty} \left| h\left(i, y\left(h_1(i)\right), \dots, y\left(h_k(i)\right)\right) \right|$$

$$- \frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty} \left| g\left(s, y\left(g_1(s)\right), \dots, y\left(g_k(s)\right)\right) \right|$$

$$- \frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty} \left[\left| f\left(t, y\left(f_1(t)\right), \dots, y\left(f_k(t)\right)\right) \right| + |d(t)| \right]$$

$$\geq L - \frac{|c(n)|}{n} - \frac{1}{T}\sum_{i=T}^{\infty} H_i - \frac{1}{T}\sum_{i=T}^{\infty}\sum_{s=i}^{\infty} G_s$$

$$- \frac{1}{T}\sum_{i=T}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty} \left[F_t + |d(t)| \right] > L - \frac{|c(n)|}{n}$$

$$- \min\left\{L - A, B + b_* B^* - L\right\} \geq A - \frac{|c(n)|}{n} = A(n),$$

$$n \geq T,$$

$$\frac{(S_L x)(n) + (G_L y)(n)}{n} = \frac{n}{T} \cdot \frac{(S_L x)(T) + (G_L y)(T)}{n}$$

$$\geq A(T), \quad \beta \leq n < T,$$

$$\left| \frac{(S_L x)(n) - (S_L y)(n)}{n} \right| = |b(n)| \frac{n-\tau}{n} \left| \frac{x(n-\tau) - y(n-\tau)}{n-\tau} \right|$$

$$\leq \left| b_* \right| \left\| x - y \right\|, \quad n \geq T,$$

$$\left| \frac{(S_L x)(n) - (S_L y)(n)}{n} \right| = \frac{n}{T} \left| \frac{(S_L x)(T) - (S_L y)(T)}{n} \right|$$

$$\leq \left| b_* \right| \left\| x - y \right\|, \quad \beta \leq n < T,$$

$$\left| \frac{(G_L y)(n)}{n} \right|$$

$$= \left| \frac{1}{n}\sum_{i=n}^{\infty} h\left(i, y\left(h_1(i)\right), \dots, y\left(h_k(i)\right)\right) \right.$$

$$- \frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty} g\left(s, y\left(g_1(s)\right), \dots, y\left(g_k(s)\right)\right)$$

$$\left. + \frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty} \left[f\left(t, y\left(f_1(t)\right), \dots, y\left(f_k(t)\right)\right) - d(t) \right] \right|$$

$$\leq \frac{1}{n}\sum_{i=n}^{\infty} \left| h\left(i, y\left(h_1(i)\right), \dots, y\left(h_k(i)\right)\right) \right|$$

$$+ \frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty} \left| g\left(s, y\left(g_1(s)\right), \dots, y\left(g_k(s)\right)\right) \right|$$

$$+ \frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty} \left[\left| f\left(t, y\left(f_1(t)\right), \dots, y\left(f_k(t)\right)\right) \right| + |d(t)| \right]$$

$$\leq \frac{1}{T}\sum_{i=T}^{\infty} H_i + \frac{1}{T}\sum_{i=T}^{\infty}\sum_{s=i}^{\infty} G_s$$

$$+ \frac{1}{T}\sum_{i=T}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty} \left[F_t + |d(t)| \right]$$

$$< \min\{L - A, B + b_* B^* - L\} \le B, \quad n \ge T,$$
$$\left|\frac{(G_L y)(n)}{n}\right| = \frac{n}{T}\left|\frac{(G_L y)(T)}{n}\right| \le B, \quad \beta \le n < T, \tag{67}$$

which yield (21) and (66). The rest of the proof is similar to that of Theorem 5. This completes the proof. □

Theorem 8. *Assume that there exist constants b_* and b^* and three nonnegative sequences $\{F_n\}_{n\in\mathbb{N}_{n_0}}$, $\{G_n\}_{n\in\mathbb{N}_{n_0}}$, and $\{H_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying (15)–(17) and*

$$b^* B + B^* + c^* < b_* A + A_* - \frac{b_* c^*}{b^*} < 0,$$
$$b_* \le b(n) \le b^* < -1, \quad n \in \mathbb{N}_{n_0}. \tag{68}$$

Then

(i) *equation (4) possesses a positive solution $x = \{x(n)\}_{n\in\mathbb{N}_\beta} \in l_\beta^\infty$ with (19) and*

$$\lim_{n\to\infty} \frac{x(n) + b(n)x(n-\tau) + c(n)}{n} \in \left(b^* B + B^* + c^*, b_* A + A_* - \frac{b_* c^*}{b^*}\right); \tag{69}$$

(ii) *equation (4) possesses uncountably many positive solutions in l_β^∞.*

Proof. (i) Let $L \in (b^* B + (B^* + c^*), b_* A + A_* - b_* c^*/b^*)$. Notice that

$$\lim_{n\to\infty}\left[b^* B + (B^* + c^*)\left(1 + \frac{\tau}{n}\right)\right] = b^* B + B^* + c^* < L < b_* A + A_* - \frac{b_* c^*}{b^*} = \lim_{n\to\infty}\left[b_* A + A_* - \frac{b_* c^*}{b^*}\left(1 + \frac{\tau}{n}\right)\right], \tag{70}$$

which means that there exists $N \in \mathbb{N}$ satisfying

$$b^* B + (B^* + c^*) < b^* B + (B^* + c^*)\left(1 + \frac{\tau}{N}\right) < L < b_* A + A_* - \frac{b_* c^*}{b^*}\left(1 + \frac{\tau}{N}\right) < b_* A + A_* - \frac{b_* c^*}{b^*}. \tag{71}$$

It follows from (16) and (17) that there exist $\theta \in (0,1)$ and $T > \max\{N, n_0 + \tau + \alpha\}$ satisfying

$$\theta = \frac{1}{|b^*|}\left(1 + \frac{\tau}{T}\right),$$
$$\frac{1}{T}\sum_{i=T+\tau}^{\infty}\left\{H_i + \sum_{s=i}^{\infty} G_s + \sum_{s=i}^{\infty}\sum_{t=s}^{\infty}[F_t + |d(t)|]\right\} < \min\left\{L - b^* B - \left(1 + \frac{\tau}{N}\right)(B^* + c^*), b^* A + \frac{b^* A_*}{b_*} - c^*\left(1 + \frac{\tau}{N}\right) - \frac{b^*}{b_*}L\right\} < \min\left\{L - b^* B - \left(1 + \frac{\tau}{T}\right)(B^* + c^*), b^* A + \frac{b^* A_*}{b_*} - c^*\left(1 + \frac{\tau}{T}\right) - \frac{b^*}{b_*}L\right\}. \tag{72}$$

Define two mappings S_L and $G_L : \Omega(A_*, B^*, T) \to l_\beta^\infty$ by (46) and (47).

Now we show that (23), (25), and (48) hold. Using (15), (46), (47), (68), and (72), we get that for any $x = \{x(n)\}_{n\in\mathbb{N}_\beta}$, $y = \{y(n)\}_{n\in\mathbb{N}_\beta} \in \Omega(A_*, B^*, T)$

$$\begin{aligned}
\frac{(S_L x)(n) + (G_L y)(n)}{n}
&= \frac{L}{b(n+\tau)} - \frac{x(n+\tau)}{nb(n+\tau)} - \frac{c(n+\tau)}{nb(n+\tau)} + \frac{1}{nb(n+\tau)} \\
&\quad \cdot \sum_{i=n+\tau}^{\infty} h(i, y(h_1(i)), \dots, y(h_k(i))) - \frac{1}{nb(n+\tau)} \\
&\quad \cdot \sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty} g(s, y(g_1(s)), \dots, y(g_k(s))) + \frac{1}{nb(n+\tau)} \\
&\quad \cdot \sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}[f(t, y(f_1(t)), \dots, y(f_k(t))) - d(t)] \\
&\le \frac{L}{b^*} - \frac{n+\tau}{nb(n+\tau)} \cdot \frac{x(n+\tau)}{(n+\tau)} - \frac{n+\tau}{nb(n+\tau)} \\
&\quad \cdot \frac{|c(n+\tau)|}{n+\tau} - \frac{1}{nb(n+\tau)} \\
&\quad \cdot \sum_{i=n+\tau}^{\infty} |h(i, y(h_1(i)), \dots, y(h_k(i)))| - \frac{1}{nb(n+\tau)} \\
&\quad \cdot \sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty} |g(s, y(g_1(s)), \dots, y(g_k(s)))| \\
&\quad - \frac{1}{nb(n+\tau)} \\
&\quad \cdot \sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}[|f(t, y(f_1(t)), \dots, y(f_k(t)))| + |d(t)|]
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{L}{b^*} - \frac{1}{b^*}\left(1+\frac{\tau}{T}\right)B^* + \frac{c^*}{b^*}\left(1+\frac{\tau}{T}\right) \\
&\quad - \frac{1}{Tb^*}\sum_{i=T+\tau}^{\infty} H_i - \frac{1}{Tb^*}\sum_{i=T+\tau}^{\infty}\sum_{s=i}^{\infty} G_s \\
&\quad - \frac{1}{Tb^*}\sum_{i=T+\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[F_t + |d(t)|\right] \\
&< \frac{L}{b^*} - \frac{1}{b^*}\left(1+\frac{\tau}{T}\right)(B^* + c^*) \\
&\quad - \frac{1}{b^*}\min\left\{L - b^*B - \left(1+\frac{\tau}{T}\right)(B^*+c^*),\right. \\
&\qquad\qquad \left. b^*A + \frac{b^*A_*}{b_*} - c^*\left(1+\frac{\tau}{T}\right) - \frac{b^*}{b_*}L\right\} \\
&\le B \le B(n), \quad n \ge T,
\end{aligned}
$$

$$
\begin{aligned}
\frac{(S_Lx)(n) + (G_Ly)(n)}{n} &= \frac{n}{T}\cdot\frac{(S_Lx)(T) + (G_Ly)(T)}{n} \\
&\le B(T), \quad \beta \le n < T,
\end{aligned}
$$

$$
\begin{aligned}
&\frac{(S_Lx)(n) + (G_Ly)(n)}{n} \\
&= \frac{L}{b(n+\tau)} - \frac{x(n+\tau)}{nb(n+\tau)} - \frac{c(n+\tau)}{nb(n+\tau)} + \frac{1}{nb(n+\tau)} \\
&\quad \cdot \sum_{i=n+\tau}^{\infty} h(i, y(h_1(i)), \dots, y(h_k(i))) \\
&\quad - \frac{1}{nb(n+\tau)} \\
&\quad \cdot \sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty} g(s, y(g_1(s)), \dots, y(g_k(s))) \\
&\quad + \frac{1}{nb(n+\tau)} \\
&\quad \cdot \sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[f(t, y(f_1(t)), \dots, y(f_k(t))) - d(t)\right] \\
&\ge \frac{L}{b_*} - \frac{n+\tau}{nb(n+\tau)}\cdot\frac{x(n+\tau)}{n+\tau} + \frac{n+\tau}{nb(n+\tau)}\cdot\frac{|c(n+\tau)|}{n+\tau} \\
&\quad + \frac{1}{nb(n+\tau)}\sum_{i=n+\tau}^{\infty}\left|h(i, y(h_1(i)), \dots, y(h_k(i)))\right| \\
&\quad + \frac{1}{nb(n+\tau)} \\
&\quad \cdot \sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty}\left|g(s, y(g_1(s)), \dots, y(g_k(s)))\right| \\
&\quad + \frac{1}{nb(n+\tau)} \\
&\quad \cdot \sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[\left|f(t, y(f_1(t)), \dots, y(f_k(t)))\right|\right. \\
&\qquad\qquad \left. + |d(t)|\right] \\
&\ge \frac{L}{b_*} - \frac{A_*}{b_*} + \frac{c^*}{b^*}\left(1+\frac{\tau}{T}\right) \\
&\quad + \frac{1}{Tb^*}\sum_{i=T+\tau}^{\infty} H_i + \frac{1}{Tb^*}\sum_{i=T+\tau}^{\infty}\sum_{s=i}^{\infty} G_s \\
&\quad + \frac{1}{Tb^*}\sum_{i=T+\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[F_t + |d(t)|\right] \\
&\ge \frac{L}{b_*} - \frac{A_*}{b_*} + \frac{c^*}{b^*}\left(1+\frac{\tau}{T}\right) \\
&\quad + \frac{1}{b^*}\min\left\{L - b^*B - \left(1+\frac{\tau}{T}\right)(B^*+c^*),\right. \\
&\qquad\qquad \left. b^*A + \frac{b^*A_*}{b_*} - c^*\left(1+\frac{\tau}{T}\right) - \frac{b^*}{b_*}L\right\} \\
&\ge A \ge A(n), \quad n \ge T,
\end{aligned}
$$

$$
\begin{aligned}
\frac{(S_Lx)(n) + (G_Ly)(n)}{n} &= \frac{n}{T}\cdot\frac{(S_Lx)(T) + (G_Ly)(T)}{n} \\
&\ge A(T), \quad \beta \le n < T,
\end{aligned}
$$

$$
\begin{aligned}
\left|\frac{(S_Lx)(n) - (S_Ly)(n)}{n}\right| &= \frac{n+\tau}{n|b(n+\tau)|}\left|\frac{x(n+\tau) - y(n+\tau)}{n+\tau}\right| \\
&\le \theta\|x-y\|, \quad n \ge T,
\end{aligned}
$$

$$
\begin{aligned}
\left|\frac{(S_Lx)(n) - (S_Ly)(n)}{n}\right| &= \frac{n}{T}\left|\frac{(S_Lx)(T) - (S_Ly)(T)}{n}\right| \\
&\le \theta\|x-y\|, \quad \beta \le n < T,
\end{aligned}
$$

$$
\begin{aligned}
&\left|\frac{(G_Ly)(n)}{n}\right| \\
&= \left|\frac{1}{nb(n+\tau)}\sum_{i=n+\tau}^{\infty} h(i, y(h_1(i)), \dots, y(h_k(i)))\right. \\
&\quad - \frac{1}{nb(n+\tau)}\sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty} g(s, y(g_1(s)), \dots, y(g_k(s)))
\end{aligned}
$$

$$\begin{aligned}
&+\frac{1}{nb(n+\tau)} \\
&\cdot \sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[f\left(t,y\left(f_1(t)\right),\ldots,y\left(f_k(t)\right)\right)-d(t)\right]\Bigg| \\
\leq{}& \frac{1}{n|b(n+\tau)|}\sum_{i=n+\tau}^{\infty}\left|h\left(i,y\left(h_1(i)\right),\ldots,y\left(h_k(i)\right)\right)\right| \\
&+\frac{1}{n|b(n+\tau)|}\sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty}\left|g\left(s,y\left(g_1(s)\right),\ldots,y\left(g_k(s)\right)\right)\right| \\
&+\frac{1}{n|b(n+\tau)|} \\
&\cdot \sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[\left|f\left(t,y\left(f_1(t)\right),\ldots,y\left(f_k(t)\right)\right)\right|+|d(t)|\right] \\
\leq{}& \frac{1}{T|b^*|}\sum_{i=T+\tau}^{\infty}H_i+\frac{1}{T|b^*|}\sum_{i=T+\tau}^{\infty}\sum_{s=i}^{\infty}G_s+\frac{1}{T|b^*|} \\
&\cdot \sum_{i=T+\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[F_t+|d(t)|\right] \\
\leq{}& -\frac{1}{b^*}\min\left\{L-b^*B-\left(1+\frac{\tau}{T}\right)\left(B^*+c^*\right),\right. \\
&\left. b^*A+\frac{b^*A_*}{b_*}-c^*\left(1+\frac{\tau}{T}\right)-\frac{b^*}{b_*}L\right\} \\
\leq{}& B,\quad n\geq T, \\
&\left|\frac{(G_Ly)(n)}{n}\right|=\frac{n}{T}\left|\frac{(G_Ly)(T)}{n}\right|\leq B,\quad \beta\leq n<T,
\end{aligned} \tag{73}$$

which yield (23), (25), and (48).

Next we show that G_L is completely continuous. Let $y^w = \{y^w(n)\}_{n\in\mathbb{N}_\beta}$ and $y = \{y(n)\}_{n\in\mathbb{N}_\beta} \in \Omega(A_*, B^*, T)$ with (27). Using (16), (17), (27), and the continuity of f, g, and h, we know that for given $\varepsilon > 0$, there exist T_1, T_2, T_3, and $T_4 \in \mathbb{N}$ with $T_4 > T_3 > T_2 > T_1 > T + \tau$ satisfying

$$\begin{aligned}
&\frac{1}{T|b^*|}\max\left\{\sum_{i=T_1+1}^{\infty}H_i+\sum_{i=T_1+1}^{\infty}\sum_{s=i}^{\infty}G_s\right. \\
&\quad+\sum_{i=T_1+1}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[F_t+|d(t)|\right], \\
&\quad\sum_{i=T+\tau}^{T_1}\sum_{s=T_2+1}^{\infty}G_s+\sum_{i=T+\tau}^{T_1}\sum_{s=T_2+1}^{\infty}\sum_{t=s}^{\infty}F_t \\
&\quad\left.+\sum_{i=T+\tau}^{T_1}\sum_{s=i}^{T_2}\sum_{t=T_3+1}^{\infty}F_t\right\}<\frac{\varepsilon}{16}, \\
&\frac{1}{T|b^*|}\max\left\{\sum_{i=T+\tau}^{T_1}\left|h\left(i,y^w\left(h_1(i)\right),\ldots,y^w\left(h_k(i)\right)\right)\right.\right. \\
&\quad\left.-h\left(i,y\left(h_1(i)\right),\ldots,y\left(h_k(i)\right)\right)\right|, \\
&\quad\sum_{i=T+\tau}^{T_1}\sum_{s=i}^{T_2}\left|g\left(s,y^w\left(g_1(s)\right),\ldots,y^w\left(g_k(s)\right)\right)\right. \\
&\quad\left.-g\left(s,y\left(g_1(s)\right),\ldots,y\left(g_k(s)\right)\right)\right|, \\
&\quad\sum_{i=T+\tau}^{T_1}\sum_{s=i}^{T_2}\sum_{t=s}^{T_3}\left|f\left(t,y^w\left(f_1(t)\right),\ldots,y^w\left(f_k(t)\right)\right)\right. \\
&\quad\left.\left.-f\left(t,y\left(f_1(t)\right),\ldots,y\left(f_k(t)\right)\right)\right|\right\} \\
&<\frac{\varepsilon}{16},\quad w\geq T_4.
\end{aligned} \tag{74}$$

Combining (15), (47), and (74), we infer that

$$\begin{aligned}
&\left\|G_Ly^w-G_Ly\right\| \\
&=\sup\left\{\frac{\left|\left(G_Ly^w\right)(n)-\left(G_Ly\right)(n)\right|}{n}:n\in\mathbb{N}_\beta\right\} \\
&=\max\left\{\sup\left\{\frac{n}{T}\cdot\frac{\left|\left(G_Ly^w\right)(T)-\left(G_Ly\right)(T)\right|}{n}:\beta\leq n<T\right\},\right. \\
&\quad\left.\sup\left\{\frac{\left|\left(G_Ly^w\right)(n)-\left(G_Ly\right)(n)\right|}{n}:n\in\mathbb{N}_T\right\}\right\} \\
&\leq\sup\left\{\frac{1}{n|b(n+\tau)|}\right. \\
&\quad\cdot\sum_{i=n+\tau}^{\infty}\left|h\left(i,y^w\left(h_1(i)\right),\ldots,y^w\left(h_k(i)\right)\right)\right. \\
&\quad\left.-h\left(i,y\left(h_1(i)\right),\ldots,y\left(h_k(i)\right)\right)\right| \\
&\quad+\frac{1}{n|b(n+\tau)|} \\
&\quad\cdot\sum_{i=n+\tau}^{\infty}\sum_{s=i}^{\infty}\left|g\left(s,y^w\left(g_1(s)\right),\ldots,y^w\left(g_k(s)\right)\right)\right. \\
&\quad\left.-g\left(s,y\left(g_1(s)\right),\ldots,y\left(g_k(s)\right)\right)\right| \\
&\quad+\frac{1}{n|b(n+\tau)|}
\end{aligned}$$

$$\cdot \sum_{i=n+\tau}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} \left| f\left(t, y^w\left(f_1(t)\right), \dots, y^w\left(f_k(t)\right)\right) - f\left(t, y\left(f_1(t)\right), \dots, y\left(f_k(t)\right)\right) \right| : n \in \mathbb{N}_T \Bigg\}$$

$$\begin{aligned}
&\le \frac{1}{T|b^*|} \sum_{i=T+\tau}^{T_1} \left| h\left(i, y^w\left(h_1(i)\right), \dots, y^w\left(h_k(i)\right)\right) - h\left(i, y\left(h_1(i)\right), \dots, y\left(h_k(i)\right)\right) \right| \\
&+ \frac{1}{T|b^*|} \sum_{i=T_1+1}^{\infty} \left| h\left(i, y^w\left(h_1(i)\right), \dots, y^w\left(h_k(i)\right)\right) - h\left(i, y\left(h_1(i)\right), \dots, y\left(h_k(i)\right)\right) \right| \\
&+ \frac{1}{T|b^*|} \cdot \sum_{i=T+\tau}^{T_1} \sum_{s=i}^{T_2} \left| g\left(s, y^w\left(g_1(s)\right), \dots, y^w\left(g_k(s)\right)\right) - g\left(s, y\left(g_1(s)\right), \dots, y\left(g_k(s)\right)\right) \right| \\
&+ \frac{1}{T|b^*|} \cdot \sum_{i=T+\tau}^{T_1} \sum_{s=T_2+1}^{\infty} \left| g\left(s, y^w\left(g_1(s)\right), \dots, y^w\left(g_k(s)\right)\right) - g\left(s, y\left(g_1(s)\right), \dots, y\left(g_k(s)\right)\right) \right| \\
&+ \frac{1}{T|b^*|} \cdot \sum_{i=T_1+1}^{\infty} \sum_{s=i}^{\infty} \left| g\left(s, y^w\left(g_1(s)\right), \dots, y^w\left(g_k(s)\right)\right) - g\left(s, y\left(g_1(s)\right), \dots, y\left(g_k(s)\right)\right) \right| \\
&+ \frac{1}{T|b^*|} \cdot \sum_{i=T+\tau}^{T_1} \sum_{s=i}^{T_2} \sum_{t=s}^{T_3} \left| f\left(t, y^w\left(f_1(t)\right), \dots, y^w\left(f_k(t)\right)\right) - f\left(t, y\left(f_1(t)\right), \dots, y\left(f_k(t)\right)\right) \right| \\
&+ \frac{1}{T|b^*|} \cdot \sum_{i=T+\tau}^{T_1} \sum_{s=i}^{T_2} \sum_{t=T_3+1}^{\infty} \left| f\left(t, y^w\left(f_1(t)\right), \dots, y^w\left(f_k(t)\right)\right) - f\left(t, y\left(f_1(t)\right), \dots, y\left(f_k(t)\right)\right) \right| \\
&+ \frac{1}{T|b^*|} \cdot \sum_{i=T+\tau}^{T_1} \sum_{s=T_2+1}^{\infty} \sum_{t=s}^{\infty} \left| f\left(t, y^w\left(f_1(t)\right), \dots, y^w\left(f_k(t)\right)\right) - f\left(t, y\left(f_1(t)\right), \dots, y\left(f_k(t)\right)\right) \right| \\
&+ \frac{1}{T|b^*|} \cdot \sum_{i=T_1+1}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} \left| f\left(t, y^w\left(f_1(t)\right), \dots, y^w\left(f_k(t)\right)\right) - f\left(t, y\left(f_1(t)\right), \dots, y\left(f_k(t)\right)\right) \right| \\
&< \frac{\varepsilon}{16} + \frac{2}{T|b^*|} \sum_{i=T_1+1}^{\infty} H_i + \frac{\varepsilon}{16} \\
&+ \frac{2}{T|b^*|} \sum_{i=T+\tau}^{T_1} \sum_{s=T_2+1}^{\infty} G_s + \frac{2}{T|b^*|} \sum_{i=T_1+1}^{\infty} \sum_{s=i}^{\infty} G_s \\
&+ \frac{\varepsilon}{16} + \frac{2}{T|b^*|} \sum_{i=T+\tau}^{T_1} \sum_{s=i}^{T_2} \sum_{t=T_3+1}^{\infty} F_t \\
&+ \frac{2}{T|b^*|} \sum_{i=T+\tau}^{T_1} \sum_{s=T_2+1}^{\infty} \sum_{t=s}^{\infty} F_t \\
&+ \frac{2}{T|b^*|} \sum_{i=T_1+1}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} F_t < \varepsilon, \quad w \ge T_4,
\end{aligned} \tag{75}$$

which implies that G_L is continuous in $\Omega(A_*, B^*, T)$.

It follows from (47) and (68) that for any $x = \{x(n)\}_{n \in \mathbb{N}_\beta} \in \Omega(A_*, B^*, T)$ and $t_1, t_2 \ge T_4$

$$\begin{aligned}
&\left| \frac{(G_L y)(t_2)}{t_2} - \frac{(G_L y)(t_1)}{t_1} \right| \\
&= \Bigg| \frac{1}{t_2 b(n+\tau)} \sum_{i=t_2+\tau}^{\infty} h\left(i, y\left(h_1(i)\right), \dots, y\left(h_k(i)\right)\right) \\
&\quad - \frac{1}{t_1 b(n+\tau)} \sum_{i=t_1+\tau}^{\infty} h\left(i, y\left(h_1(i)\right), \dots, y\left(h_k(i)\right)\right)
\end{aligned}$$

$$\begin{aligned}
&- \frac{1}{t_2 b(n+\tau)} \sum_{i=t_2+\tau}^{\infty} \sum_{s=i}^{\infty} g(s, y(g_1(s)), \ldots, y(g_k(s))) \\
&+ \frac{1}{t_1 b(n+\tau)} \sum_{i=t_1+\tau}^{\infty} \sum_{s=i}^{\infty} g(s, y(g_1(s)), \ldots, y(g_k(s))) \\
&+ \frac{1}{t_2 b(n+\tau)} \\
&\cdot \sum_{i=t_2+\tau}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} [f(t, y(f_1(t)), \ldots, y(f_k(t))) - d(t)] \\
&- \frac{1}{t_1 b(n+\tau)} \\
&\cdot \sum_{i=t_1+\tau}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} [f(t, y(f_1(t)), \ldots, y(f_k(t))) - d(t)] \Bigg| \\
\le{}& \frac{2}{T_4 |b^*|} \\
&\cdot \left(\sum_{i=T_4+\tau}^{\infty} H_i + \sum_{i=T_4+\tau}^{\infty} \sum_{s=i}^{\infty} G_s + \sum_{i=T_4+\tau}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} [F_t + |d(t)|] \right) \\
<{}& \varepsilon,
\end{aligned} \tag{76}$$

which means that $G_L(\Omega(A_*, B^*, T))$ is uniformly Cauchy, which together with (25) and Lemma 1 yields that $G_L(\Omega(A_*, B^*, T))$ is relatively compact. That is, G_L is completely continuous in $\Omega(A_*, B^*, T)$. Thus (25), (48), and Lemma 2 ensure that the mapping $S_L + G_L$ has a fixed point $x = \{x(n)\}_{n\in\mathbb{N}_\beta} \in \Omega(A_*, B^*, T)$, which together with (46) and (47) implies that

$$\begin{aligned}
x(n) ={}& \frac{nL}{b(n+\tau)} - \frac{x(n+\tau)}{b(n+\tau)} - \frac{c(n+\tau)}{b(n+\tau)} + \frac{1}{b(n+\tau)} \\
&\cdot \sum_{i=n+\tau}^{\infty} h(i, x(h_1(i)), \ldots, x(h_k(i))) - \frac{1}{b(n+\tau)} \\
&\cdot \sum_{i=n+\tau}^{\infty} \sum_{s=i}^{\infty} g(s, x(g_1(s)), \ldots, x(g_k(s))) + \frac{1}{b(n+\tau)} \\
&\cdot \sum_{i=n+\tau}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} [f(t, x(f_1(t)), \ldots, x(f_k(t))) - d(t)], \\
& n \ge T,
\end{aligned} \tag{77}$$

That is, $x = \{x(n)\}_{n\in\mathbb{N}_\beta}$ is a positive solution of (4). The rest of the proof is similar to that of Theorem 6 and is omitted. This completes the proof. □

Theorem 9. *Assume that there exist three nonnegative sequences $\{F_n\}_{n\in\mathbb{N}_{n_0}}$, $\{G_n\}_{n\in\mathbb{N}_{n_0}}$, and $\{H_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying (15),*

$$\lim_{n\to\infty} \frac{1}{n} \max\left\{ \sum_{t=n}^{\infty} t H_t, \sum_{t=n}^{\infty} t^2 G_t \right\} = 0, \tag{78}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{t=n}^{\infty} t^3 \max\{F_t, |d(t)|\} = 0, \tag{79}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{\infty} |c(n+i\tau)| = 0, \tag{80}$$

$$b(n) = -1, \quad n \in \mathbb{N}_{n_0}. \tag{81}$$

Then

(i) *equation (4) possesses a positive solution $x = \{x(n)\}_{n\in\mathbb{N}_\beta} \in l_\beta^\infty$ with (19) and*

$$\lim_{n\to\infty} \frac{x(n) - x(n-\tau) + c(n)}{n} = 0; \tag{82}$$

(ii) *equation (4) possesses uncountably many positive solutions in l_β^∞.*

Proof. (i) Let $L \in (A, B)$. It follows from (78)–(80) that there exists $T \ge n_0 + \tau + \alpha$ satisfying

$$\frac{1}{n} \sum_{i=1}^{\infty} |c(n+i\tau)| < \frac{1}{2} \min\{B - L, L - A\}, \quad n \in \mathbb{N}_T, \tag{83}$$

$$\begin{aligned}
&\frac{1}{T} \sum_{t=T}^{\infty} t H_t + \frac{1}{T} \sum_{t=T}^{\infty} t^2 G_t + \frac{1}{T} \sum_{t=T}^{\infty} t^3 [F_t + |d(t)|] \\
&\quad < \frac{1}{2} \min\{B - L, L - A\}.
\end{aligned} \tag{84}$$

Define a mapping $S_L : \Omega(A_*, B^*, T) \to l_\beta^\infty$ by

$$(S_L x)(n) = \begin{cases}
nL + \displaystyle\sum_{i=1}^{\infty} c(n+i\tau) \\
\quad - \displaystyle\sum_{i=1}^{\infty} \sum_{t=n+i\tau}^{\infty} h(t, x(h_1(t)), \ldots, x(h_k(t))) \\
\quad + \displaystyle\sum_{i=1}^{\infty} \sum_{s=n+i\tau}^{\infty} \sum_{t=s}^{\infty} g(t, x(g_1(t)), \ldots, x(g_k(t))) \\
\quad - \displaystyle\sum_{p=1}^{\infty} \sum_{i=n+p\tau}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} [f(t, x(f_1(t)), \ldots, x(f_k(t))) \\
\qquad - d(t)], \\
\qquad n \ge T, \\
\dfrac{n}{T} (S_L x)(T), \quad \beta \le n < T,
\end{cases} \tag{85}$$

for any $x = \{x(n)\}_{n\in\mathbb{N}_\beta} \in \Omega(A_*, B^*, T)$.

Now we show that

$$S_L x \in \Omega\left(A_*, B^*, T\right), \quad x \in \Omega\left(A_*, B^*, T\right); \tag{86}$$

$$\left\|S_L x\right\| \leq B, \quad x \in \Omega\left(A_*, B^*, T\right). \tag{87}$$

It follows from (15), (83)–(85), and Lemma 4 that for any $x = \{x(n)\}_{n\in\mathbb{N}_\beta} \in \Omega(A_*, B^*, T)$

$$\begin{aligned}
&\left|\frac{(S_L x)(n)}{n} - L\right| \\
&= \left|\frac{1}{n}\sum_{i=1}^{\infty} c(n+i\tau)\right. \\
&\quad - \frac{1}{n}\sum_{i=1}^{\infty}\sum_{t=n+i\tau}^{\infty} h\left(t, x\left(h_1(t)\right), \ldots, x\left(h_k(t)\right)\right) \\
&\quad + \frac{1}{n}\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty} g\left(t, x\left(g_1(t)\right), \ldots, x\left(g_k(t)\right)\right) \\
&\quad - \frac{1}{n}\sum_{p=1}^{\infty}\sum_{i=n+p\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[f\left(t, x\left(f_1(t)\right), \ldots, x\left(f_k(t)\right)\right)\right. \\
&\qquad\qquad \left.\left. - d(t)\right]\right| \\
&\leq \frac{1}{n}\sum_{i=1}^{\infty}\left|c(n+i\tau)\right| \\
&\quad + \frac{1}{n}\sum_{i=1}^{\infty}\sum_{t=n+i\tau}^{\infty}\left|h\left(t, x\left(h_1(t)\right), \ldots, x\left(h_k(t)\right)\right)\right| \\
&\quad + \frac{1}{n}\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty}\left|g\left(t, x\left(g_1(t)\right), \ldots, x\left(g_k(t)\right)\right)\right| \\
&\quad + \frac{1}{n}\sum_{p=1}^{\infty}\sum_{i=n+p\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[\left|f\left(t, x\left(f_1(t)\right), \ldots, x\left(f_k(t)\right)\right)\right|\right. \\
&\qquad\qquad \left. + \left|d(t)\right|\right] \\
&< \frac{1}{2}\min\{B-L, L-A\} \\
&\quad + \frac{1}{T}\sum_{t=T}^{\infty} tH_t + \frac{1}{T}\sum_{t=T}^{\infty} t^2 G_t + \frac{1}{T}\sum_{t=T}^{\infty} t^3\left[F_t + |d(t)|\right] \\
&< \min\{B-L, L-A\}, \quad n \geq T, \\
&\left|\frac{(S_L x)(n)}{n} - L\right| = \left|\frac{n}{T}\cdot\frac{(S_L x)(T)}{n} - L\right| \\
&\qquad < \min\{B-L, L-A\}, \quad \beta \leq n < T,
\end{aligned} \tag{88}$$

which yields that

$$\begin{aligned}
A(n) &\leq A \leq L - \min\{B-L, L-A\} < \frac{(S_L x)(n)}{n} \\
&< L + \min\{B-L, L-A\} \leq B \leq B(n), \quad n \in \mathbb{N}_\beta;
\end{aligned} \tag{89}$$

that is, (86) and (87) hold.

Next we show that S_L is continuous and $S_L(\Omega(A_*, B^*, T))$ is uniformly Cauchy. Let $x^w = \{x^w(n)\}_{n\in\mathbb{N}_\beta}$ and $x = \{x(n)\}_{n\in\mathbb{N}_\beta} \in \Omega(A_*, B^*, T)$ with

$$\lim_{w\to\infty} x^w = x. \tag{90}$$

Using (15), (78), and (80) the continuity of f, g, and h, we know that for given $\varepsilon > 0$, there exist $T_2 > T_1 > T$ satisfying

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{\infty}\left|c(n+i\tau)\right| < \frac{\varepsilon}{16}, \quad \forall n \in \mathbb{N}_{T_1}, \\
&\frac{1}{T}\max\left\{\sum_{t=T}^{T_1} t\left|h\left(t, x^w\left(h_1(t)\right), \ldots, x^w\left(h_k(t)\right)\right)\right.\right. \\
&\qquad\qquad \left. - h\left(t, x\left(h_1(t)\right), \ldots, x\left(h_k(t)\right)\right)\right|, \\
&\qquad \sum_{t=T}^{T_1} t^2\left|g\left(t, x^w\left(g_1(t)\right), \ldots, x^w\left(g_k(t)\right)\right)\right. \\
&\qquad\qquad \left. - g\left(t, x\left(g_1(t)\right), \ldots, x\left(g_k(t)\right)\right)\right|, \\
&\qquad \sum_{t=T}^{T_1} t^3\left|f\left(t, x^w\left(f_1(t)\right), \ldots, x^w\left(f_k(t)\right)\right)\right. \\
&\qquad\qquad \left.\left. - f\left(t, x\left(f_1(t)\right), \ldots, x\left(f_k(t)\right)\right)\right|\right\} \\
&\quad < \frac{\varepsilon}{16}, \quad w \geq T_2, \\
&\frac{1}{T}\left(\sum_{t=T_1+1}^{\infty} tH_t + \sum_{t=T_1+1}^{\infty} t^2 G_t + \sum_{t=T_1+1}^{\infty} t^3\left[F_t + |d(t)|\right]\right) \\
&\qquad < \frac{\varepsilon}{16}.
\end{aligned} \tag{91}$$

Combining (15), (91), and Lemma 4, we infer that

$$\begin{aligned}
&\left\|S_L x^w - S_L x\right\| \\
&= \sup\left\{\left|\frac{(S_L x^w)(n) - (S_L x)(n)}{n}\right| : n \in \mathbb{N}_\beta\right\} \\
&= \max\left\{\sup\left\{\left|\frac{n}{T}\frac{(S_L x^w)(T) - (S_L x)(T)}{n}\right| : \beta \leq n < T\right\},\right.
\end{aligned}$$

$$\sup\left\{\left|\frac{(S_Lx^w)(n)-(S_Lx)(n)}{n}\right| : n\in\mathbb{N}_T\right\}$$
$$\le \sup\left\{\frac{1}{n}\sum_{i=1}^{\infty}\sum_{t=n+i\tau}^{\infty}\left|h\left(t,x^w\left(h_1(t)\right),\ldots,x^w\left(h_k(t)\right)\right) -h\left(t,x\left(h_1(t)\right),\ldots,x\left(h_k(t)\right)\right)\right| + \frac{1}{n}\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty}\left|g\left(t,x^w\left(g_1(t)\right),\ldots,x^w\left(g_k(t)\right)\right) -g\left(t,x\left(g_1(t)\right),\ldots,x\left(g_k(t)\right)\right)\right| + \frac{1}{n}\sum_{p=1}^{\infty}\sum_{i=n+p\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left|f\left(t,x^w\left(f_1(t)\right),\ldots, x^w\left(f_k(t)\right)\right) - f\left(t,x\left(f_1(t)\right),\ldots, x\left(f_k(t)\right)\right)\right| : n\in\mathbb{N}_T\right\}$$
$$\le \frac{1}{T}\sum_{t=T+\tau}^{\infty} t\left|h\left(t,x^w\left(h_1(t)\right),\ldots,x^w\left(h_k(t)\right)\right) -h\left(t,x\left(h_1(t)\right),\ldots,x\left(h_k(t)\right)\right)\right| + \frac{1}{T}\sum_{t=T+\tau}^{\infty} t^2\left|g\left(t,x^w\left(g_1(t)\right),\ldots,x^w\left(g_k(t)\right)\right) -g\left(t,x\left(g_1(t)\right),\ldots,x\left(g_k(t)\right)\right)\right| + \frac{1}{T}\sum_{t=T+\tau}^{\infty} t^3\left|f\left(t,x^w\left(f_1(t)\right),\ldots,x^w\left(f_k(t)\right)\right) -f\left(t,x\left(f_1(t)\right),\ldots,x\left(f_k(t)\right)\right)\right|$$
$$\le \frac{1}{T}\sum_{t=T}^{T_1} t\left|h\left(t,x^w\left(h_1(t)\right),\ldots,x^w\left(h_k(t)\right)\right) -h\left(t,x\left(h_1(t)\right),\ldots,x\left(h_k(t)\right)\right)\right| + \frac{1}{T}\sum_{t=T_1+1}^{\infty} t\left|h\left(t,x^w\left(h_1(t)\right),\ldots,x^w\left(h_k(t)\right)\right) -h\left(t,x\left(h_1(t)\right),\ldots,x\left(h_k(t)\right)\right)\right|$$
$$+ \frac{1}{T}\sum_{t=T}^{T_1} t^2\left|g\left(t,x^w\left(g_1(t)\right),\ldots,x^w\left(g_k(t)\right)\right) -g\left(t,x\left(g_1(t)\right),\ldots,x\left(g_k(t)\right)\right)\right| + \frac{1}{T}\sum_{t=T_1+1}^{\infty} t^2\left|g\left(t,x^w\left(g_1(t)\right),\ldots,x^w\left(g_k(t)\right)\right) -g\left(t,x\left(g_1(t)\right),\ldots,x\left(g_k(t)\right)\right)\right|$$
$$+ \frac{1}{T}\sum_{t=T}^{T_1} t^3\left|f\left(t,x^w\left(f_1(t)\right),\ldots,x^w\left(f_k(t)\right)\right) -f\left(t,x\left(f_1(t)\right),\ldots,x\left(f_k(t)\right)\right)\right| + \frac{1}{T}\sum_{t=T_1+1}^{\infty} t^3\left|f\left(t,x^w\left(f_1(t)\right),\ldots,x^w\left(f_k(t)\right)\right) -f\left(t,x\left(f_1(t)\right),\ldots,x\left(f_k(t)\right)\right)\right|$$
$$< \frac{\varepsilon}{16} + \frac{2}{T}\sum_{t=T_1+1}^{\infty} tH_t + \frac{\varepsilon}{16} + \frac{2}{T}\sum_{t=T_1+1}^{\infty} t^2G_t + \frac{\varepsilon}{16} + \frac{2}{T}\sum_{i=T_1+1}^{\infty} t^3F_t < \varepsilon, \quad w\ge T_2, \tag{92}$$

which implies that S_L is continuous in $\Omega(A_*, B^*, T)$. It follows from (80), (85), and Lemma 4 that for any $x=\{x(n)\}_{n\in\mathbb{N}_\beta}\in\Omega(A_*, B^*, T)$ and $t_1>t_2\ge T_2$

$$\left|\frac{(S_Lx)(t_1)}{t_1}-\frac{(S_Lx)(t_2)}{t_2}\right|$$
$$= \left|\frac{1}{t_1}\sum_{i=1}^{\infty}c\left(t_1+i\tau\right)-\frac{1}{t_2}\sum_{i=1}^{\infty}c\left(t_2+i\tau\right) - \frac{1}{t_1}\sum_{i=1}^{\infty}\sum_{t=t_1+i\tau}^{\infty}h\left(t,x\left(h_1(t)\right),\ldots,x\left(h_k(t)\right)\right) + \frac{1}{t_2}\sum_{i=1}^{\infty}\sum_{t=t_2+i\tau}^{\infty}h\left(t,x\left(h_1(t)\right),\ldots,x\left(h_k(t)\right)\right) + \frac{1}{t_1}\sum_{i=1}^{\infty}\sum_{s=t_1+i\tau}^{\infty}\sum_{t=s}^{\infty}g\left(t,x\left(g_1(t)\right),\ldots,x\left(g_k(t)\right)\right) - \frac{1}{t_2}\sum_{i=1}^{\infty}\sum_{s=t_2+i\tau}^{\infty}\sum_{t=s}^{\infty}g\left(t,x\left(g_1(t)\right),\ldots,x\left(g_k(t)\right)\right)\right.$$

$$
\begin{aligned}
&- \frac{1}{t_1}\sum_{p=1}^{\infty}\sum_{i=t_1+p\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[f\left(t, x\left(f_1(t)\right), \ldots, x\left(f_k(t)\right)\right) - d(t)\right] \\
&+ \frac{1}{t_2}\sum_{p=1}^{\infty}\sum_{i=t_2+p\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[f\left(t, x\left(f_1(t)\right), \ldots, x\left(f_k(t)\right)\right) - d(t)\right]\Bigg| \\
&\le \frac{1}{t_1}\sum_{i=1}^{\infty}\left|c\left(t_1+i\tau\right)\right| + \frac{1}{t_2}\sum_{i=1}^{\infty}\left|c\left(t_2+i\tau\right)\right| \\
&+ \frac{1}{t_1}\sum_{t=t_1}^{\infty} t\left|h\left(t, x\left(h_1(t)\right), \ldots, x\left(h_k(t)\right)\right)\right| \\
&+ \frac{1}{t_2}\sum_{t=t_2}^{\infty} t\left|h\left(t, x\left(h_1(t)\right), \ldots, x\left(h_k(t)\right)\right)\right| \\
&+ \frac{1}{t_1}\sum_{t=t_1}^{\infty} t^2\left|g\left(t, x\left(g_1(t)\right), \ldots, x\left(g_k(t)\right)\right)\right| \\
&+ \frac{1}{t_2}\sum_{t=t_2}^{\infty} t^2\left|g\left(t, x\left(g_1(t)\right), \ldots, x\left(g_k(t)\right)\right)\right| \\
&+ \frac{1}{t_1}\sum_{t=t_1}^{\infty} t^3\left|f\left(t, x\left(f_1(t)\right), \ldots, x\left(f_k(t)\right)\right) - d(t)\right| \\
&+ \frac{1}{t_2}\sum_{t=t_2}^{\infty} t^3\left|f\left(t, x\left(f_1(t)\right), \ldots, x\left(f_k(t)\right)\right) - d(t)\right| \\
&< \frac{2\varepsilon}{16} + \frac{2}{T_2}\sum_{t=T_2}^{\infty} tH_t + \frac{2}{T_2}\sum_{t=T_2}^{\infty} t^2 G_t \\
&+ \frac{2}{T_2}\sum_{t=T_2}^{\infty} t^3\left[F_t + |d(t)|\right] < \varepsilon,
\end{aligned} \tag{93}
$$

which means that $S_L(\Omega(A_*, B^*, T))$ is uniformly Cauchy, which together with (87) and Lemma 1 yields that $S_L(\Omega(A_*, B^*, T))$ is relatively compact. It follows from Lemma 3 that the mapping S_L has a fixed point $x = \{x(n)\}_{n\in\mathbb{N}_\beta} \in \Omega(A_*, B^*, T)$; that is,

$$
\begin{aligned}
x(n) &= nL + \sum_{i=1}^{\infty} c(n+i\tau) \\
&- \sum_{i=1}^{\infty}\sum_{t=n+i\tau}^{\infty} h\left(t, x\left(h_1(t)\right), \ldots, x\left(h_k(t)\right)\right) \\
&+ \sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty} g\left(t, x\left(g_1(t)\right), \ldots, x\left(g_k(t)\right)\right) \\
&- \sum_{p=1}^{\infty}\sum_{i=n+p\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[f\left(t, x\left(f_1(t)\right), \ldots, x\left(f_k(t)\right)\right) - d(t)\right], \quad n \ge T,
\end{aligned} \tag{94}
$$

which gives that

$$
\begin{aligned}
&x(n) - x(n-\tau) \\
&= \tau L - c(n) + \sum_{t=n}^{\infty} h\left(t, x\left(h_1(t)\right), \ldots, x\left(h_k(t)\right)\right) \\
&- \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} g\left(t, x\left(g_1(t)\right), \ldots, x\left(g_k(t)\right)\right) \\
&+ \sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[f\left(t, x\left(f_1(t)\right), \ldots, x\left(f_k(t)\right)\right) - d(t)\right], \\
&\qquad n \ge T + \tau.
\end{aligned} \tag{95}
$$

It is easy to verify that (95) implies that

$$
\begin{aligned}
&\Delta\left(x(n) - x(n-\tau) + c(n)\right) \\
&= -h\left(n, x\left(h_1(n)\right), \ldots, x\left(h_k(n)\right)\right) \\
&+ \sum_{t=n}^{\infty} g\left(t, x\left(g_1(t)\right), \ldots, x\left(g_k(t)\right)\right) \\
&- \sum_{s=n}^{\infty}\sum_{t=s}^{\infty}\left[f\left(t, x\left(f_1(t)\right), \ldots, x\left(f_k(t)\right)\right) - d(t)\right], \\
&\qquad n \ge T + \tau, \\
&\Delta^2\left(x(n) - x(n-\tau) + c(n)\right) \\
&= -\Delta h\left(n, x\left(h_1(n)\right), \ldots, x\left(h_k(n)\right)\right) \\
&- g\left(n, x\left(g_1(n)\right), \ldots, x\left(g_k(n)\right)\right) \\
&+ \sum_{t=n}^{\infty}\left[f\left(t, x\left(f_1(t)\right), \ldots, x\left(f_k(t)\right)\right) - d(t)\right], \\
&\qquad n \ge T + \tau,
\end{aligned} \tag{96}
$$

which yields that

$$\begin{aligned}\Delta^3(x(n)-x(n-\tau)+c(n)) &= -\Delta^2 h(n,x(h_1(n)),\dots,x(h_k(n))) \\ &\quad -\Delta g(n,x(g_1(n)),\dots,x(g_k(n))) \\ &\quad -[f(n,x(f_1(n)),\dots,x(f_k(n)))-d(n)], \\ &\qquad n\ge T+\tau,\end{aligned} \tag{97}$$

which together with (81) gives that $x=\{x(n)\}_{n\in\mathbb{N}_\beta}\in\Omega(A_*,B^*,T)$ is a positive solution of (4). It follows from (78), (79), (95), and Lemma 4 that

$$\begin{aligned}\frac{|x(n)-x(n-\tau)+c(n)|}{n} &= \Bigg|\frac{\tau L}{n}+\frac{1}{n}\sum_{t=n}^{\infty}h(t,x(h_1(t)),\dots,x(h_k(t))) \\ &\quad -\frac{1}{n}\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}g(t,x(g_1(t)),\dots,x(g_k(t))) \\ &\quad +\frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}[f(t,x(f_1(t)),\dots,x(f_k(t))) \\ &\quad -d(t)]\Bigg| \\ &\le \frac{\tau L}{n}+\frac{1}{n}\sum_{t=n}^{\infty}H_t+\frac{1}{n}\sum_{s=n}^{\infty}\sum_{t=s}^{\infty}G_t \\ &\quad +\frac{1}{n}\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}[H_t+|d(t)|] \\ &\le \frac{\tau L}{n}+\frac{1}{n}\sum_{t=n}^{\infty}H_t+\frac{1}{n}\sum_{t=n}^{\infty}tG_t \\ &\quad +\frac{1}{n}\sum_{t=n}^{\infty}t^2[H_t+|d(t)|]\longrightarrow 0 \quad \text{as } n\longrightarrow\infty;\end{aligned} \tag{98}$$

that is, (82) holds. The proof of (19) is similar to that of Theorem 5 and is omitted.

(ii) Let $L_1,L_2\in(A,B)$ and $L_1\ne L_2$. Similarly we conclude that for each $l\in\{1,2\}$, there exist a constant $T_l\ge n_1+\tau+|\alpha|$ and a mapping $S_{L_l}:\Omega(A_*,B^*,T_l)\to l_\beta^\infty$ satisfying (83)–(87), where T, L, and S_L are replaced by T_l, L_l, and S_{L_l}, respectively, and S_{L_l} possesses a fixed point $x_l=\{x_l(n)\}_{n\in\mathbb{N}_\beta}\in\Omega(A_*,B^*,T_l)$, which is a positive solution of (4); that is,

$$\begin{aligned}x_l(n) &= nL_l+\sum_{i=1}^{\infty}c(n+i\tau) \\ &\quad -\sum_{i=1}^{\infty}\sum_{t=n+i\tau}^{\infty}h(t,x_l(h_1(t)),\dots,x_l(h_k(t))) \\ &\quad +\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty}g(t,x_l(g_1(t)),\dots,x_l(g_k(t))) \\ &\quad -\sum_{p=1}^{\infty}\sum_{i=n+p\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}[f(t,x_l(f_1(t)),\dots,x_l(f_k(t))) \\ &\quad -d(t)], \quad n\ge T_l.\end{aligned} \tag{99}$$

Note that (79) and (80) imply that there exists $T_*>\max\{T_1,T_2\}$ satisfying

$$\frac{1}{T_*}\left(\sum_{t=T_*}^{\infty}tH_t+\sum_{t=T_*}^{\infty}t^2G_t+\sum_{t=T_*}^{\infty}t^3F_t\right) < \frac{|L_1-L_2|}{4}. \tag{100}$$

In view of (15), (99), (100), and Lemma 4, we infer that for any $n\ge T_*$

$$\begin{aligned}\left|\frac{x_1(n)}{n}-\frac{x_2(n)}{n}\right| &= \Bigg|L_1-L_2 \\ &\quad -\frac{1}{n}\sum_{i=1}^{\infty}\sum_{t=n+i\tau}^{\infty}[h(t,x_1(h_1(t)),\dots,x_1(h_k(t))) \\ &\qquad -h(t,x_2(h_1(t)),\dots,x_2(h_k(t)))] \\ &\quad +\frac{1}{n}\sum_{i=1}^{\infty}\sum_{s=n+i\tau}^{\infty}\sum_{t=s}^{\infty}[g(t,x_1(g_1(t)),\dots,x_1(g_k(t))) \\ &\qquad -g(t,x_2(g_1(t)),\dots,x_2(g_k(t)))] \\ &\quad -\frac{1}{n}\sum_{p=1}^{\infty}\sum_{i=n+p\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}[f(t,x_1(f_1(t)),\dots,x_1(f_k(t))) \\ &\qquad -f(t,x_2(f_1(t)),\dots,x_2(f_k(t)))]\Bigg|\end{aligned}$$

$$\begin{aligned}
&\geq |L_1 - L_2| \\
&\quad - \frac{1}{T_*}\sum_{i=1}^{\infty}\sum_{t=T_*+i\tau}^{\infty}\left|h(t, x_1(h_1(t)), \ldots, x_1(h_k(t)))\right. \\
&\qquad\qquad \left. - h(t, x_2(h_1(t)), \ldots, x_2(h_k(t)))\right| \\
&\quad - \frac{1}{T_*}\sum_{i=1}^{\infty}\sum_{s=T_*+i\tau}^{\infty}\sum_{t=s}^{\infty}\left|g(t, x_1(g_1(t)), \ldots, x_1(g_k(t)))\right. \\
&\qquad\qquad \left. - g(t, x_2(g_1(t)), \ldots, x_2(g_k(t)))\right| \\
&\quad - \frac{1}{T_*}\sum_{p=1}^{\infty}\sum_{i=T_*+p\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left|f(t, x_1(f_1(t)), \ldots, x_1(f_k(t)))\right. \\
&\qquad\qquad - f(t, x_2(f_1(t)), x_2(f_2(t)), \ldots, \\
&\qquad\qquad \left. x_2(f_k(t)))\right| \\
&\geq |L_1 - L_2| \\
&\quad - \frac{2}{T_*}\left(\sum_{t=T_*}^{\infty} tH_t + \sum_{t=T_*}^{\infty} t^2 G_t + \sum_{t=T_*}^{\infty} t^3 F_t\right) \\
&> \frac{|L_1 - L_2|}{2} > 0,
\end{aligned} \tag{101}$$

which yields that $x_1 \neq x_2$. Thus (4) possesses uncountably many positive solutions in $\Omega(A_*, B^*, T)$. This completes the proof. □

Theorem 10. *Assume that there exist three nonnegative sequences $\{F_n\}_{n\in\mathbb{N}_{n_0}}$, $\{G_n\}_{n\in\mathbb{N}_{n_0}}$, and $\{H_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying (15)–(17), (80), and*

$$b(n) = 1, \quad n \in \mathbb{N}_{n_0}. \tag{102}$$

Then

(i) *equation (4) possesses uncountably many positive solutions $x = \{x(n)\}_{n\in\mathbb{N}_\beta} \in l_\beta^\infty$ with (19) and*

$$\lim_{n\to\infty} \frac{x(n) + x(n-\tau) + c(n)}{n} \in (2A, 2B); \tag{103}$$

(ii) *equation (4) possesses uncountably many positive solutions in l_β^∞.*

Proof. (i) Let $L \in (A, B)$. It follows from (15)–(17) and (80) that there exists $T \geq n_0 + \tau + \alpha$ satisfying (83) and

$$\begin{aligned}
&\frac{1}{T}\sum_{t=T}^{\infty} H_t + \frac{1}{T}\sum_{s=T}^{\infty}\sum_{t=s}^{\infty} G_t \\
&\quad + \frac{1}{T}\sum_{i=T}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[F_t + |d(t)|\right] \\
&< \frac{1}{2}\min\{L - A, B - L\}.
\end{aligned} \tag{104}$$

Define a mapping $S_L : \Omega(A_*, B^*, T) \to l_\beta^\infty$ by

$$(S_L x)(n) = \begin{cases}
nL + \sum_{i=1}^{\infty}(-1)^i c(n+i\tau) \\
\quad + \sum_{s=1}^{\infty}\sum_{t=n+(2s-1)\tau}^{n+2s\tau-1} h(t, x(h_1(t)), \ldots, x(h_k(t))) \\
\quad - \sum_{i=1}^{\infty}\sum_{s=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{t=s}^{\infty} g(t, x(g_1(t)), \ldots, x(g_k(t))) \\
\quad + \sum_{p=1}^{\infty}\sum_{i=n+(2p-1)\tau}^{n+2p\tau-1}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[f(t, x(f_1(t)), \ldots, x(f_k(t)))\right. \\
\qquad \left. - d(t)\right], \quad n \geq T, \\
\frac{n}{T}(S_L x)(T), \quad \beta \leq n < T
\end{cases} \tag{105}$$

for any $x = \{x(n)\}_{n\in\mathbb{N}_\beta} \in \Omega(A_*, B^*, T)$.

Now we show that (86) and (87) hold. It follows from (15), (83), (104), and (105) that for any $x = \{x(n)\}_{n\in\mathbb{N}_\beta} \in \Omega(A_*, B^*, T)$

$$\begin{aligned}
&\left|\frac{(S_L x)(n)}{n} - L\right| \\
&= \left|\frac{1}{n}\sum_{i=1}^{\infty}(-1)^i c(n+i\tau)\right. \\
&\quad + \frac{1}{n}\sum_{s=1}^{\infty}\sum_{t=n+(2s-1)\tau}^{n+2s\tau-1} h(t, x(h_1(t)), \ldots, x(h_k(t))) \\
&\quad - \frac{1}{n}\sum_{i=1}^{\infty}\sum_{s=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{t=s}^{\infty} g(t, x(g_1(t)), \ldots, x(g_k(t))) \\
&\quad + \frac{1}{n}\sum_{p=1}^{\infty}\sum_{i=n+(2p-1)\tau}^{n+2p\tau-1}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[f(t, x(f_1(t)), \ldots, x(f_k(t)))\right. \\
&\qquad \left.\left. - d(t)\right]\right| \\
&\leq \frac{1}{n}\sum_{i=1}^{\infty}|c(n+i\tau)| \\
&\quad + \frac{1}{n}\sum_{s=1}^{\infty}\sum_{t=n+(2s-1)\tau}^{n+2s\tau-1}\left|h(t, x(h_1(t)), \ldots, x(h_k(t)))\right| \\
&\quad + \frac{1}{n}\sum_{i=1}^{\infty}\sum_{s=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{t=s}^{\infty}\left|g(t, x(g_1(t)), \ldots, x(g_k(t)))\right|
\end{aligned}$$

$$+\frac{1}{n}\sum_{p=1}^{\infty}\sum_{i=n+(2p-1)\tau}^{n+2p\tau-1}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[\left|f\left(t,x\left(f_1(t)\right),\ldots,x\left(f_k(t)\right)\right)\right|+|d(t)|\right]$$
$$<\frac{1}{2}\min\{L-A,B-L\}+\frac{1}{T}\sum_{t=T}^{\infty}\left|h\left(t,x\left(h_1(t)\right),\ldots,x\left(h_k(t)\right)\right)\right|+\frac{1}{T}\sum_{s=T}^{\infty}\sum_{t=s}^{\infty}\left|g\left(t,x\left(g_1(t)\right),\ldots,x\left(g_k(t)\right)\right)\right|+\frac{1}{T}\sum_{i=T}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[\left|f\left(t,x\left(f_1(t)\right),\ldots,x\left(f_k(t)\right)\right)\right|+|d(t)|\right]$$
$$\leq\frac{1}{2}\min\{L-A,B-L\}+\frac{1}{T}\sum_{t=T}^{\infty}H_t+\frac{1}{T}\sum_{s=T}^{\infty}\sum_{t=s}^{\infty}G_t+\frac{1}{T}\sum_{i=T}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[F_t+|d(t)|\right]$$
$$<\min\{L-A,B-L\},\quad n\geq T,$$
$$\left|\frac{(S_Lx)(n)}{n}-L\right|=\left|\frac{n}{T}\cdot\frac{(S_Lx)(T)}{n}-L\right|<\min\{L-A,B-L\},\quad\beta\leq n<T,$$
$$\left|\frac{(S_Lx)(n)}{n}\right|=\left|L+\frac{1}{n}\sum_{i=1}^{\infty}(-1)^i c(n+i\tau)+\frac{1}{n}\sum_{s=1}^{\infty}\sum_{t=n+(2s-1)\tau}^{n+2s\tau-1}h\left(t,x\left(h_1(t)\right),\ldots,x\left(h_k(t)\right)\right)-\frac{1}{n}\sum_{i=1}^{\infty}\sum_{s=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{t=s}^{\infty}g\left(t,x\left(g_1(t)\right),\ldots,x\left(g_k(t)\right)\right)+\frac{1}{n}\sum_{p=1}^{\infty}\sum_{i=n+(2p-1)\tau}^{n+2p\tau-1}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[f\left(t,x\left(f_1(t)\right),\ldots,x\left(f_k(t)\right)\right)-d(t)\right]\right|$$
$$\leq L+\frac{1}{n}\sum_{i=1}^{\infty}|c(n+i\tau)|+\frac{1}{n}\sum_{s=1}^{\infty}\sum_{t=n+(2s-1)\tau}^{n+2s\tau-1}\left|h\left(t,x\left(h_1(t)\right),\ldots,x\left(h_k(t)\right)\right)\right|+\frac{1}{n}\sum_{i=1}^{\infty}\sum_{s=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{t=s}^{\infty}\left|g\left(t,x\left(g_1(t)\right),\ldots,x\left(g_k(t)\right)\right)\right|+\frac{1}{n}\sum_{p=1}^{\infty}\sum_{i=n+(2p-1)\tau}^{n+2p\tau-1}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[\left|f\left(t,x\left(f_1(t)\right),\ldots,x\left(f_k(t)\right)\right)\right|+|d(t)|\right]$$
$$<L+\frac{1}{2}\min\{L-A,B-L\}+\frac{1}{T}\sum_{t=T}^{\infty}\left|h\left(t,x\left(h_1(t)\right),\ldots,x\left(h_k(t)\right)\right)\right|+\frac{1}{T}\sum_{s=T}^{\infty}\sum_{t=s}^{\infty}\left|g\left(t,x\left(g_1(t)\right),\ldots,x\left(g_k(t)\right)\right)\right|+\frac{1}{T}\sum_{i=T}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[\left|f\left(t,x\left(f_1(t)\right),\ldots,x\left(f_k(t)\right)\right)\right|+|d(t)|\right]$$
$$\leq L+\frac{1}{2}\min\{L-A,B-L\}+\frac{1}{T}\sum_{t=T}^{\infty}H_t+\frac{1}{T}\sum_{s=T}^{\infty}\sum_{t=s}^{\infty}G_t+\frac{1}{T}\sum_{i=T}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left[F_t+|d(t)|\right]$$
$$<L+\min\{L-A,B-L\}\leq B,\quad n\geq T,$$
$$\left|\frac{(S_Lx)(T)}{T}\right|=\frac{n}{T}\left|\frac{(S_Lx)(T)}{n}\right|\leq B,\quad\beta\leq n<T,\tag{106}$$

which yield that (86) and (87) hold.

Next we show that S_L is continuous and $S_L(\Omega(A_*,B^*,T))$ is uniformly Cauchy. Let $x^w=\{x^w(n)\}_{n\in\mathbb{N}_\beta}$ and $x=\{x(n)\}_{n\in\mathbb{N}_\beta}\in\Omega(A_*,B^*,T)$ with (90). Using (16), (17), (80), and the continuity of f, g, and h, we know that for given $\varepsilon>0$, there exist T_1, T_2, T_3, and $T_4\in\mathbb{N}$ with $T_4>T_3>T_2>T_1>T+\tau$ satisfying

$$\frac{1}{T}\max\left\{\sum_{t=T+\tau}^{T_1}\left|h\left(t,x^w\left(h_1(t)\right),\ldots,x^w\left(h_k(t)\right)\right)-h\left(t,x\left(h_1(t)\right),\ldots,x\left(h_k(t)\right)\right)\right|,\right.$$

$$\sum_{s=T+\tau}^{T_1}\sum_{t=s}^{T_2}\left|g\left(t,x^w\left(g_1(t)\right),\ldots,x^w\left(g_k(t)\right)\right)-g\left(t,x\left(g_1(t)\right),\ldots,x\left(g_k(t)\right)\right)\right|,$$
$$\left.\sum_{i=T+\tau}^{T_1}\sum_{s=i}^{T_2}\sum_{t=s}^{T_3}\left|f\left(t,x^w\left(f_1(t)\right),\ldots,x^w\left(f_k(t)\right)\right)-f\left(t,x\left(f_1(t)\right),\ldots,x\left(f_k(t)\right)\right)\right|\right\}<\frac{\varepsilon}{16},\quad w\geq T_4, \tag{107}$$

$$\frac{1}{T}\max\left\{\sum_{t=T_1+1}^{\infty}H_t+\sum_{s=T_1+1}^{\infty}\sum_{t=s}^{\infty}G_t+\sum_{i=T_1+1}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}F_t,\right.$$
$$\left.\sum_{s=T+\tau}^{T_1}\sum_{t=T_2+1}^{\infty}G_t+\sum_{i=T+\tau}^{T_1}\sum_{s=i}^{T_2}\sum_{t=T_3+1}^{\infty}F_t+\sum_{i=T+\tau}^{T_1}\sum_{s=T_2+1}^{\infty}\sum_{t=s}^{\infty}F_t\right\}<\frac{\varepsilon}{16}, \tag{108}$$

$$\frac{1}{n}\sum_{i=1}^{\infty}\left|c\left(n+i\tau\right)\right|<\frac{\varepsilon}{16},\quad n\geq T_4. \tag{109}$$

Combining (15) and (105)–(108), we infer that

$$\begin{aligned}
&\left\|S_Lx^w-S_Lx\right\|\\
&\leq\frac{1}{n}\sum_{s=1}^{\infty}\sum_{t=n+(2s-1)\tau}^{n+2s\tau-1}\left|h\left(t,x^w\left(h_1(t)\right),\ldots,x^w\left(h_k(t)\right)\right)-h\left(t,x\left(h_1(t)\right),\ldots,x\left(h_k(t)\right)\right)\right|\\
&+\frac{1}{n}\sum_{i=1}^{\infty}\sum_{s=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{t=s}^{\infty}\left|g\left(t,x^w\left(g_1(t)\right),\ldots,x^w\left(g_k(t)\right)\right)-g\left(t,x\left(g_1(t)\right),\ldots,x\left(g_k(t)\right)\right)\right|\\
&+\frac{1}{n}\sum_{p=1}^{\infty}\sum_{i=n+(2p-1)\tau}^{n+2p\tau-1}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left|f\left(t,x^w\left(f_1(t)\right),\ldots,x^w\left(f_k(t)\right)\right)-f\left(t,x\left(f_1(t)\right),\ldots,x\left(f_k(t)\right)\right)\right|\\
&\leq\frac{1}{T}\sum_{t=T+\tau}^{T_1}\left|h\left(t,x^w\left(h_1(t)\right),\ldots,x^w\left(h_k(t)\right)\right)-h\left(t,x\left(h_1(t)\right),\ldots,x\left(h_k(t)\right)\right)\right|\\
&+\frac{1}{T}\sum_{t=T_1+1}^{\infty}\left|h\left(t,x^w\left(h_1(t)\right),\ldots,x^w\left(h_k(t)\right)\right)-h\left(t,x\left(h_1(t)\right),\ldots,x\left(h_k(t)\right)\right)\right|\\
&+\frac{1}{T}\sum_{s=T+\tau}^{T_1}\sum_{t=s}^{T_2}\left|g\left(t,x^w\left(g_1(t)\right),\ldots,x^w\left(g_k(t)\right)\right)-g\left(t,x\left(g_1(t)\right),\ldots,x\left(g_k(t)\right)\right)\right|\\
&+\frac{1}{T}\sum_{s=T+\tau}^{T_1}\sum_{t=T_2+1}^{\infty}\left|g\left(t,x^w\left(g_1(t)\right),\ldots,x^w\left(g_k(t)\right)\right)-g\left(t,x\left(g_1(t)\right),\ldots,x\left(g_k(t)\right)\right)\right|\\
&+\frac{1}{T}\sum_{s=T_1+1}^{\infty}\sum_{t=s}^{\infty}\left|g\left(t,x^w\left(g_1(t)\right),\ldots,x^w\left(g_k(t)\right)\right)-g\left(t,x\left(g_1(t)\right),\ldots,x\left(g_k(t)\right)\right)\right|\\
&+\frac{1}{T}\sum_{i=T+\tau}^{T_1}\sum_{s=i}^{T_2}\sum_{t=s}^{T_3}\left|f\left(t,x^w\left(f_1(t)\right),\ldots,x^w\left(f_k(t)\right)\right)-f\left(t,x\left(f_1(t)\right),\ldots,x\left(f_k(t)\right)\right)\right|\\
&+\frac{1}{T}\sum_{i=T+\tau}^{T_1}\sum_{s=i}^{T_2}\sum_{t=T_3+1}^{\infty}\left|f\left(t,x^w\left(f_1(t)\right),\ldots,x^w\left(f_k(t)\right)\right)-f\left(t,x\left(f_1(t)\right),\ldots,x\left(f_k(t)\right)\right)\right|\\
&+\frac{1}{T}\sum_{i=T+\tau}^{T_1}\sum_{s=T_2+1}^{\infty}\sum_{t=s}^{\infty}\left|f\left(t,x^w\left(f_1(t)\right),\ldots,x^w\left(f_k(t)\right)\right)-f\left(t,x\left(f_1(t)\right),\ldots,x\left(f_k(t)\right)\right)\right|\\
&+\frac{1}{T}\sum_{i=T_1+1}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\left|f\left(t,x^w\left(f_1(t)\right),\ldots,x^w\left(f_k(t)\right)\right)-f\left(t,x\left(f_1(t)\right),\ldots,x\left(f_k(t)\right)\right)\right|\\
&<\frac{\varepsilon}{16}+\frac{2}{T}\sum_{t=T_1+1}^{\infty}H_t+\frac{\varepsilon}{16}+\frac{2}{T}\sum_{s=T+\tau}^{T_1}\sum_{t=T_2+1}^{\infty}G_t\\
&+\frac{2}{T}\sum_{s=T_1+1}^{\infty}\sum_{t=s}^{\infty}G_t+\frac{\varepsilon}{16}+\frac{2}{T}\sum_{i=T+\tau}^{T_1}\sum_{s=i}^{T_2}\sum_{t=T_3+1}^{\infty}F_t
\end{aligned}$$

$$+\frac{2}{T}\sum_{i=T+\tau}^{T_1}\sum_{s=T_2+1}^{\infty}\sum_{t=s}^{\infty}F_t + \frac{2}{T}\sum_{i=T_1+1}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}F_t < \varepsilon, \quad w \ge T_4, \tag{110}$$

which implies that S_L is continuous in $\Omega(A_*, B^*, T)$. It follows from (15), (108), and (109) that for any $x = \{x(n)\}_{n\in\mathbb{N}_\beta} \in \Omega(A_*, B^*, T)$ and $t_1, t_2 \ge T_4$

$$\begin{aligned}
&\left|\frac{(S_Lx)(t_1)}{t_1} - \frac{(S_Lx)(t_2)}{t_2}\right| \\
&= \Bigg|\frac{1}{t_1}\sum_{i=1}^{\infty}(-1)^i c(t_1+i\tau) - \frac{1}{t_2}\sum_{i=1}^{\infty}(-1)^i c(t_2+i\tau) \\
&\quad + \frac{1}{t_1}\sum_{s=1}^{\infty}\sum_{t=t_1+(2s-1)\tau}^{t_1+2s\tau-1} h(t, x(h_1(t)), \ldots, x(h_k(t))) \\
&\quad - \frac{1}{t_2}\sum_{s=1}^{\infty}\sum_{t=t_2+(2s-1)\tau}^{t_2+2s\tau-1} h(t, x(h_1(t)), \ldots, x(h_k(t))) \\
&\quad - \frac{1}{t_1}\sum_{i=1}^{\infty}\sum_{s=t_1+(2i-1)\tau}^{t_1+2i\tau-1}\sum_{t=s}^{\infty} g(t, x(g_1(t)), \ldots, x(g_k(t))) \\
&\quad + \frac{1}{t_2}\sum_{i=1}^{\infty}\sum_{s=t_2+(2i-1)\tau}^{t_2+2i\tau-1}\sum_{t=s}^{\infty} g(t, x(g_1(t)), \ldots, x(g_k(t))) \\
&\quad + \frac{1}{t_1}\sum_{p=1}^{\infty}\sum_{i=t_1+(2p-1)\tau}^{t_1+2p\tau-1}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}[f(t, x(f_1(t)), \ldots, x(f_k(t))) - d(t)] \\
&\quad - \frac{1}{t_2}\sum_{p=1}^{\infty}\sum_{i=t_2+(2p-1)\tau}^{t_2+2p\tau-1}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}[f(t, x(f_1(t)), \ldots, x(f_k(t))) - d(t)]\Bigg| \\
&\le \frac{1}{t_1}\sum_{i=1}^{\infty}|c(t_1+i\tau)| + \frac{1}{t_2}\sum_{i=1}^{\infty}|c(t_2+i\tau)| + \frac{2}{T_4}\sum_{t=T_4+\tau}^{\infty}H_t \\
&\quad + \frac{2}{T_4}\sum_{s=T_4+\tau}^{\infty}\sum_{t=s}^{\infty}G_t \\
&\quad + \frac{2}{T_4}\sum_{i=T_4+\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}[F_t + |d(t)|] < \varepsilon,
\end{aligned} \tag{111}$$

which means that $S_L(\Omega(A_*, B^*, T))$ is uniformly Cauchy, which together with (87) and Lemma 1 yields that $S_L(\Omega(A_*, B^*, T))$ is relatively compact. It follows from Lemma 3 that the mapping S_L has a fixed point $x = \{x(n)\}_{n\in\mathbb{N}_\beta} \in \Omega(A_*, B^*, T)$; that is,

$$\begin{aligned}
x(n) &= nL + \sum_{i=1}^{\infty}(-1)^i c(n+i\tau) \\
&\quad + \sum_{s=1}^{\infty}\sum_{t=n+(2s-1)\tau}^{n+2s\tau-1} h(t, x(h_1(t)), \ldots, x(h_k(t))) \\
&\quad - \sum_{i=1}^{\infty}\sum_{s=n+(2i-1)\tau}^{n+2i\tau-1}\sum_{t=s}^{\infty} g(t, x(g_1(t)), \ldots, x(g_k(t))) \\
&\quad + \sum_{p=1}^{\infty}\sum_{i=n+(2p-1)\tau}^{n+2p\tau-1}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}[f(t, x(f_1(t)), \ldots, x(f_k(t))) - d(t)], \quad n \ge T,
\end{aligned} \tag{112}$$

which gives that

$$\begin{aligned}
x(n) + x(n-\tau) &= (2n-\tau)L - c(n) \\
&\quad + \sum_{t=n}^{\infty} h(t, x(h_1(t)), \ldots, x(h_k(t))) \\
&\quad - \sum_{s=n}^{\infty}\sum_{t=s}^{\infty} g(t, x(g_1(t)), \ldots, x(g_k(t))) \\
&\quad + \sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}[f(t, x(f_1(t)), \ldots, x(f_k(t))) - d(t)], \\
&\qquad n \ge T+\tau.
\end{aligned} \tag{113}$$

It follows from (113) that

$$\begin{aligned}
\Delta(x(n) + x(n-\tau) + c(n)) &= 2L - h(n, x(h_1(n)), \ldots, x(h_k(n))) \\
&\quad + \sum_{t=n}^{\infty} g(t, x(g_1(t)), \ldots, x(g_k(t))) \\
&\quad - \sum_{s=n}^{\infty}\sum_{t=s}^{\infty}[f(t, x(f_1(t)), \ldots, x(f_k(t))) - d(t)], \\
&\qquad n \ge T+\tau,
\end{aligned}$$

$$\Delta^2 (x(n) + x(n-\tau) + c(n)) = -\Delta h(n, x(h_1(n)), \dots, x(h_k(n))) - g(n, x(g_1(n)), \dots, x(g_k(n))) + \sum_{t=n}^{\infty} [f(t, x(f_1(t)), \dots, x(f_k(t))) - d(t)], \quad n \geq T + \tau, \tag{114}$$

which yields that

$$\Delta^3 (x(n) + x(n-\tau) + c(n)) = -\Delta^2 h(n, x(h_1(n)), \dots, x(h_k(n))) - \Delta g(n, x(g_1(n)), \dots, x(g_k(n))) - [f(n, x(f_1(n)), \dots, x(f_k(n))) - d(n)], \quad n \geq T + \tau, \tag{115}$$

which together with (102) means that $x = \{x(n)\}_{n \in \mathbb{N}_\beta} \in \Omega(A_*, B^*, T)$ is a positive solution of (4). In view of (15)–(17) and (113), we get that

$$\begin{aligned} &\left| \frac{x(n) + x(n-\tau) + c(n)}{n} - 2L \right| \\ &= \left| -\frac{\tau L}{n} + \frac{1}{n} \sum_{t=n}^{\infty} h(t, x(h_1(t)), \dots, x(h_k(t))) - \frac{1}{n} \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} g(t, x(g_1(t)), \dots, x(g_k(t))) + \frac{1}{n} \sum_{i=n}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} [f(t, x(f_1(t)), \dots, x(f_k(t))) - d(t)] \right| \\ &\leq \frac{\tau L}{n} + \frac{1}{n} \sum_{t=n}^{\infty} H_t + \frac{1}{n} \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} G_t + \frac{1}{n} \sum_{i=n}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} [F_t + |d(t)|] \longrightarrow 0 \quad \text{as } n \longrightarrow \infty; \end{aligned} \tag{116}$$

that is, (103) holds. Similar to the proof of Theorem 5, we deduce that (19) holds.

(ii) Let $L_1, L_2 \in (A, B)$ and $L_1 \neq L_2$. Similarly we obtain that for each $l \in \{1, 2\}$, there exist a constant $T_l \geq n_0 + \tau + \alpha$ and two mappings $S_{L_l} : \Omega(A_*, B^*, T_l) \to l_\beta^\infty$ satisfying (83), (104), and (105), where T, L, and S_L are replaced by T_l, L_l, and S_{L_l}, respectively, and S_{L_l} possesses a fixed point $x_l = \{x_l(n)\}_{n \in \mathbb{N}_\beta} \in \Omega(A_*, B^*, T_l)$, which is a positive solution of (4); that is,

$$\begin{aligned} x_l(n) &= nL_l + \sum_{i=1}^{\infty} (-1)^i c(n + i\tau) + \sum_{i=1}^{\infty} \sum_{t=n+(2s-1)\tau}^{n+2s\tau-1} h(t, x_l(h_1(t)), \dots, x_l(h_k(t))) \\ &\quad - \sum_{i=1}^{\infty} \sum_{s=n+(2i-1)\tau}^{n+2i\tau-1} \sum_{t=s}^{\infty} g(t, x_l(g_1(t)), \dots, x_l(g_k(t))) \\ &\quad + \sum_{p=1}^{\infty} \sum_{i=n+(2p-1)\tau}^{n+2p\tau-1} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} [f(t, x_l(f_1(t)), \dots, x_l(f_k(t))) - d(t)], \quad n \geq T_l. \end{aligned} \tag{117}$$

Note that (16) and (17) imply that there exists $T_* > \max\{T_1, T_2\}$ satisfying

$$\frac{1}{T_*} \left(\sum_{t=T_*+\tau}^{\infty} H_t + \sum_{s=T_*+\tau}^{\infty} \sum_{t=s}^{\infty} G_t + \sum_{i=T_*+\tau}^{\infty} \sum_{s=i}^{\infty} \sum_{t=s}^{\infty} F_t \right) < \frac{|L_1 - L_2|}{4}. \tag{118}$$

In view of (15), (117), and (118), we infer that for any $n \geq T_*$

$$\begin{aligned} &\left| \frac{x_1(n)}{n} - \frac{x_2(n)}{n} \right| \\ &= \Bigg| L_1 - L_2 + \frac{1}{n} \sum_{s=1}^{\infty} \sum_{t=n+(2s-1)\tau}^{n+2s\tau-1} [h(t, x_1(h_1(t)), \dots, x_1(h_k(t))) - h(i, x_2(h_1(t)), \dots, x_2(h_k(t)))] \\ &\quad - \frac{1}{n} \sum_{i=1}^{\infty} \sum_{s=n+(2i-1)\tau}^{n+2i\tau-1} \sum_{t=s}^{\infty} [g(t, x_1(g_1(t)), \dots, x_1(g_k(t))) - g(t, x_2(g_1(t)), \dots, x_2(g_k(t)))] \end{aligned}$$

$$+\frac{1}{n}\sum_{p=1}^{\infty}\sum_{i=n+(2p-1)\tau}^{n+2p\tau-1}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}[f(t,x_1(f_1(t)),x_1(f_2(t)),\ldots,x_1(f_k(t)))-f(t,x_2(f_1(t)),\ldots,x_2(f_k(t)))]\Bigg|$$

$$\begin{aligned}
&\geq |L_1-L_2|\\
&\quad-\frac{1}{T_*}\sum_{t=T_*+\tau}^{\infty}|h(t,x_1(h_1(t)),\ldots,x_1(h_k(t)))-h(t,x_2(h_1(t)),\ldots,x_2(h_k(t)))|\\
&\quad-\frac{1}{T_*}\sum_{s=T_*+\tau}^{\infty}\sum_{t=s}^{\infty}|g(t,x_1(g_1(t)),\ldots,x_1(g_k(t)))-g(t,x_2(g_1(t)),\ldots,x_2(g_k(t)))|\\
&\quad-\frac{1}{T_*}\sum_{i=T_*+\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}|f(t,x_1(f_1(t)),\ldots,x_1(f_k(t)))-f(t,x_2(f_1(t)),\ldots,x_2(f_k(t)))|\\
&\geq |L_1-L_2|\\
&\quad-\frac{2}{T_*}\left(\sum_{t=T_*+\tau}^{\infty}H_t+\sum_{s=T_*+\tau}^{\infty}\sum_{t=s}^{\infty}G_t+\sum_{i=T_*+\tau}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}F_t\right)\\
&>\frac{|L_1-L_2|}{2}>0,
\end{aligned}\tag{119}$$

which yields that $x_1 \neq x_2$. Therefore (4) possesses uncountably many positive solutions in l_β^∞. This completes the proof. □

3. Illustrative Examples

Now we suggest six examples to explain the results presented in Section 2. Notice that none of the known results can be applied to these examples.

Example 1. Consider the third order nonlinear neutral delay difference equation:

$$\begin{aligned}
&\Delta^3\left(x(n)+\frac{n-2}{2n}x(n-\tau)+(-1)^n\frac{n+1}{n}\right)\\
&+\Delta^2\left(\frac{1}{n^2+\sqrt{|x(n-1)|}}\right)+\Delta\left(\frac{1}{n^3+2x^2(n^2-n)}\right)\\
&+\frac{1}{n^4+x^4(n-2)}\\
&=\frac{(-1)^n}{n^6+n^4-1},\quad n\geq 3,
\end{aligned}\tag{120}$$

where $\tau\in\mathbb{N}\setminus\{3\}$ is fixed. Let $n_0=3$, $k=1$, $\beta=\min\{|3-\tau|,1\}=1$, $A=3$, $B=12$, $b^*=1/2$, $c^*=3$, $B^*=14$, $A_*=1$, and

$$\begin{aligned}
&b(n)=\frac{n-2}{2n},\qquad c(n)=(-1)^n\frac{n+1}{n},\\
&f(n,u)=\frac{1}{n^4+u^4},\qquad g(n,u)=\frac{1}{n^3+2u^2},\\
&h(n,u)=\frac{1}{n^2+\sqrt{|u|}},\qquad d(n)=\frac{(-1)^n}{n^6+n^4+1},\\
&h_1(n)=n-1,\qquad g_1(n)=n^2-n,\\
&f_1(n)=n-2,\qquad F_n=\frac{1}{n^4},\\
&G_n=\frac{1}{n^3},\qquad H_n=\frac{1}{n^2},\\
&\forall (n,u)\in\mathbb{N}_{n_0}\times\mathbb{R}.
\end{aligned}\tag{121}$$

Note that for any $p>2$ and $q>3$

$$\begin{aligned}
0&\leq\frac{1}{n}\max\left\{\sum_{i=n}^{\infty}\sum_{t=i}^{\infty}\frac{1}{t^p},\sum_{i=n}^{\infty}\sum_{s=i}^{\infty}\sum_{t=s}^{\infty}\frac{1}{t^q}\right\}\\
&\leq\frac{1}{n}\max\left\{\sum_{t=n}^{\infty}\frac{1}{t^{p-1}},\sum_{t=n}^{\infty}\frac{1}{t^{q-2}}\right\}\longrightarrow 0\quad\text{as } n\longrightarrow\infty.
\end{aligned}\tag{122}$$

It is easy to see that (14)–(17) are satisfied. It follows from Theorem 5 that (120) possesses a positive solution $x=\{x(n)\}_{n\in\mathbb{N}_\beta}\in l_\beta^\infty$ satisfying (18) and (19). Moreover, (120) possesses uncountably many positive solutions in l_β^∞.

Example 2. Consider the third order nonlinear neutral delay difference equation:

$$\begin{aligned}
&\Delta^3\left(x(n)+\left(5+\frac{1}{2n}\right)x(n-\tau)+2+\frac{1}{2n}\right)\\
&+\Delta^2\left(\frac{1}{n^3+(n+1)x^6(2n-3)}\right)\\
&+\Delta\left(\frac{2}{2n^4+|x(n+5)|+2}\right)
\end{aligned}$$

$$+\frac{\sin\left[n^3 x\left(n^2-n\right)\right]}{n^6+x^2\left(n^2-n\right)}$$

$$=\frac{n^2-1}{n^6+n^3+2}, \quad n\geq 2, \tag{123}$$

where $\tau\in\mathbb{N}\setminus\{2\}$ is fixed. Let $n_0=2$, $k=1$, $\beta=\min\{|2-\tau|,1\}=1$, $A=5$, $B=200$, $b^*=6$, $b_*=5$, $c^*=4$, $B^*=204$, $A_*=1$, and

$$b(n)=5+\frac{1}{2n}, \qquad c(n)=2+\frac{1}{2n},$$

$$f(n,u)=\frac{\sin\left(n^3 u\right)}{n^6+u^2}, \qquad g(n,u)=\frac{2}{2n^4+|u|+2},$$

$$h(n,u)=\frac{1}{n^3+(n+1)u^6}, \qquad d(n)=\frac{n^2-1}{n^6+n^3+2},$$

$$f_1(n)=n^2-n, \qquad g_1(n)=n+5,$$

$$h_1(n)=2n-3, \qquad F_n=\frac{1}{n^6},$$

$$G_n=\frac{1}{n^4}, \qquad H_n=\frac{1}{n^3},$$

$$(n,u)\in\mathbb{N}_{n_0}\times\mathbb{R}. \tag{124}$$

It follows from (122) that (15)–(17) and (40) hold. Thus Theorem 6 ensures that (123) possesses a positive solution $x=\{x(n)\}_{n\in\mathbb{N}_\beta}\in l_\beta^\infty$ satisfying (19) and (41). Moreover, (123) possesses uncountably many positive solutions in l_β^∞.

Example 3. Consider the third order nonlinear neutral delay difference equation:

$$\Delta^3\left(x(n)+\frac{2-n}{2n}x(n-\tau)+\frac{64}{64-15n}\right)$$

$$+\Delta^2\left(\frac{4\cos x\left(n^2-2\right)}{2n^3+|x(2n-1)|}\right)$$

$$\cdot\Delta\left(\frac{1}{n^6+\sqrt{\left|x\left(n^2+n\right)\right|}x^4\left(n^2-2n\right)}\right) \tag{125}$$

$$+\frac{\sin x^2(2n-7)}{n^4+x^4(3n-8)}$$

$$=\frac{\sqrt{n+1}-\ln n}{n^8+n^5+3}, \quad n\geq 4,$$

where $\tau\in\mathbb{N}\setminus\{4\}$ is fixed. Let $n_0=4$, $k=2$, $\beta=\min\{|4-\tau|,1\}=1$, $A=30$, $B=300$, $b^*=-1/4$, $b_*=-1/2$, $c^*=20$, $A_*=10$, $B^*=320$, and

$$b(n)=\frac{2-n}{2n}, \qquad c(n)=\frac{64}{64-15n},$$

$$f(n,u,v)=\frac{\sin u^2}{n^4+v^4}, \qquad g(n,u,v)=\frac{1}{n^6+\sqrt{|u|}v^4},$$

$$h(n,u,v)=\frac{4\cos v}{2n^3+|u|}, \qquad d(n)=\frac{\sqrt{n+1}-\ln n}{n^8+n^5+3},$$

$$f_1(n)=2n-7, \qquad f_2(n)=3n-8,$$

$$g_1(n)=n^2+n, \qquad g_2(n)=n^2-2n,$$

$$h_1(n)=2n-1, \qquad h_2(n)=n^2-2,$$

$$F_n=\frac{1}{n^4}, \qquad G_n=\frac{1}{n^6}, \qquad H_n=\frac{2}{n^3},$$

$$(n,u,v)\in\mathbb{N}_{n_0}\times\mathbb{R}^2. \tag{126}$$

It follows from (122) that (15)–(17) and (63) hold. Thus Theorem 7 ensures that (125) possesses a positive solution $x=\{x(n)\}_{n\in\mathbb{N}_\beta}\in l_\beta^\infty$ satisfying (19) and (64). Moreover, (125) possesses uncountably many positive solutions in l_β^∞.

Example 4. Consider the third order nonlinear neutral delay difference equation

$$\Delta^3\left(x(n)+\frac{1-10n^2-10n}{n^2+n}x(n-\tau)+\frac{2n+2}{n^2}\right)$$

$$+\Delta^2\left(\frac{\sin x\left(2n^2-1\right)}{n^4+2\left|x\left(n^2+2n\right)\right|}\right)$$

$$+\Delta\left(\frac{1}{n^7+|x(n+10)|^3+x^2(5n-4)}\right) \tag{127}$$

$$+\frac{\cos 2x\left(n^2+3\right)}{n^5+x^2\left(4n^2-1\right)}$$

$$=\frac{1}{n^6+2n^3+8}, \quad n\geq 1,$$

where $\tau \in \mathbb{N} \setminus \{1\} = 1$ is fixed. Let $n_0 = 1$, $k = 2$, $\beta = \min\{|1-\tau|, 1\}$, $A = 10$, $B = 200$, $b^* = -4$, $b_* = -5$, $c^* = 5$, $A_* = 5$, $B^* = 205$, and

$$\begin{aligned} b(n) &= \frac{1-10n^2-10n}{n^2+n}, \qquad c(n) = \frac{2n+2}{n^2}, \\ f(n,u,v) &= \frac{\cos 2u}{n^5+v^2}, \qquad g(n,u,v) = \frac{3}{n^7+|u|^3+v^2}, \\ h(n,u,v) &= \frac{\sin v}{n^4+2|u|}, \qquad d(n) = \frac{1}{n^6+2n^3+8}, \\ f_1(n) &= n^2+3, \qquad f_2(n) = 4n^2, \\ g_1(n) &= n+10, \qquad g_2(n) = 5n-4, \\ h_1(n) &= n^2+2n, \qquad h_2(n) = 2n^2-1, \\ F_n &= \frac{1}{n^5}, \qquad G_n = \frac{1}{n^7}, \qquad H_n = \frac{1}{n^4}, \\ & (n,u,v) \in \mathbb{N}_{n_0} \times \mathbb{R}^2. \end{aligned} \tag{128}$$

It follows from (122) that (15)–(17) and (68) hold. Thus Theorem 8 ensures that (127) possesses a positive solution $x = \{x(n)\}_{n\in\mathbb{N}_\beta} \in l_\beta^\infty$ satisfying (19) and (69). Moreover, (127) possesses uncountably many positive solutions in l_β^∞.

Example 5. Consider the third order nonlinear neutral delay difference equation:

$$\begin{aligned} &\Delta^3\left(x(n) - x(n-\tau) + \frac{n+1}{n^3}\right) \\ &\quad + \Delta^2\left(\frac{1}{n^3+x^2(5n-4)}\right) + \Delta\left(\frac{1}{n^6+2x^8(n^2-n+1)}\right) \\ &\quad + \frac{\sin x^3(3n-2)}{n^8+3} = \frac{(-1)^{n(n+1)/2}}{n^{10}+n^2+3}, \quad n \geq 1, \end{aligned} \tag{129}$$

where $\tau \in \mathbb{N} \setminus \{1\}$ is fixed. Let $n_0 = 1$, $k = 1$, $\beta = 1$, $A = 3$, $B = 5$, $c^* = 2$, $A_* = 1$, $B^* = 7$, and

$$\begin{aligned} b(n) &= -1, \qquad c(n) = \frac{n+1}{n^3}, \\ f(n,u) &= \frac{\sin u^3}{n^8+3}, \qquad g(n,u) = \frac{1}{n^6+2u^8}, \\ h(n,u) &= \frac{1}{n^3+u^2}, \qquad d(n) = \frac{(-1)^{n(n+1)/2}}{n^{10}+n^2+3}, \\ f_1(n) &= 3n-2, \qquad g_1(n) = n^2-n+1, \\ h_1(n) &= 5n-4, \qquad F_n = \frac{1}{n^8}, \\ G_n &= \frac{1}{n^6}, \qquad H_n = \frac{1}{n^3}, \\ & \forall (n,u) \in \mathbb{N}_{n_0} \times \mathbb{R}^+ \setminus \{0\}. \end{aligned} \tag{130}$$

It follows from (122) that (15) and (78)–(81) are satisfied. Thus Theorem 9 ensures that (129) possesses a positive solution $x = \{x(n)\}_{n\in\mathbb{N}_\beta} \in l_\beta^\infty$ satisfying (19) and (82). Moreover, (129) possesses uncountably many positive solutions in l_β^∞.

Example 6. Consider the third order nonlinear neutral delay difference equation:

$$\begin{aligned} &\Delta^3\left(x(n) + x(n-\tau) + \frac{2n+4}{n^3}\right) \\ &\quad + \Delta^2\left(\frac{1}{n^4+2x^2(3n-4)}\right) + \Delta\left(\frac{1}{n^8+|x^3(n-2)|}\right) \\ &\quad + \frac{\sin\left[5x\left(n^2-3\right)\right]}{n^5+8} = \frac{1}{n^8+n^5+5}, \quad n \geq 2, \end{aligned} \tag{131}$$

where $\tau \in \mathbb{N} \setminus \{2\}$ is fixed. Let $n_0 = 2$, $k = 1$, $\beta = 1$, $A = 100$, $B = 101$, $c^* = 1$, $A_* = 99$, $B^* = 102$, and

$$\begin{aligned} b(n) &= 1, \qquad c(n) = \frac{2n+4}{n^3}, \\ f(n,u) &= \frac{\sin(5u)}{n^5+8}, \qquad g(n,u) = \frac{1}{n^8+|u|^3}, \\ h(n,u) &= \frac{1}{n^4+2u^2}, \qquad d(n) = \frac{1}{n^8+n^5+5}, \\ f_1(n) &= n^2-3, \qquad g_1(n) = n-2, \\ h_1(n) &= 3n-4, \qquad F_n = \frac{1}{n^5}, \\ G_n &= \frac{1}{n^8}, \qquad H_n = \frac{1}{n^4}, \\ & (n,u) \in \mathbb{N}_{n_0} \times \mathbb{R}. \end{aligned} \tag{132}$$

It follows from (122) that (15)–(17), (80), and (100) hold. Thus Theorem 10 ensures that (131) possesses a positive solution $x = \{x(n)\}_{n\in\mathbb{N}_\beta} \in l_\beta^\infty$ satisfying (19) and (103). Moreover, (131) possesses uncountably many positive solutions in l_β^∞.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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