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Research Article

Exp-Function Method for a Generalized MKdV Equation

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Under investigation in this paper is a generalized MKdV equation, which describes the propagation of shallow water in fluid mechanics. In this paper, we have derived the exact solutions for the generalized MKdV equation including the bright soliton, dark soliton, two-peak bright soliton, two-peak dark soliton, shock soliton and periodic wave solution via Exp-function method. By figures and symbolic computations, we have discussed the propagation characteristics of those solitons under different values of those coefficients in the generalized MKdV equation. The method constructing soliton solutions in this paper may be useful for the investigations on the other nonlinear mathematical physics model and the conclusions of this paper can give theory support for the study of dynamic features of models in the shallow water.

1. Introduction

Nonlinear science studies nonlinear phenomena of the world, which is cross-disciplinary [1–4]. In fact, any science, whether natural science or social science, has its own nonlinear phenomena and problems. To study these phenomena and problems, many branches of nonlinear science have been promoted to be built and developed [5–7]. Obviously, most of the phenomena are not linear in the natural sciences and engineering practice; thus many problems cannot be researched and solved by the linear methods, which makes the study of nonlinear science very significant. Actually, nonlinear science, which mainly consists of soliton, chaos, and fractal [3, 4, 7], is not the simple superposition and comprehensive of these nonlinear branches but a comprehensive subject to study the various communist rules in nonlinear phenomena. With the rapid development of nonlinear science, the study of exact solutions of nonlinear evolution equations has attracted much attention of many mathematicians and physicists.

In 1834, the solitary wave phenomenon was observed by Huang et al. [8] and Russell [9]. Later, people named the isolation water peak, which kept moving with constant shape and speed on the surface of the water, as a solitary wave [8]. The discovery of a solitary wave turned people into a new field in the study of the waves of the convection. In 1895, Holland's Professor Kortewrg and his disciples Vries derived the famous KdV equation from the research of shallow water wave motion [10]. The analysis to the KdV equation has improved to recover inverse scattering method, on which people expanded the new research directions of algebra and geometry [8]:

$$u_t \pm 6uu_x + u_{xxx} = 0. \tag{1}$$

MKdV equation plays an important role in describing the plasmas and phonon in anharmonic lattice. In the nineteen fifties, physicist Fermi, Pasta, and Ulam made a famous FPU experiment, connecting the 64 particles by nonlinear spring, thereby forming a nonlinear vibrating string. Although the FPU experiment did not gain the solutions of solitary wave, it will expand the study of the solitary wave to the field outside the mechanics [11]. Later, Fermi et al. studied the nonlinear vibration problem of FPU model and obtained the solitary wave solutions [11], which is the right answer for the question of FPU. It is the first time to find solitary wave solutions in the field outside mechanics after the solution was found in the KdV equation, which give rise to the scientists' interest in researching the solitary wave phenomenon. In 1965, Zabusky and Kruskal studied solitary waves in plasma [12]; they found that the waveform does not change nature before and after collision of nonlinear solitary waves, which is similar to particle collisions, so Zabusky and Kruskal named the solitary wave with the impact properties of collisions as soliton [12]. Soliton concept is an important milepost on the history of the development of the soliton theory. The next few decades, the soliton theory had a rapid development and penetrated into many areas, such as fluid mechanics, nonlinear optical fiber communication, plasma physics, fluid physics, chemistry, life science, and marine science [13–15].

Recently many new approaches to nonlinear wave equations have been proposed, such as Tanh-function method [16–18], F-expansion method [19], Jacobian elliptic function method [20], Darboux transformation method [21–26], adomian method [27–29], variational approach [30], and homotopy perturbation method [31]. All methods mentioned above have their limitations in their applications. We will apply Exp-function method to a generalized MKdV equation to gain exact solutions:

$$u_t + \alpha u^2 u_x + \beta u_{xxx} = 0, \tag{2}$$

where α and β are real parameters. When $u(x,t) = c$ (c is a constant) as $t \to \pm\infty$, solitons can be obtained.

This paper will be organized as follows. In Section 2, the basic idea of Exp-function method is introduced. In Section 3, carrying on calculating and illustrating by the mathematical software MATHEMATIC, we will solve the solitary wave solutions of (2) based on the Exp-function method. In Section 4, we will obtain shock soliton, bright soliton, two-peak bright soliton, dark soliton, two-peak dark soliton, and periodic wave solutions and analyze the dynamic features of soliton solutions by using some figures. Finally, our conclusions will be addressed in Section 5.

2. Basic Idea of Exp-Function Method

In order to illustrate the basic idea of the suggested method, we consider firstly the following general partial differential equation:

$$F\left(u, u_t, u_x, u_{xt}, u_{xx}, \ldots\right) = 0. \tag{3}$$

We aim at its exact solutions, so we introduce a complex variable, ξ, defined as

$$\xi = kx + \omega t, \tag{4}$$

where ω and k are constants unknown to be further determined. Therefore we can convert (3) into an ordinary differential equation with respect to ξ:

$$F\left(u, u_\xi, u_{\xi\xi}, \ldots\right) = 0. \tag{5}$$

Very simple and straightforward, the Exp-function method is based on the assumption that traveling wave solutions can be expressed in the following form:

$$u(\xi) = \frac{\sum_{n=-d}^{c} a_n \exp(n\xi)}{\sum_{m=-q}^{p} b_m \exp(m\xi)}, \tag{6}$$

where c, d, p, and q are positive integers which are unknown to be further determined and a_n and b_m are unknown constants.

We suppose that $u(\xi)$ which is the solution of (5) can be expressed as

$$u(\xi) = \frac{a_c \exp(c\xi) + \cdots + a_{-d} \exp(-d\xi)}{b_p \exp(p\xi) + \cdots + b_{-q} \exp(-q\xi)}. \tag{7}$$

To determine the values of c and p, we balance the linear term of highest order in (5) with the highest order nonlinear term. Similarly we balance the lowest orders to confirm d and q. For simplicity, we set some particular values for c, p, d, and q and then change the left side of (5) into the polynomial of $\exp(n\xi)$. Equating the coefficients of $\exp(n\xi)$ to be zero results in a set of algebraic equations. Then solving the algebraic system with symbolic computation system, we can gain the solution. Substituting it into (3), we have the general form of the exact solution expressed as the form of $\exp(n\xi)$. Taking the form of the solutions into consideration, we can have more extensive solutions via Exp-function method.

3. Solving Generalized MKdV Equation via Exp-Function Method

Now we consider the follwing equation [32, 33]:

$$u_t + u^2 u_x + u_{xxx} = 0. \tag{8}$$

This equation is called modified KdV equation, which arises in the process of understanding the role of nonlinear dispersion and in the formation of structures like liquid drops. The KdV equation is one of the most familiar models for solitons and the foundation which studies other equations. Developing upon this foundation, the isolated theories are treated as the milestone of mathematics physical method. During researching the isolated theories, the remarkable application should be laser shooting practice and fiber-optic communication.

For researching the variety of the solutions for the modified KdV equation, we carry on the MKdV equation to expand. Thus, we have

$$u_t + \alpha u^2 u_x + \beta u_{xxx} = 0, \tag{9}$$

where α and β are real parameters. In the following, we consider (9).

Introduce a complex variable, η, defined as

$$\begin{gathered} u = u(\eta), \\ \eta = kx + \omega t. \end{gathered} \tag{10}$$

By (10), (9) becomes

$$\omega u' + \alpha k u^2 u' + \beta k^3 u''' = 0, \tag{11}$$

which denotes the differential with respect to η.

The Exp-function method is based on the assumption that traveling wave solutions can be expressed in the following form:

$$u(\eta) = \frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)}, \tag{12}$$

where c, d, p, and q are positive integers unknown to be further determined and a_n and b_m are unknown constants.

We suppose that the solution of (11) can be expressed as

$$u(\eta) = \frac{a_c \exp(c\eta) + \cdots + a_{-d} \exp(-d\eta)}{b_p \exp(p\eta) + \cdots + b_{-q} \exp(-q\eta)}. \tag{13}$$

To determine values of c and p, we balance the linear term of highest order in (11) with the highest order nonlinear term. By simple calculation, we have

$$u''' = \frac{c_1 \exp[(7p+c)\eta] + \cdots}{c_2 \exp[8p\eta] + \cdots},$$
$$u^2 u' = \frac{c_3 \exp[(p+3c)\eta] + \cdots}{c_4 \exp[4p\eta] + \cdots} = \frac{c_3 \exp[(5p+3c)\eta] + \cdots}{c_4 \exp[8p\eta] + \cdots}, \tag{14}$$

where c_i $(i = 1, 2, 3, 4)$ are coefficients only for simplicity.

Balancing highest order of exponential function in (14), we have

$$5p + c = 7p + c, \tag{15}$$

which leads to the result

$$p = c. \tag{16}$$

Similarly to make sure of the values of d and q, we balance the linear term of lowest order in (11):

$$u''' = \frac{\cdots + d_1 \exp[-(7q+d)\eta]}{\cdots + d_2 \exp[-8q\eta]},$$
$$u^2 u' = \frac{\cdots + d_3 \exp[-(q+3d)\eta]}{\cdots + d_4 \exp[-8q\eta]} = \frac{\cdots + d_3 \exp[-(5q+3d)\eta]}{\cdots + d_4 \exp[-8q\eta]}, \tag{17}$$

where d_i $(i = 1, 2, 3, 4)$ are determined to be coefficients only for simplicity.

Balancing lowest order of exponential function in (17), we have

$$-(7q + d) = -(5q + 3d), \tag{18}$$

which leads to the result

$$q = d. \tag{19}$$

For simplicity, we set $p = c = 1$ and $q = d = 1$, so (13) reduces to

$$u(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{\exp(\eta) + b_0 + b_{-1} \exp(-\eta)}. \tag{20}$$

Substituting (20) into (11), and by the help of Mathematica, we have

$$\begin{aligned}\frac{1}{A}\big[&C_3 \exp(3\eta) + C_2 \exp(2\eta) \\ &+ C_1 \exp(\eta) + C_0 + C_{-1} \exp(-\eta) \\ &+ C_{-2} \exp(-2\eta) + C_{-3} \exp(-3\eta)\big] = 0,\end{aligned} \tag{21}$$

where

$$\begin{aligned}
A &= \left(b_{-1} + e^{2\eta} + b_0 e^{\eta}\right)^4, \\
C_3 &= \left(h + k\alpha a_1^2\right)\left(-a_0 + a_1 b_0\right), \\
C_2 &= -2k\alpha a_0^2 a_1 - 2a_{-1}\left(3k^3\beta + h + k\alpha a_1^2\right) \\
&\quad + 2a_0\left(3k^3\beta - h + k\alpha a_1^2\right) b_0 \\
&\quad + 2a_1\left[\left(3k^3\beta + h + k\alpha a_1^2\right) b_{-1} + \left(-3k^3\beta + h\right) b_0^2\right], \\
C_1 &= a_0\left[-k\alpha\left(a_0^2 + 6a_{-1}a_1\right)\right. \\
&\qquad \left. + \left(24k^3\beta - h + 5k\alpha a_1^2\right) b_{-1}\right] \\
&\quad - hb_0^2\left(a_0 - a_1 b_0\right) \\
&\quad + \left\{a_{-1}\left(-5h + k\alpha a_1^2\right)\right. \\
&\qquad \left. + a_1\left[k\alpha a_0^2 + 6\left(-4k^3\beta + h\right) b_{-1}\right]\right\} b_0, \\
C_0 &= -4\left(a_{-1} - a_1 b_{-1}\right) \\
&\quad \times \left[k\alpha\left(a_0^2 + a_{-1}a_1\right) - \left(9k^3\beta - h\right) b_{-1} + hb_0^2\right], \\
C_{-1} &= a_{-1}\left\{6k\alpha a_0 a_1 b_{-1}\right. \\
&\qquad \left. + \left[-k\alpha a_0^2 + 6\left(4k^3\beta - h\right) b_{-1} - hb_0^2\right] b_0\right\} \\
&\quad - k\alpha a_{-1}^2\left(5a_0 + a_1 b_0\right) \\
&\quad + b_{-1}\left\{k\alpha a_0^3 + 5ha_1 b_{-1} b_0\right. \\
&\qquad \left. + a_0\left[\left(-24k^3\beta + h\right) b_{-1} + hb_0^2\right]\right\}, \\
C_{-2} &= 2b_{-1}^2\left(a_{-1} - a_1 b_{-1}\right)\left(3k^3\beta + h\right) \\
&\quad + 2b_0 b_{-1}\left(a_{-1}b_0 - a_0 b_{-1}\right)\left(-3k^3\beta + h\right) \\
&\quad - 2k\alpha a_{-1} \\
&\quad \times \left[a_{-1}\left(a_{-1} + a_0 b_0\right) - b_{-1}\left(a_0^2 + a_1 a_{-1}\right)\right], \\
C_{-3} &= -\left(k\alpha a_{-1}^2 + hb_{-1}^2\right)\left(-a_0 b_{-1} + a_{-1} b_0\right),
\end{aligned} \tag{22}$$

with $h = k^3\beta + \omega$.

Equating the coefficients of $\exp(n\eta)$ to be zero, we have

$$\begin{gathered} C_3 = 0, \quad C_2 = 0, \quad C_1 = 0, \quad C_0 = 0, \\ C_{-3} = 0, \quad C_{-2} = 0, \quad C_{-1} = 0. \end{gathered} \tag{23}$$

Solving (23), simultaneously, we obtain

$$\begin{gathered} \omega = -k^3\beta - k\alpha a_1^2, \qquad a_{-1} = \frac{3k^2\beta b_0^2 + 2\alpha a_1^2 b_0^2}{8\alpha a_1}, \\ a_0 = \frac{3k^2\beta b_0 + \alpha a_1^2 b_0}{\alpha a_1}, \qquad b_{-1} = \frac{3k^2\beta b_0^2 + 2\alpha a_1^2 b_0^2}{8\alpha a_1^2}. \end{gathered} \tag{24}$$

Substituting (24) into (20) results in a compact-like solution, which reads

$$u(x,t) = a_1\left(1 + \frac{24yk^2\beta b_0}{3k^2\beta b_0^2 + 2\alpha a_1^2(2y+b_0)^2}\right), \tag{25}$$

where a_1, b_0, α, β, and k are real parameters and $y = e^{kx+t(-k^3\beta - k\alpha a_1^2)}$. Besides, the obtained solution equation (25) is a generalized soliton solution of (9).

In case, k is an imaginary parameter, the obtained soliton solution can be converted into the periodic solution or compact-like solution. We write

$$k = iK, \tag{26}$$

where K is a real parameter.

Use the transformations

$$\begin{aligned}
&\exp\left(kx + \sqrt{k^2 + 6a_1 + 1}kt\right)\\
&\quad = \cos\left(Kx + \sqrt{-K^2 + 6a_1 + 1}Kt\right)\\
&\qquad + i\sin\left(Kx + \sqrt{-K^2 + 6a_1 + 1}Kt\right),\\
&\exp\left(-kx - \sqrt{k^2 + 6a_1 + 1}kt\right)\\
&\quad = \cos\left(Kx + \sqrt{-K^2 + 6a_1 + 1}Kt\right)\\
&\qquad - i\sin\left(Kx + \sqrt{-K^2 + 6a_1 + 1}Kt\right).
\end{aligned} \tag{27}$$

Equation (25) becomes

$$\begin{aligned}
&u(x,t)\\
&\quad = a_1\Big(1 + \Big(- 24K^2\beta\left(\cos(y_1) + i\sin(y_1)\right)b_0\\
&\qquad \times\Big(- 3K^2\beta b_0^2 + 2\alpha a_1^2\\
&\qquad\quad \times\left(2\left(\cos(y_1) + i\sin(y_1)\right) + b_0\right)^2\Big)^{-1}\Big)\Big),
\end{aligned} \tag{28}$$

where a_1, b_0, α, β, and K are real parameters and $y_1 = Kx - (\alpha K a_1^2 - \beta K^3)t$. Simplifying the above formula, we have

$$u(x,t) = a_1 + \frac{-3K^2 b_0\beta/\alpha a_1}{b_0 + (1+\rho)\cos(y_1) + i(1-\rho)\sin(y_1)}, \tag{29}$$

where a_1, b_0, α, β, and K are real parameters and $\rho = b_0^2(-3\beta K^2 + 2\alpha a_1^2)/8\alpha a_1^2$.

If we search for a periodic solution or compact-like solution, the imaginary part in the denominator of (29) must be zero, which requires that

$$1 - \rho = 1 - \frac{b_0^2\left(-3\beta K^2 + 2\alpha a_1^2\right)}{8\alpha a_1^2} = 0. \tag{30}$$

Solving b_0 from (30), we obtain

$$b_0 = \pm\sqrt{\frac{8\alpha a_1^2}{-3\beta K^2 + 2\alpha a_1^2}}. \tag{31}$$

Substituting (31) into (29) results in a compact-like solution, which reads

$$\begin{aligned}
&u(x,t)\\
&\quad = a_1 - \frac{6\sqrt{2}K^2\beta g}{2\sqrt{2}\alpha a_1 g \pm \cos(y_1)\left(-\alpha a_1 + g^2\left(3K^2\beta - 2\alpha a_1^2\right)\right)},
\end{aligned} \tag{32}$$

where a_1 and K are free parameters, and $y_1 = Kx - (\alpha K a_1^2 - \beta K^3)t$, $g = \sqrt{\alpha a_1^2/(-3K^2\beta + 2\alpha a_1^2)}$, and it requires that $\alpha a_1^2/(-3K^2\beta + 2\alpha a_1^2) > 0$.

4. Discussing the Forms of the Solutions

In the above, we gain two cases of solution equations (25) and (32). In order to get the richness of solutions of (9), we have these two cases discussed in the following.

4.1. The Solutions from (25).

Through discussions, we obtain the following results.

4.1.1. When $\alpha = 0$ and $\beta = 1$. when $\alpha = 0$ and $\beta = 1$, (9) becomes $u_t + u_{xxx} = 0$. Then we can obtain shock solitons. The solution formula is shown as follows:

$$\begin{aligned}
u(x,t) &= a_1\left(1 + \frac{8e^{-k^3t+kx}}{b_0}\right)\\
&= a_1 + \frac{8a_1\cosh\left(k^3t - kx\right)}{b_0} - \frac{8a_1\sinh\left(k^3t - kx\right)}{b_0},
\end{aligned} \tag{33}$$

where a_1, b_0, and k are real parameters.

Case 1 (when $a_1 = 1, b_0 = 2, k = 3$). Consider

$$\begin{aligned}
u_1(x,t) &= 1 + 4e^{-27t+3x}\\
&= 1 + 4\cosh(27t - 3x) - 4\sinh(27t - 3x).
\end{aligned} \tag{34}$$

Description is as shown in Figure 1(a).

Case 2 (when $a_1 = 3$, $b_0 = -2$, and $k = 4$). Consider

$$\begin{aligned}
u_2(x,t) &= 3\left(1 - 4e^{-64t+4x}\right)\\
&= 3 - 12\cosh(64t - 4x) + 12\sinh(64t - 4x).
\end{aligned} \tag{35}$$

Description is as shown in Figure 1(b).

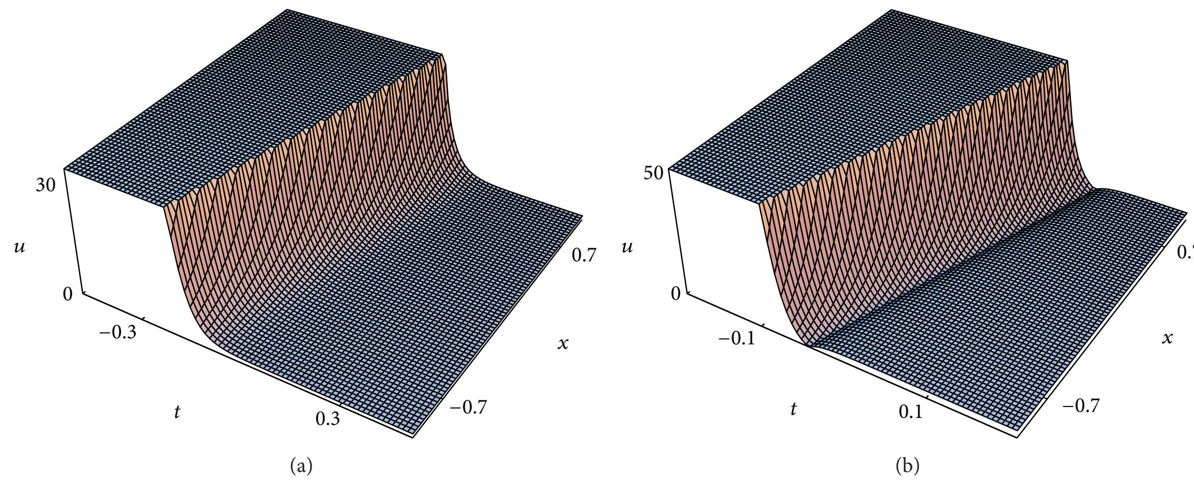


FIGURE 1: Evolution of the solutions of (25). (a) Shock soliton: parameters are $\alpha = 0$, $\beta = 1$, $a_1 = 1$, $b_0 = 2$, and $k = 3$. (b) A shock soliton in collision with a dark soliton: parameters are $\alpha = 0$, $\beta = 1$, $a_1 = 3$, $b_0 = -2$, and $k = 4$.

4.1.2. When $\alpha = 1$ and $\beta = 1$. when $\alpha = 1$ and $\beta = 1$, (9) becomes $u_t + u^2 u_x + u_{xxx} = 0$. Then we can obtain one-peak bright solitons. The solution formula is shown as follows:

$$u(x,t) = a_1\left(1 + \frac{24e^{y}k^2 b_0}{3k^2 b_0^2 + 2a_1^2(2e^{y} + b_0)^2}\right) = a_1\left(1 + \frac{24k^2(\cosh(y) + \sinh(y))b_0}{3k^2 b_0^2 + 2a_1^2(2\cosh(y) + 2\sinh(y) + b_0)^2}\right), \tag{36}$$

where a_1, b_0, and k are real parameters and $y = kx + t(-k^3 - ka_1^2)$.

Case 3 (when $a_1 = 1$, $b_0 = 2$, and $k = 2$). Consider

$$u_3(x,t) = 1 + \frac{192e^{-10t+2x}}{48 + 2(2 + 2e^{-10t+2x})^2} = 1 + \frac{192(\cosh(10t - 2x) - \sinh(10t - 2x))}{48 + 2(2 + 2\cosh(10t - 2x) - 2\sinh(10t - 2x))^2}. \tag{37}$$

Description is as shown in Figure 2(a).

Case 4 (when $a_1 = 1$, $b_0 = -1$, and $k = -3$). Consider

$$u_4(x,t) = 1 - \frac{216e^{30t-3x}}{27 + 2(-1 + 2e^{30t-3x})^2} = 1 - \frac{216(\cosh(30t - 3x) + \sinh(30t - 3x))}{27 + 2(-1 + 2\cosh(30t - 3x) + 2\sinh(30t - 3x))^2}. \tag{38}$$

Description is as shown in Figure 2(b).

4.1.3. When $\alpha = 1$ and $\beta = -1$. when $\alpha = 1$ and $\beta = -1$, (9) becomes $u_t + u^2 u_x - u_{xxx} = 0$. Then we can obtain bright solitons and dark solitons. The solution formula is shown as follows:

$$u(x,t) = a_1\left(1 - \frac{24e^{y}k^2 b_0}{-3k^2 b_0^2 + 2a_1^2(2e^{y} + b_0)^2}\right) = a_1\left(1 - \frac{24k^2(\cosh(y) + \sinh(y))b_0}{-3k^2 b_0^2 + 2a_1^2(2\cosh(y) + 2\sinh(y) + b_0)^2}\right), \tag{39}$$

where a_1, b_0, and k are real parameters and $y = kx + t(k^3 - ka_1^2)$.

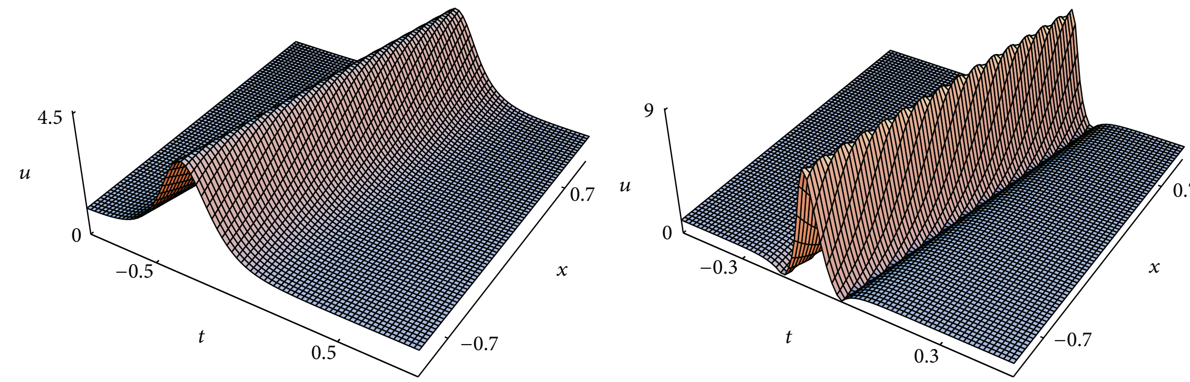
Case 5 (when $a_1 = -5$, $b_0 = -2$, and $k = 3$). Consider

$$u_5(x,t) = -5\left(1 + \frac{432e^{-48t+3x}}{-108 + 50(-2 + 2e^{-48t+3x})^2}\right) = -5\Big(1 + \big(432(\cosh(48t - 3x) + \sinh(48t - 3x)) \times \big(-108 + 50(-2 + 2\cosh(48t - 3x) - 2\sinh(48t - 3x))^2\big)^{-1}\big)\Big). \tag{40}$$

Description is as shown in Figure 3(a).

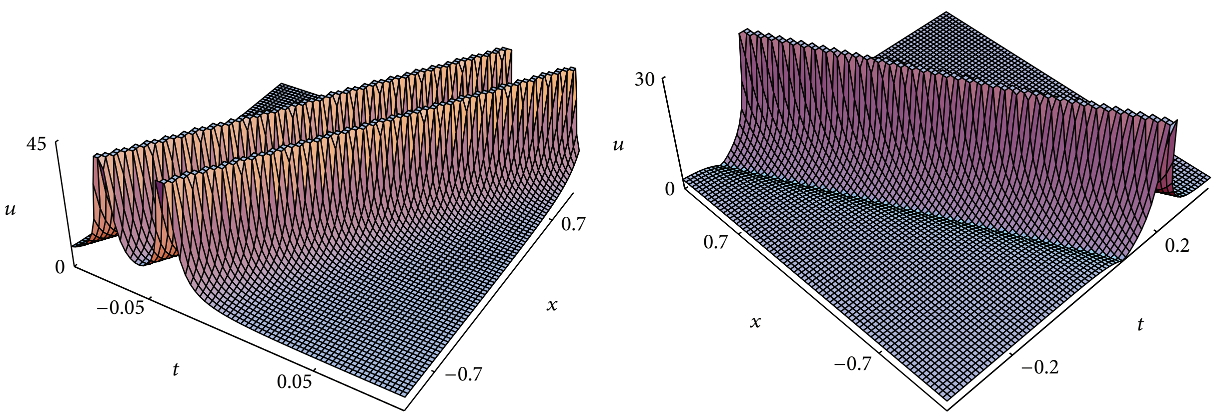
Case 6 (when $a_1 = 3$, $b_0 = -1$, and $k = 4$). Consider

$$u_7(x,t) = 3\left(1 + \frac{384e^{28t+4x}}{-48 + 18(-1 + 2e^{28t+4x})^2}\right)$$



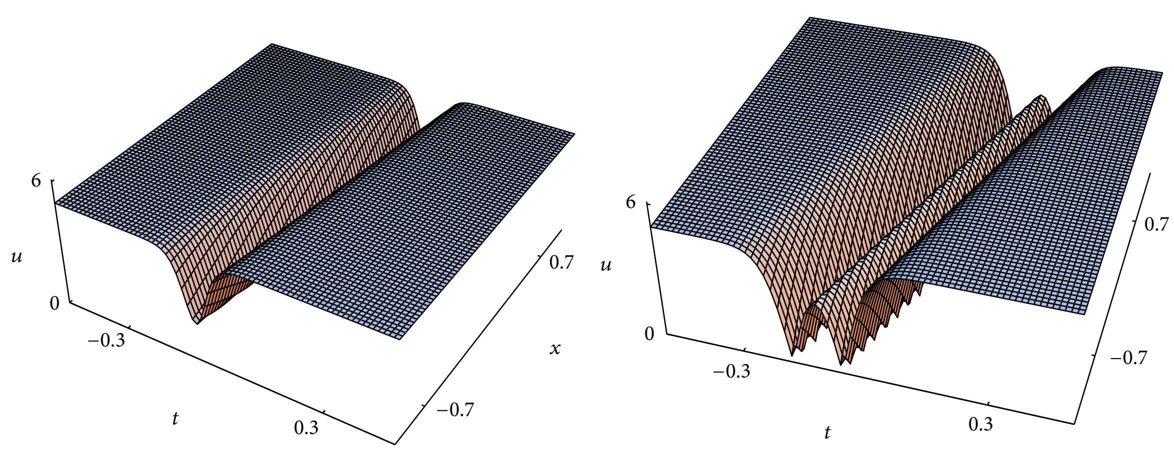
(a) (b)

Figure 2: Evolution of the solutions of (25). (a) Bright soliton: parameters are $\alpha = 1$, $\beta = 1$, $a_1 = 1$, $b_0 = 2$, and $k = 2$. (b) A bright soliton in collision with two dark solitons: parameters are $\alpha = 1$, $\beta = 1$, $a_1 = 1$, $b_0 = -1$, and $k = -3$.



(a) (b)

Figure 3: Evolution of the solutions of (25). (a) Two-peak bright soliton: parameters are $\alpha = 1$, $\beta = -1$, $a_1 = -5$, $b_0 = -2$, and $k = 3$. (b) A bright soliton in collision with a dark soliton: parameters are $\alpha = 1$, $\beta = -1$, $a_1 = 3$, $b_0 = -1$, and $k = 4$.



(a) (b)

Figure 4: Evolution of the solutions of (25). (a) Dark soliton: parameters are $\alpha = 1$, $\beta = -1$, $a_1 = -5$, $b_0 = 2$, and $k = -3$. (b) Two-peak dark soliton: parameters are $\alpha = 1$, $\beta = -1$, $a_1 = -5$, $b_0 = 3$, and $k = 4$.

$$= 3\left(1 + \left(384\left(\cosh\left(28t + 4x\right) + \sinh\left(28t + 4x\right)\right)\right.\right.$$
$$\times\left(-48 + 18\left(-1 + 2\cosh\left(28t + 4x\right)\right.\right.$$
$$\left.\left.\left.\left.+2\sinh\left(28t + 4x\right)\right)^2\right)^{-1}\right)\right). \tag{41}$$

Description is as shown in Figure 3(b).

Case 7 (when $a_1 = -5$, $b_0 = 2$, and $k = -3$). Consider

$$u_6(x,t) = -5\left(1 - \frac{432e^{48t-3x}}{-108 + 50(2 + 2e^{48t-3x})^2}\right)$$
$$= -5\left(1 - \left(432\left(\cosh\left(48t - 3x\right) + \sinh\left(48t - 3x\right)\right)\right.\right.$$
$$\times\left(-108 - 50\left(2 + 2\cosh\left(48t - 3x\right)\right.\right.$$
$$\left.\left.\left.\left.+2\sinh\left(48t - 3x\right)\right)^2\right)^{-1}\right)\right). \tag{42}$$

Description is as shown in Figure 4(a).

Case 8 (when $a_1 = -5$, $b_0 = 3$, and $k = 4$). Consider

$$u_8(x,t) = -5\left(1 - \frac{1152e^{-36t+4x}}{-432 + 50(3 + 2e^{-36t+4x})^2}\right)$$
$$= -5\left(1 - \left(1152\left(\cosh\left(36t - 4x\right) - \sinh\left(36t - 4x\right)\right)\right.\right.$$
$$\times\left(-432 + 50\left(3 + 2\cosh\left(36t - 4x\right)\right.\right.$$
$$\left.\left.\left.\left.-2\sinh\left(36t - 4x\right)\right)^2\right)^{-1}\right)\right). \tag{43}$$

Description is as shown in Figure 4(b).

4.2. The Solutions from (32). Based on the form of (32), if we set $\alpha = 0$ and $\beta = 0$ at the same time, the solution should be zero. Therefore, it is not significative to study and analyse. So, in the following, we consider α and β under the circumstance that $\alpha = 0$ and $\beta = 0$ are not zero in the meantime.

When $\alpha = 1$ and $\beta = 1$, (9) becomes $u_t + u^2u_x + u_{xxx} = 0$. Then we can obtain periodic wave solutions. The solution formula is shown as follows:

$$u(x,t) = a_1 - \frac{6\sqrt{2}K^2 g}{2\sqrt{2}a_1 g \pm \cos(y)\left(-a_1 + g^2\left(3K^2 - 2a_1^2\right)\right)}, \tag{44}$$

where $y = Kx - (Ka_1^2 - K^3)t$ and $g = \sqrt{a_1^2/(-3K^2 + 2a_1^2)}$.

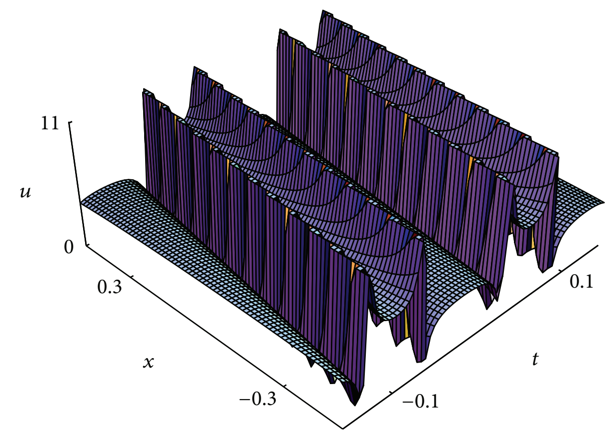


Figure 5: Evolution of the solutions of (32). Periodic wave solution: parameters are $\alpha = 1$, $\beta = 1$, $a_1 = 5$, and $K = -2$.

To illustrate the analysis, we obtain the following results.

Case 9 (when $a_1 = 5$, $K = -2$). Consider

$$u_9(x,t) = 5 - \frac{24}{\sqrt{19}\left(10/\sqrt{19} + 6\cos\left(2\left(21t - x\right)\right)\right)}. \tag{45}$$

Description is as shown in Figure 5.

5. Conclusions

In this paper, our main attention has been focused on seeking the soliton solutions of (9) via Exp-function method. By applying Exp-function method, we have obtained shock soliton, bright soliton, two-peak bright soliton, dark soliton, two-peak dark soliton, and periodic wave solution. In addition, with figures and symbolic computations, we have described the propagation characteristics of those solitons under different values of those coefficients in the generalized MKdV equation. Furthermore, the problem solving process and the algorithm, by the help of Mathematica, can be easily extended to all kinds of nonlinear equations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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