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Hindawi Publishing Corporation
Advances in Fuzzy Systems
Volume 2016, Article ID 3839265, 4 pages
http://dx.doi.org/10.1155/2016/3839265

Research Article

Fuzzy Logic versus Classical Logic: An Example in Multiplicative Ideal Theory

Olivier A. Heubo-Kwegna

Department of Mathematical Sciences, Saginaw Valley State University, 7400 Bay Road, University Center, MI 48710-0001, USA

Correspondence should be addressed to Olivier A. Heubo-Kwegna; oheubokw@svsu.edu

Received 6 October 2016; Accepted 24 November 2016

Academic Editor: Erich Peter Klement

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We discuss a fuzzy result by displaying an example that shows how a classical argument fails to work when one passes from classical logic to fuzzy logic. Precisely, we present an example to show that, in the fuzzy context, the fact that the supremum is naturally used in lieu of the union can alter an argument that may work in the classical context.

1. Introduction

Rosenfeld in 1971 was the first classical algebraist to introduce fuzzy algebra by writing a paper on fuzzy groups [1]. The introduction of fuzzy groups then motivated several researchers to shift their interest to the extension of the seminal work of Zadeh [2] on fuzzy subsets of a set to algebraic structures such as rings and modules [3–7]. In that regard, Lee and Mordeson in [3, 4] introduced the notion of fractionary fuzzy ideal and the notion of invertible fractionary fuzzy ideal and used these notions to characterize Dedekind domains in terms of the invertibility of certain fractionary fuzzy ideals, leading to the fuzzification of one of the main results in multiplicative ideal theory. Other significant introduced notions to tackle the fuzzification of multiplicative ideal theory are the notion of fuzzy star operation [8] and the notion of fuzzy semistar operation [9, 10] on integral domains. This paper is concerned with the fuzzification of multiplicative ideal theory in commutative algebra (see, e.g., [1, 3, 5, 8–11]).

In the field of commutative ring, it is customary to use star operations not only to generalize classical domains, but also to produce a common treatment and deeper understanding of those domains. Some of the instances are the notion of Prüfer $\star$-multiplication domain which generalizes the notion of Prüfer domain [12] and the notion of $\star$-completely integrally closed domain which generalizes the notion of completely integrally closed domain [13, 14]. The importance of star operations in the classical theory has led scholars to be interested in fuzzy star operations introduced in [8] and this has been generalized to fuzzy semistar operations in [10]; this generalization has led to more fuzzification of main results in multiplicative ideal theory.

In this note, we focus on some classical arguments of multiplicative ideal theory that do not hold in the fuzzy context. The example chosen is to infer that what appears to be pretty simple and even rather easy in the context of classical logic may not be true in the fuzzy context. So, one challenge of fuzzification is to detect any defect or incongruous statement that may first appear benign but is a real poison in the argument used to prove fuzzy statements. Precisely, in our example, we display the difficulty in how the natural definition in the fuzzy context may make it a little bit more challenging to work with in comparison with its equivalent classical definition. For an overview of all definitions of fuzzy submodules, fuzzy ideals, and fuzzy (semi)star operations (of finite character), the reader may refer to [8–10, 15].

2. Preliminaries and Notations

Recall that an integral domain R is a commutative ring with identity and no-zero divisors. Hence, its quotient ring L is a field. A group $(M, +)$ is an *R-module* if there is a mapping $R \times M \to M, (r, x) \mapsto rx$, satisfying the following conditions:

$1x = x$; $r(x - y) = rx - ry$; and $(rt)x = r(tx)$ for all $r, t \in R$ and $x, y \in M$, where 1 is the identity of R. Note that the quotient field K of an integral domain R is an R-module. An *R-submodule* N of an R-module M is a subgroup of M such that $rx \in N$ for all $r \in R$ and $x \in N$. For more reading on integral domains and modules, the reader may refer to [7, 15]. Recall also that a *star operation* on R is a mapping $A \to A^\star$ of $F(R)$ into $F(R)$ such that, for all $A, B \in F(R)$ and for all $a \in L \setminus \{0\}$,

(i) $(a)^\star = (a)$ and $(aA)^\star = aA^\star$;

(ii) $A \subseteq B \Rightarrow A^\star \subseteq B^\star$;

(iii) $A \subseteq A^\star$ and $A^{\star\star} := (A^\star)^\star = A^\star$.

For an overview of star operations, the reader may refer to [15, Sections 32 and 34].

A *fuzzy subset* of L is a function from L into the real closed interval $[0, 1]$. We say $\alpha \subseteq \beta$ if $\alpha(x) \leq \beta(x)$ for all $x \in L$. The *intersection* $\bigcap_{i\in I} \alpha_i$ of the fuzzy subsets α_i's is defined as $\bigcap_{i\in I} \alpha_I(x) = \bigwedge_i \alpha(x)$ and the *union* $\bigcup_{i\in I} \alpha_I$ of the fuzzy subsets α_i's is defined as $\bigcup_{i\in I} \alpha_i(x) = \bigvee_i \alpha(x)$ for every $x \in L$. Let $\beta_t = \{x \in R : \beta(x) \geq t\}$; then, β_t is called a *level subset* of β. We let χ_A denote the characteristic function of the subset A of R. A fuzzy subset of L is a fuzzy R-submodule of L if $\beta(x - y) \geq \beta(x) \wedge \beta(y)$, $\beta(rx) \geq \beta(x)$, and $\beta(0) = 1$, for every $x, y \in L$ and every $r \in R$. Note that a fuzzy subset β of L is a fuzzy R-submodule of L if and only if $\beta(0) = 1$ and β_t is an R-submodule of L for every real number t in $[0, 1]$. Let d_t denote the fuzzy subset of L defined as follows: for each x in L, $d_t(x) = t$ if $x = d$ and $d_t(x) = 0$ otherwise. We call d_t a *fuzzy singleton*. A fuzzy R-submodule β of L is *finitely generated* if β is generated by some finite fuzzy singletons; that is, it is the smallest fuzzy R-submodule of L containing those fuzzy singletons. Throughout this paper, $F_z(R)$ denotes the set of all fuzzy R-submodules of L and $f_z(R)$ denotes the set of all finitely generated fuzzy R-submodules of L.

Definition 1 (see [9]). A *fuzzy semistar operation* on R is a mapping $\star : \overline{F}_z(R) \to \overline{F}_z(R)$, $\beta \mapsto \beta^\star$, which satisfies the following three properties for all $\alpha, \beta \in \overline{F}_z(R)$, and $0 \neq d \in K$:

$(\star_1)$ $(d_1 \circ \beta)^\star = d_1 \circ \beta^\star$;

$(\star_2)$ $\alpha \subseteq \beta \Rightarrow \alpha^\star \subseteq \beta^\star$;

$(\star_3)$ $\beta \subseteq \beta^\star$ and $\beta^{\star\star} := (\beta^\star)^\star = \beta^\star$.

Recall from [9] that a fuzzy semistar operation $\star$ on R is *union preserving* if $(\bigcup_{n\in\mathbb{Z}_+} \beta_n)^\star = \bigcup_{n\in\mathbb{Z}_+} \beta_n^\star$. Note that the preservation of union on $\star$ is over a countable set. Now, define a mapping $\star_f$ from $\overline{F}_z(R)$ into $\overline{F}_z(R)$ as follows:

$$\beta \longmapsto \beta^{\star_f} := \bigcup \{\alpha^\star \mid \alpha \in f_z(R),\ \alpha \subseteq \beta\}. \tag{1}$$

Then, if $\star$ is a union preserving fuzzy semistar operation on R, then $\star_f$ is a fuzzy semistar operation on R [9, Theorem 3.5]. This leads to the following definition.

Definition 2 (see [9]). Let $\star$ be a fuzzy semistar operation on R.

(1) If $\star_f$ is a fuzzy semistar operation on R, then $\star_f$ is called *the fuzzy semistar operation of finite character (or finite type) associated with* $\star$.

(2) $\star$ is called a *fuzzy semistar operation of finite character* if $\star = \star_f$.

Example 3. (1) It is clear by definition that $(\star_f)_f = \star_f$; that is, $\star_f$ is of finite character whenever $\star_f$ is a fuzzy semistar operation on R for any fuzzy semistar operation $\star$ on R.

(2) The constant map $\beta \mapsto \chi_L$ is also trivially a fuzzy semistar operation on R that is not of finite character.

(3) Let $\mathbb{Z}$ denote the set of all integers with quotient field $\mathbb{Q}$ of all rational numbers. Let $L = [0, 1]$ be the unit interval (note that the unit interval is a completely distributive lattice). Define $\star : \overline{F}_z(\mathbb{Z}) \to \overline{F}_z(\mathbb{Z})$ by

$$\beta^\star(x) = \begin{cases} 1, & \text{if } x = 0; \\ \displaystyle\bigvee_{y\in\mathbb{Q}\setminus\{0\}} \beta(y), & \text{if } \beta(x) \neq 0,\ x \neq 0; \\ 0, & \text{if } \beta(x) = 0, \end{cases} \tag{2}$$

for any $\beta \in \overline{F}_z(\mathbb{Z})$. Then, $\star$ is a fuzzy semistar operation on $\mathbb{Z}$ of finite character (the reader may refer to [9, Example 3.8. (2)] for the proof of this fact).

3. Fuzzy Logic versus Classical Logic: An Example

Recall from [9] that a fuzzy semistar operation $\star$ on R is said to be *union preserving* if $(\bigcup_{n\in\mathbb{Z}_+} \beta_n)^\star = \bigcup_{n\in\mathbb{Z}_+} \beta_n^\star$. Note that the preservation of union on $\star$ is over a countable set. Also, recall the following result in [9].

Theorem 4 (see [9, Theorem 3.5]). *Let $\star$ be a union preserving fuzzy semistar operation on R. Then, $\star_f$ is a fuzzy semistar operation on R.*

Let R be an integral domain with quotient field L. Recall that $F_z(R)$ denotes the set of all fuzzy R-submodules of L and $f_z(R)$ denotes the set of finitely generated fuzzy R-submodules of L. Now, we claim that we could not get rid of the assumption in Theorem 4 because we could not use the fuzzy counterpart of the following classical argument below.

3.1. The Fuzzy and Classical Statements

A Classical Argument. Let B be a submodule of R in L and let I be a finitely generated submodule of R in L such that $I \subseteq \bigcup\{A^\star \mid A \in f(R) \text{ and } A \subseteq B\}$, where $\star$ is a classical semistar operation on R. Then, I is contained in some $A_i^\star$ with $A_i \in f(R)$ and $A_i \subseteq B$. This classical argument is a well-known simple argument in multiplicative ideal theory. In fact, suppose we set $I = \langle x_1, \ldots, x_s\rangle$ as a finitely generated ideal such that $I \subseteq \bigcup\{A^\star \mid A \in f(R) \text{ and } A \subseteq B\}$. Then, for each $i = 1, \ldots, s$, $x_i \in A_i^\star$ with $A_i \in f(R)$ and $A_i \subseteq B$. So, $I = \langle x_1, \ldots, x_s\rangle \subseteq \sum_{i=1}^s A_i^\star$. Now, using the well-known facts that $A \subseteq A^\star$ and $(\sum_{i=1}^s A_i^\star)^\star = (\sum_{i=1}^s A_i)^\star$, for a classical

semistar operation $\star$ on R, we obtain that $I \subseteq (\sum_{i=1}^{s} A_i)^\star$. Now, since each A_i is finitely generated, the finite sum of A_i's is also finitely generated and this completes the proof.

The Fuzzy Counterpart of the Above Classical Argument. Let β be a fuzzy R-submodule of L and let α be a finitely generated fuzzy R-submodule of L such that $\alpha \subseteq \bigcup\{\gamma^\star \mid \gamma \in f_z(R) \text{ and } \gamma \subseteq \beta\}$, where $\star$ is a fuzzy semistar operation on R. Then, α is contained in some $\gamma_i^\star$, with $\gamma_i \in f_z(R)$ and $\gamma_i \subseteq \beta$.

3.2. A Counterexample to Negate the Fuzzy Counterpart. We now produce an example to prove that the fuzzy counterpart statement is false. Note that the reason why the counterpart may be false is clearly the fact that the union in the fuzzy context is the supremum. So, the real challenge here is to construct a counterexample that will clearly justify the wrongness of the argument.

The Counterexample. Let $\star$ be the fuzzy semistar operation of finite character as defined in Example 3(4):

$\star : \overline{F}_z(\mathbb{Z}) \to \overline{F}_z(\mathbb{Z})$ by

$$\beta^\star(x) = \begin{cases} 1, & \text{if } x = 0; \\ \bigvee\limits_{y \in \mathbb{Q}\setminus\{0\}} \beta(y), & \text{if } \beta(x) \neq 0,\ x \neq 0; \\ 0, & \text{if } \beta(x) = 0, \end{cases} \tag{3}$$

for any $\beta \in \overline{F}_z(\mathbb{Z})$.

Let $\mathbb{Q}$ denote the quotient field of all rational numbers. We define $\beta : \mathbb{Q} \to [0,1]$ (note that the unit interval is a completely distributive lattice), and we use the known fact that β is a fuzzy $\mathbb{Z}$-submonoid of $\mathbb{Q}$ if and only if β_t is a $\mathbb{Z}$-submonoid of $\mathbb{Q}$ for any $t \in [0,1]$ and $\beta(0) = 1$. Let $f : \mathbb{Z} \to [0,1]$ be defined by

$$f(n) = \frac{1}{2}\left(1 + \operatorname{sgn}(n)\frac{|n|}{|n|+1}\right), \tag{4}$$

where sgn is the signature function and $|n|$ denotes the absolute value of n. It is easy to see that $f(n) \to 1$ for $n \to \infty$ and $f(n) \to 0$ for $n \to -\infty$. Consider an infinite sequence of $\mathbb{Z}$-submodules of $\mathbb{Q}$ as follows:

$$\begin{aligned} \cdots 2^n\mathbb{Z} \subseteq 2^{n-1}\mathbb{Z} \subseteq \cdots 2^0\mathbb{Z} \subseteq \cdots \frac{1}{2^{n-1}}\mathbb{Z} \subseteq \frac{1}{2^n}\mathbb{Z} \\ \subseteq \cdots \mathbb{Q}. \end{aligned} \tag{5}$$

Obviously, $2^n\mathbb{Z}$ is a $\mathbb{Z}$-submodule of $\mathbb{Q}$ for any $n \in \mathbb{Z}$, since $r2^n - s2^n = (r-s)2^n$ and $r(s2^n) = (rs)2^n$. Moreover, $2^n\mathbb{Z} \subseteq 2^m\mathbb{Z}$, whenever $m < n$. Indeed, $r2^n = (r2^{n-m})2^m$, which implies $r2^n \in 2^m\mathbb{Z}$. Then, one can define β for any $x \in \mathbb{Q}$ by

$$\beta(x) = \bigvee\{f(n) \mid x \in 2^n\mathbb{Z},\ n \in \mathbb{Z}\}. \tag{6}$$

Note that if $x \notin 2^n\mathbb{Z}$ for any $n \in \mathbb{Z}$, then $\beta(x) = 0$ (e.g., $\beta(\sqrt{2}) = 0$). On the other hand,

$$\begin{aligned} \beta(1) = \beta(3) &= \frac{1}{2}, \\ \beta\left(\frac{1}{2}\right) = \beta\left(\frac{3}{2}\right) &= \frac{1}{4}, \\ \beta(2) = \beta(6) &= \frac{3}{4}. \end{aligned} \tag{7}$$

Since $0 \in 2^n\mathbb{Z}$ for any $n \in \mathbb{Z}$ (and 0 is the unique element having this property), a consequence of the supremum is $\beta(0) = 1$. Thus, $\beta(x) = 1$ if and only if $x = 0$. It should be noted that if $0 < \beta(x) < 1$, then there exists $n \in \mathbb{Z}$ such that $\beta(x) = f(n)$. Indeed, it is easy to see from definition of f and *beta* that $x \in \bigcup_{i\in\mathbb{Z}} 2^i\mathbb{Z}$ and there exist $m, m' \in \mathbb{Z}$ such that $0 < f(m') < \beta(x) < f(m) < 1$ (see the remark above about the convergence of f to zero and one); therefore, $x \in 2^{m'}\mathbb{Z}$ and $x \notin 2^m\mathbb{Z}$. Hence, $\beta(x) = f(n)$ for a suitable $n \in \mathbb{Z}$ for which $m' < n < m$, since the supremum is calculated over only a finite set of linearly ordered values.

To demonstrate that β is a fuzzy $\mathbb{Z}$-submodule of $\mathbb{Q}$, let us first consider the cases $t = 0$ and $t = 1$. One can simply check that $\beta_0 = \mathbb{Q}$ and $\beta_1 = \{0\}$. If $t \in (0,1)$, then there exists exactly one $n \in \mathbb{Z}$ such that $f(n-1) < t \leq f(n)$ and $\beta_t = 2^n\mathbb{Z}$. Therefore, each t-level subset of β is a $\mathbb{Z}$-submonoid of $\mathbb{Q}$. Since $\beta(0) = 1$, $\beta \in \overline{F_z(\mathbb{Z})}$.

Now, let us consider

$$\beta^\star = \beta^{\star_f} = \{\gamma^\star \mid \gamma \in f_z(R),\ \gamma \subseteq \beta\}, \tag{8}$$

where β and $\star$ have been defined above. Put $\mathbb{K} = \bigcup_{i\in\mathbb{Z}} 2^i\mathbb{Z}$. Let us show that $\beta^\star = \chi_{\mathbb{K}}$. Since $\beta(x) = 0$ for any $x \in \mathbb{Q} \setminus \mathbb{K}$, we have $\beta^\star(x) = 0$; therefore, $\beta^\star \subseteq \chi_{\mathbb{K}}$. To show the opposite inequality, let $x \in \mathbb{K}$ and without loss of generality let $x \in 2^{n_0}\mathbb{Z}$ and $x \notin 2^{n_0+1}\mathbb{Z}$. Consider $x_n \in 2^n\mathbb{Z}$ for $n = n_0 + 1, n_0 + 2, \ldots$. It is easy to see that $\alpha_n = \langle (x)_{\beta(x)}, (x_n)_{\beta(x_n)} \rangle \subseteq \beta$ and α_n is finitely generated $\mathbb{Z}$-submodule of $\mathbb{Q}$ for any $n > n_0$. Moreover, $\alpha_n(x) = \beta(x)$ and $\alpha_n(x_n) = \beta(x_n)$, where $\beta(xn) > \beta(x) > 0$. Then, by definition of $\star$, we obtain

$$\begin{aligned} \beta^{\star_f} \geq \left(\bigcup_{n=n_0+1}^{\infty} \alpha_n^\star\right)(x) \geq \bigvee_{n=n_0+1}^{\infty} \beta(x) = \bigvee_{n=n_0+1}^{\infty} f(n) = 1 \\ = \chi_{\mathbb{K}}(x). \end{aligned} \tag{9}$$

Now, let us demonstrate the false argument. Let us consider $\alpha = \langle x_1 \rangle$ for some $x \in \mathbb{K} \setminus \{0\}$. Obviously, $\alpha \in f_z(\mathbb{Z})$ and $\alpha \subseteq \beta^{\star_f}$. According to our false argument, it should be true that "since α is finitely generated, α is contained in finitely many $\gamma_i^\star$ with $\gamma_i \in f_z(\mathbb{Z})$ and $\gamma_i \subseteq \beta$." But this is impossible, because for any choice of finitely many $\gamma_1, \ldots, \gamma_n \in f_z(\mathbb{Z})$ we can find $x_1, \ldots, x_m \in \mathbb{K}$ (it is sufficient to consider elements of $\mathbb{K}$ that are used for generating $\gamma_1, \ldots, \gamma_n$) such that $(\sum_{i=1}^{n} \gamma_i^\star)(x) \leq \bigvee_{j=1}^{m} \beta(x_j) < 1 = \alpha(x)$.

3.3. Final Remark. The proof of the classical argument holds due to the fact that the classical union is involved allowing the choice of a finitely generated R-submodule of L for each element of I. However, in the fuzzy counterpart statement, the fuzzy union is defined in terms of the supremum and the technique used in the proof of the classical argument cannot apply in the fuzzy context since clearly $a = \bigvee_{i\in I} a_i$ does not imply the existence of a_{i_0}, $i_0 \in I$, with $a \leq a_{i_0}$.

We must also note that the fuzzy counterpart statement is the natural one that grasps some thoughts about the context in which the crisp result can be extended. In fact, the condition of union preserving of fuzzy star operation, that is, $(\bigcup_{n\in\mathbb{Z}_+} \beta_n)^\star = \bigcup_{n\in\mathbb{Z}_+} \beta_n^\star$, which does not always hold in the fuzzy context is not needed in the crisp case to get a classical finite character semistar operation. This additional condition of union preserving of fuzzy star operation will make our fuzzy counterpart statement true.

Competing Interests

The author declares that there are no competing interests regarding the publication of this paper.

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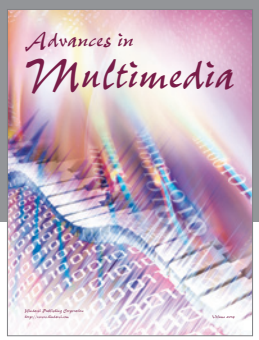

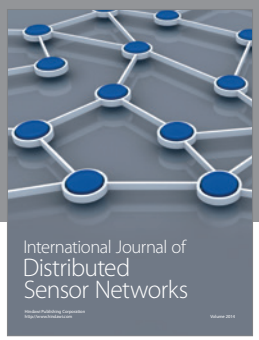

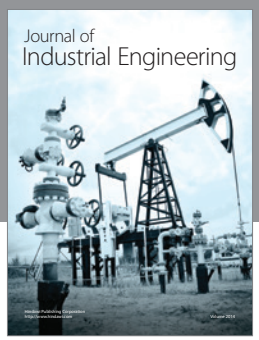

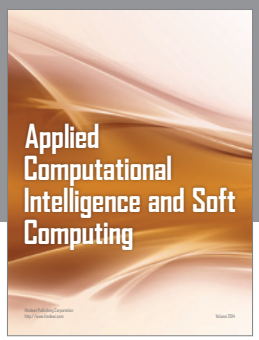

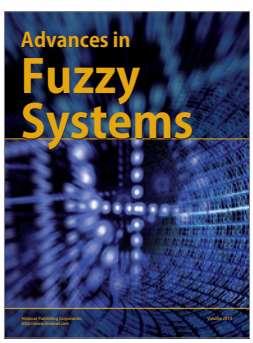

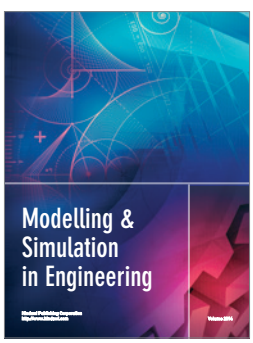

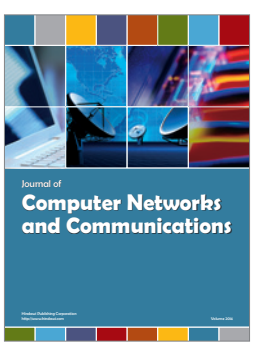
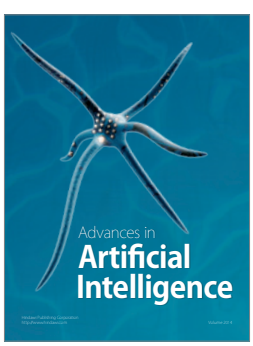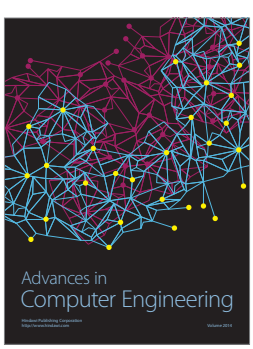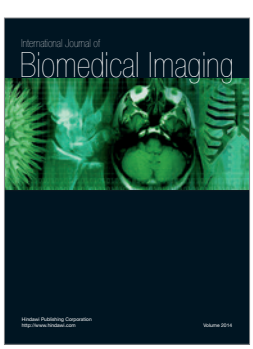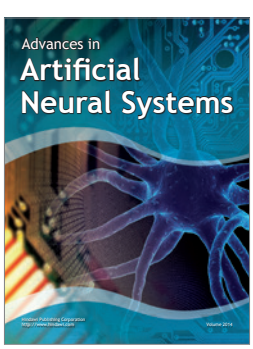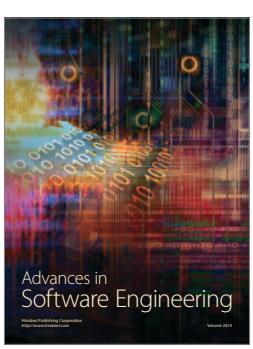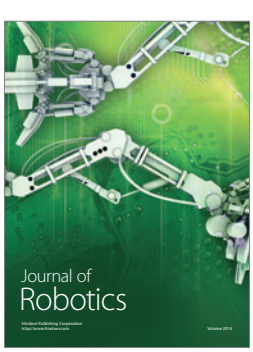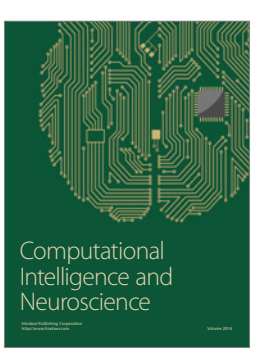
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