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Research Article

Fixed-Term Homotopy

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A new tool for the solution of nonlinear differential equations is presented. The Fixed-Term Homotopy (FTH) delivers a high precision representation of the nonlinear differential equation using only a few linear algebraic terms. In addition to this tool, a procedure based on Laplace-Padé to deal with the truncate power series resulting from the FTH method is also proposed. In order to assess the benefits of this proposal, two nonlinear problems are solved and compared against other semianalytic methods. The obtained results show that FTH is a power tool capable of generating highly accurate solutions compared with other methods of literature.

1. Introduction

Many physical phenomena are commonly modelled using nonlinear differential equations, which is a straightforward way to describe the behaviour of their dynamics. Among these methods, the most commonly used is the Homotopy Perturbation Method (HPM) [1–49]. This method is based in the use of a power series of the homotopy parameter, which transforms the original nonlinear differential equation into a series of linear differential equations. In this paper, a generalization of this concept using a product of two power series of the homotopy parameter called Fixed Term Homotopy (FTH) method is proposed. FTH method transforms the nonlinear differential equation into a series of linear differential equations, generating high precision expressions with fewer algebraic terms, reducing the computing cost. Furthermore, in order to deal with the truncate power series obtained with FTH method, the use of Laplace-Pade after-treatment is also proposed. To assess the potential of the proposed methodology, two nonlinear problems, Van Der Pol Oscillator [6, 50] and Troesch's equation [51–57], will be solved and compared using similar methodologies.

This paper is organized as follows. In Section 2, the fundamental idea of the FTH method is described. Section 3 presents a study of convergence for the proposed method. Section 4 introduces the Laplace-Padé after-treatment. In Sections 5 and 6, the solution procedure of two nonlinear problems is presented. Additionally, a discussion of the obtained results and the finds of this work are summarized in Section 7. Finally, the conclusions are presented in Section 8.

2. Basic Concept of FTH Method

The FTH and HPM methods share common foundations. Both methods consider that a nonlinear differential equation can be expressed as

$$A(u) - f(r) = 0, \quad \text{where } r \in \Omega \tag{1}$$

which has as boundary condition

$$B\left(u, \frac{\partial u}{\partial \eta}\right) = 0, \quad \text{where } r \in \Gamma, \tag{2}$$

where A is a general differential operator, $f(r)$ is a known analytic function, B is a boundary operator, Γ is the boundary of domain Ω, and $\partial u/\partial\eta$ denotes differentation along the normal drawn outwards from Ω [58]. In general, the A operator can be divided into two operators L and N, which are the corresponding linear and nonlinear operators, respectively. Hence, (1) can be rewritten as

$$L(u) + N(u) - f(r) = 0. \tag{3}$$

Now, a possible homotopy formulation is given by the expression

$$\begin{aligned} H(v,p) = {} & (1-p)\left[L(v) - L(u_0)\right] \\ & + p\left(L(v) + N(v) - f(r)\right) = 0, \quad p \in [0,1], \end{aligned} \tag{4}$$

where u_0 is the trial function (initial approximation) for (3) which satises the boundary conditions, and p is known as the perturbation homotopy parameter. From analyzing (4), it can be concluded that

$$H(v,0) = L(v) - L(u_0) = 0, \tag{5}$$

$$H(v,1) = L(v) + N(v) - f(r) = 0. \tag{6}$$

For $p \to 0$, the homotopy map (4) is reduced to the problem (5) that possesses a trivial solution u_0. Moreover, for $p \to 1$, the homotopy map (4) is transformed into the original nonlinear problem (6) that possesses the sought solution.

For the HPM method [9–12], we assume that the solution for (4) can be written as a power series of p, such that

$$v = p^0 v_0 + p^1 v_1 + p^2 v_2 + \cdots. \tag{7}$$

Considering that $p \to 1$, it results that the approximate solution for (1) is

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots. \tag{8}$$

The series (8) is convergent on most cases [9, 12].

In [59], a homotopy which uses the auxiliary term $\alpha u p(1-p)$ was reported. Then, modifying that version, it results in the following proposed homotopy:

$$\begin{aligned} H(v,p) = {} & (1-p)\left[L(v) - L(u_0)\right] + p\left(L(v) + N(v) - f(r)\right) \\ & + p(1-p)G(r) = 0, \quad p \in [0,1], \end{aligned} \tag{9}$$

where $G(r)$ is an arbitrary function.

When $p \to 0$ or $p \to 1$, the auxiliary term $p(1-p)G(r)$ is set to zero. Then, $G(r)$ does not affect or change the initial solution when $p \to 0$ or the sought solution at $p \to 1$. Moreover, a properly selection of $G(r)$ can be useful to improve convergence of the homotopy. Now, for the FTH method, (7) can be rewritten as the product of two power series, such that

$$v = \left(p^0 v_0 + p^1 v_1 + p^2 v_2 + \cdots\right)\left(p^0 w_0 + p^1 w_1 + p^2 w_2 + \cdots\right), \tag{10}$$

where $v_0, v_1, \ldots$ are unknown functions to be determined by the FTH method, and $w_0, w_1, \ldots$ are arbitrarily chosen fixed terms.

After substituting (10) into (9), and equating terms with the same order of p, we obtain a set of linear equations that lead us to calculate $v_0, v_1, \ldots$ The limit of (10), when $p \to 1$, provides an approximate solution for (3) in the form of

$$u = \lim_{p\to 1} v = \left(v_0 + v_1 + v_2 + \cdots\right)\left(w_0 + w_1 + w_2 + \cdots\right). \tag{11}$$

The upper limit exists in the event that both limits exists so

$$\lim_{p\to 1}\left(\sum_{i=0}^{\infty} v_i\right), \tag{12}$$

and fixed term

$$\lim_{q\to 1}\left(\sum_{i=0}^{\infty} w_i\right), \quad \text{where } \sum_{i=0}^{\infty} w_i \neq 0, \ \text{(to avoid trivial solution)}, \tag{13}$$

exist.

3. Convergence of FTH Method

To analyze the convergence of FTH, (9) is rewritten as

$$\begin{aligned} L(v) = {} & L(u_0) + p\left[f(r) - N(v) - L(u_0)\right] \\ & + p(1-p)G(r) = 0. \end{aligned} \tag{14}$$

Applying the inverse operator L^{-1} to both sides of (14), we obtain

$$v = u_0 + p\left[L^{-1}f(r) - L^{-1}N(v) - u_0\right] + p(1-p)L^{-1}G(r). \tag{15}$$

Assuming that (see (10))

$$v = \left(\sum_{i=0}^{\infty} p^i v_i\right)\left(\sum_{i=0}^{\infty} p^i w_i\right), \tag{16}$$

and substituting (16) into the right-hand side of (15), we obtain

$$v = u_0 + p\left[L^{-1}f(r) - \left(L^{-1}N\right)\left[\left(\sum_{i=0}^{\infty}p^i v_i\right)\left(\sum_{i=0}^{\infty}p^i w_i\right)\right]\right. \\ \left. - u_0\right] + p(1-p)L^{-1}G(r). \tag{17}$$

The exact solution of (3) is obtained when $p \to 1$ of (17), resulting in

$$\begin{aligned} u &= \lim_{p\to 1}\left(pL^{-1}f(r) - p\left(L^{-1}N\right)\left[\left(\sum_{i=0}^{\infty}p^i v_i\right)\left(\sum_{i=0}^{\infty}p^i w_i\right)\right]\right. \\ &\quad \left. + u_0 - pu_0 + p(1-p)L^{-1}G(r)\right) \\ &= L^{-1}f(r) - \left[\sum_{i=0}^{\infty}\left(L^{-1}N\right)(v_i\beta)\right], \quad \beta = \sum_{i=0}^{\infty}w_i. \end{aligned} \tag{18}$$

For the convergence analysis of the FTH method, we used the Banach Theorem as reported in [1, 2, 5, 60]. Such theorem relates the solution of (3) and the fixed point problem of the nonlinear operator N.

Theorem 1 (Sufficient Condition of Convergence). *Suppose that X and Y are Banach spaces and $N : X \to Y$ is a contractive nonlinear mapping, then*

$$\forall w, w^* \in X; \quad \left\|N(w) - N(w^*)\right\| \le \gamma\left\|w - w^*\right\|; \quad 0 < \gamma < 1. \tag{19}$$

According to Banach Fixed Point Theorem, N has a unique fixed point u such that $N(u) = u$. Assume that the sequence generated by the FTH method can be written as

$$W_n = N(W_{n-1}), \quad W_{n-1} = \sum_{i=0}^{n-1}(v_i\beta), \quad n = 1, 2, 3\ldots, \tag{20}$$

If one assumes that $W_0 = v_0\beta \in B_r(u)$, where $B_r(u) = \{w^* \in X \mid \|w^* - u\| < r\}$. Then, under these conditions:

(i) $W_n \in B_r(u)$,

(ii) $\lim_{n\to\infty}W_n = u$.

Proof. (i) By inductive approach, we have for $n = 1$

$$\left\|W_1 - u\right\| = \left\|N(W_0) - N(u)\right\| \le \gamma\left\|w_0 - u\right\|. \tag{21}$$

Assuming as induction hypothesis that $\|W_{n-1} - u\| \le \gamma^{n-1}\|w_0 - u\|$, then

$$\begin{aligned} \left\|W_n - u\right\| &= \left\|N(W_{n-1}) - N(u)\right\| \le \gamma\left\|W_{n-1} - u\right\| \\ &\le \gamma^n\left\|w_0 - u\right\|. \end{aligned} \tag{22}$$

Using (i), we have

$$\left\|W_n - u\right\| \le \gamma^n\left\|w_0 - u\right\| \le \gamma^n r < r \Longrightarrow W_n \in B_r(u). \tag{23}$$

(ii) Because of $\|W_n - u\| \le \gamma^n\|w_0 - u\|$ and $\lim_{n\to\infty}\gamma^n = 0$, $\lim_{n\to\infty}\|W_n - u\| = 0$, that is,

$$\lim_{n\to\infty}W_n = u. \tag{24}$$

□

4. Laplace-Padé after Treatment for FTH Series

The coupling of Laplace transform and Padé approximant [61] is used in order to recover part of the lost information due to the truncated power series [60, 62–70]. The process can be recast as follows.

(1) First, Laplace transformation is applied to power series obtained by FTH method.

(2) Next, s is substituted by $1/t$ in the resulting equation.

(3) After that, we convert the transformed series into a meromorphic function by forming its Padé approximant of order $[N/M]$. N and M are arbitrarily chosen, but they should be of smaller value than the order of the power series. In this step, the Padé approximant extends the domain of the truncated series solution to obtain better accuracy and convergence.

(4) Then, t is substituted by $1/s$.

(5) Finally, by using the inverse Laplace s transformation, we obtain the modified approximate solution.

We will denominate to this process as the Laplace-Padé fixed term homotopy (LPFTH) method.

5. Van Der Pol Oscillator

Consider the Van der Pol Oscillator problem [6, 50]

$$\begin{aligned} u'' + u' + u + u^2u' - 2\cos(t) + \cos^3(t) &= 0, \\ u(0) = 0, \quad u'(0) &= 1, \end{aligned} \tag{25}$$

which have the exact solution

$$u(t) = \sin(t). \tag{26}$$

In order to find an approximate solution for (25) by means of LPFTH, we obtain the Taylor series of the trigonometric terms, resulting is

$$-2\cos(t) + \cos^3(t) = -1 - \frac{1}{2}t^2 + \frac{19}{24}t^4 - \frac{181}{720}t^6, \tag{27}$$

where the expansion order is 7.

From (9), we establish the following homotopy equation

$$(1-p)\left(v''\right)+p\left(v''+v'+v+v^2v'-1-\frac{1}{2}t^2+\frac{19}{24}t^4-\frac{181}{720}t^6\right)+bp(1-p)t^3=0, \tag{28}$$

where b is an adjustment parameter due to the auxiliary term $bp(1-p)t^3$.

From (10), we assume that solution for (28) has the following form:

$$v=\left(v_0+v_1p+v_2p^2\right)\left(1+at^2p\right), \tag{29}$$

where a is an adjustment parameter of the fixed term of the homotopy map.

Substituting (29) into (28), and rearranging the terms of the same order of p,

$$\begin{aligned}
p^0 &: v_0''=0, \quad v_0(0)=0,\ v_0'(0)=1\\
p^1 &: v_1''-1+av_0''t^2+4atv_0'+2av_0+bt^3\\
&\quad -\frac{1}{2}t^2+\frac{19}{24}t^4+v_0+v_0'-\frac{181}{720}t^6+v_0^2v_0'=0,\\
&\qquad v_1(0)=0,\quad v_1'(0)=0,\\
p^2 &: v_2''+av_1''t^2+3av_0^2v_0't^2+2av_0^3t+2v_0v_1v_0'+4atv_1'\\
&\quad -bt^3+v_0^2v_1'+2atv_0+v_0at^2+v_1+av_0't^2+v_1'+2av_1=0,\\
&\qquad v_2(0)=0,\quad v_2'(0)=0.\\
&\quad\vdots
\end{aligned} \tag{30}$$

By solving (30), we obtain

$$\begin{aligned}
v_0 &= t,\\
v_1 &= \frac{181}{40320}t^8-\frac{1}{20}bt^5-\frac{19}{720}t^6-\frac{1}{24}t^4+\left(-a-\frac{1}{6}\right)t^3,\\
v_2 &= \frac{181}{443520}t^{11}+\left(-\frac{181}{40320}a-\frac{181}{3628800}\right)t^{10}+\frac{883}{362880}t^9\\
&\quad+\left(\frac{19}{720}a+\frac{1}{160}b+\frac{19}{40320}\right)t^8\\
&\quad+\left(\frac{1}{20}ab+\frac{1}{840}b+\frac{7}{720}\right)t^7+\left(\frac{1}{24}a+\frac{1}{120}b+\frac{7}{240}\right)t^6\\
&\quad+\left(a^2+\frac{1}{6}a+\frac{1}{20}b+\frac{1}{60}\right)t^5+\frac{1}{24}t^4.\\
&\quad\vdots
\end{aligned} \tag{31}$$

Substituting (31) into (29), and calculating the limit when $p\to 1$, we obtain the second order approximated solution

$$u(t)=\lim_{p\to 1}v=\left(v_0+v_1+v_2\right)\left(1+at^2\right). \tag{32}$$

Then, we select the adjustment parameters as: $a = 0.4431998183E\text{-}3$, $b=-2.1761220576$; where the parameters are calculated using the NonlinearFit command from Maple Release 13 [5, 32–34]. Moreover, NonlinearFit command finds values of the approximate model parameters such that the sum of the squared k-residuals is minimized.

In order to guarantee the validity of the approximate solution (32) for large t, the series solution is transformed by the Laplace-Padé after-treatment. First, Laplace transformation is applied to (32) and then $1/t$ is written in place of s in the equation. Then, the Padé approximant [2/2] is applied and $1/s$ is written in place of t. Finally, by using the inverse Laplace s transformation, we obtain the modified approximate solution

$$u(t)=0.9986730464\sin(1.0013287167t). \tag{33}$$

6. Troesch's Problem

The Troesch's equation is a boundary value problem (BVP) that arises in the investigation of confinement of a plasma column by a radiation pressure [71] and also in the theory of gas porous electrodes [72, 73]. The problem is expressed as

$$y''=n\sinh(ny),\qquad y(0)=0,\qquad y(1)=1, \tag{34}$$

where prime denotes differentiation with respect to x and n is known as Troesch's parameter.

Straightforward application of FTH to solve (34) is not possible due to the hyperbolic sin term of dependent variable. Nevertheless, the polynomial type nonlinearities are easier to handle by the FTH method. Therefore, in order to apply FTH successfully, we convert the hyperbolic-type nonlinearity in Troesch's problem into a polynomial type nonlinearity, using the variable transformation reported in [51, 52]

$$u(x)=\tanh\left(\frac{n}{4}y(x)\right). \tag{35}$$

After using (35), we obtain the following transformed problem:

$$\left(1-u^2\right)u''+2u\left(u'\right)^2-n^2u\left(1+u^2\right)=0, \tag{36}$$

where conditions are obtained by using variable transformation (35).

Substituting original boundary conditions $y(0)=0$ and $y(1)=1$ into (35) results in

$$u(0)=0,\qquad u(1)=\tanh\left(\frac{n}{4}\right). \tag{37}$$

Table 1: Comparison between exact solution (26) for (25) and the results of approximations: LPFTH (33), HPM [6], and VIM [6] (A.E. means absolute error).

t	Exact (26)	LPFTH (33)	HPM [6]	VIM [6]	A.E. of LPFTH	A.E. of HPM	A.E. of VIM
1.0	0.84147098	0.84107061	0.83819112	0.82124433	0.00040038	0.00327987	0.02022665
2.0	0.90929743	0.90698321	0.78090770	0.23121704	0.00231422	0.12838972	0.67808039
3.0	0.14112001	0.13699062	−0.19760326	−3.14507873	0.00412939	0.33872327	3.28619874
4.0	−0.75680250	−0.75925698	−0.94834234	−8.80326033	0.00245449	0.19153985	8.04645783
5.0	−0.95892427	−0.95574867	−1.02405737	−16.02894628	0.00317560	0.06513310	15.07002201
6.0	−0.27941550	−0.27139133	−0.11322005	−24.21046104	0.00802417	0.16619545	23.93104554
7.0	0.65698660	0.66308908	0.88967763	−34.18610854	0.00610248	0.23269103	34.84309514
8.0	0.98935825	0.98644505	1.23972759	−46.77116918	0.00291320	0.25036935	47.76052743
9.0	0.41211849	0.40066121	0.43840592	−62.53129090	0.01145728	0.02628744	62.94340939
10.0	−0.54402111	−0.55438502	−0.70915755	−80.67754120	0.01036391	0.16513644	80.13352009

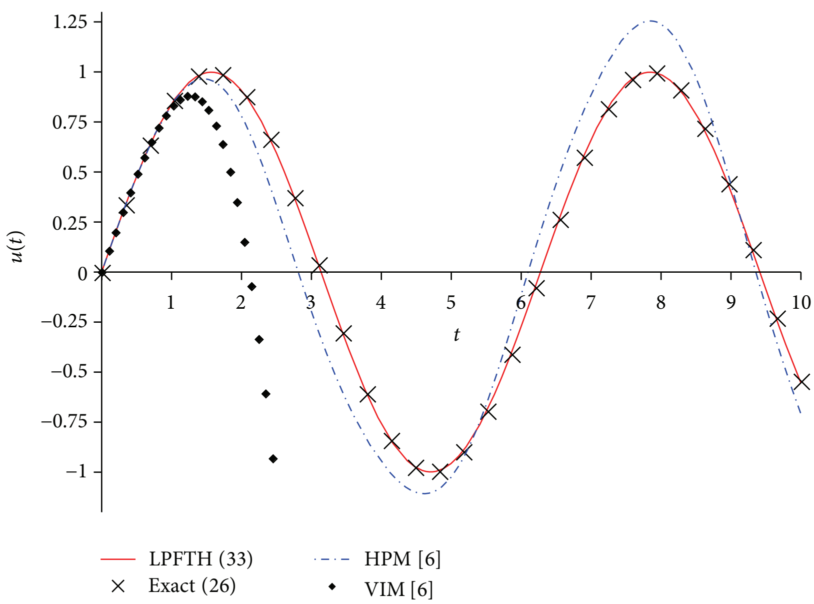

Figure 1: Exact solution (26) (diagonal cross) for (25) and its approximate solutions (33) (solid line), HPM [6] (dash-dot), and VIM [6] (solid diamond).

From (9) and (36), we can formulate the following homotopy [7–9]:

$$\begin{aligned} H(v,p) = {} & (1-p)v'' \\ & + p\left(\left(1-v^2\right)v'' + 2v\left(v'\right)^2 - n^2 v\left(1+v^2\right)\right) \\ & + pc(1-p)x^6 = 0, \end{aligned} \tag{38}$$

where p is the homotopy parameter and c is an adjustment parameter due to the auxiliary term $pc(1-p)x^6$.

From (10), we assume that solution for (38) has the following form:

$$v = \left(v_0 + v_1 p + v_2 p^2\right)\left(1 + pax + p^2 bx^2\right), \tag{39}$$

where a, b are the adjustment parameters due to the fixed term $(1 + pax + p^2bx^2)$ of the homotopy map.

Substituting (39) into (38) and equating identical powers of p terms, we obtain

$$\begin{aligned} & p^0 : v_0'' = 0, \quad v_0(0) = 0, \ v_0(1) = \gamma, \\ & p^1 : v_1'' - v_0^2 v_0'' - n^2 v_0^3 + 2v_0 v_0'^2 + v_0'' ax + 2v_0' a - n^2 v_0 \\ & \qquad + cx^6 = 0, \qquad v_1(0) = 0, \qquad v_1(1) = 0, \\ & p^2 : v_2'' + n^2 v_1 + 2v_1' a + 6v_0 v_0'^2 ax - v_0^2 v_1'' + 2v_0^2 v_0' a \\ & \qquad - 3n^2 v_0^3 ax - n^2 v_0 ax + 2bv_0 + 4v_0' bx + 4v_0 v_1' v_0' \\ & \qquad - cx^6 + v_0'' bx^2 - 3n^2 v_1 v_0^2 - 2v_0 v_1 v_0'' + 2v_1 v_0'^2 \\ & \qquad + v_1'' ax - 3v_0^2 ax v_0'' = 0, \quad v_2(0) = 0, \ v_2(1) = 0, \end{aligned} \tag{40}$$

where $\gamma = \tanh(n/4)$.

Table 2: Comparison between (43), exact solution [74, 75], and other reported approximate solutions, using $n = 0.5$.

x	Exact [74, 75]	This work (43)	HPM [52]	DMA [53]	HPM [54]	HPM [55]	HAM [56]	LDTM [57]
0.1	0.0959443493	0.0959443679	0.0959443155	0.0959383534	0.0959395656	0.095948026	0.0959446190	0.0959443520
0.2	0.1921287477	0.1921287510	0.1921286848	0.1921180592	0.1921193244	0.192135797	0.1921292845	0.1921287539
0.3	0.2887944009	0.2887943893	0.2887943176	0.2887803297	0.2887806940	0.288804238	0.2887952148	0.2887944107
0.4	0.3861848464	0.3861848355	0.3861847539	0.3861687095	0.3861675428	0.386196642	0.3861859313	0.3861848612
0.5	0.4845471647	0.4845471662	0.4845470753	0.4845302901	0.4845274183	0.4845599	0.4845485110	0.4845471832
0.6	0.5841332484	0.5841332589	0.5841331729	0.5841169798	0.5841127822	0.584145785	0.5841348222	0.5841332650
0.7	0.6852011483	0.6852011536	0.6852010943	0.6851868451	0.6851822495	0.685212297	0.6852028604	0.6852011675
0.8	0.7880165227	0.7880165163	0.7880164925	0.7880055691	0.7880018367	0.788025104	0.7880181729	0.7880165463
0.9	0.8928542161	0.8928542136	0.8928542059	0.8928480234	0.8928462193	0.892859085	0.8928553997	0.8928542363
	Order	2	2	6	2	2	6	3
	A.A.R.E.	3.5462E(-08)	1.83327E(-07)	3.47802E(-05)	3.57932E(-05)	2.44418E(-05)	2.51374E(-06)	3.10957E(-08)

We solve (40) by using Maple software, resulting in

$$\begin{aligned}
v_0 &= \gamma x,\\
v_1 &= \left(\frac{1}{56}c - \frac{1}{20}\gamma^3 n^2 + \frac{1}{3}\gamma^3 - \frac{1}{6}\gamma n^2 + \gamma a\right)x - \gamma a x^2\\
&\quad + \left(-\frac{1}{3}\gamma^3 + \frac{1}{6}\gamma n^2\right)x^3 + \frac{1}{20}\gamma^3 n^2 x^5 - \frac{1}{56}c x^8,\\
v_2 &= \left(-\frac{1}{56}c + b\gamma + \frac{1}{45}\gamma^2 c - \frac{31}{420}\gamma^5 n^2 + \frac{17}{840}n^4\gamma^3 + \frac{7}{360}n^4\gamma\right.\\
&\qquad - \frac{5}{36}\gamma^3 n^2 - \frac{1}{360}c n^2 - \frac{3}{20}a\gamma^3 n^2 - \frac{1}{6}a\gamma n^2 - \frac{1}{440}n^2\gamma^2 c\\
&\qquad \left. + \frac{13}{2400}n^4\gamma^5 + a\gamma^3 + \frac{1}{5}\gamma^5\right)x\\
&\quad + \left(\frac{1}{20}a\gamma^3 n^2 + \frac{1}{6}a\gamma n^2 - \frac{1}{56}ac - \frac{1}{3}a\gamma^3 - \gamma a^2\right)x^2\\
&\quad + \left(\frac{1}{20}\gamma^5 n^2 - a\gamma^3 - \frac{1}{3}\gamma^5 + \frac{2}{9}\gamma^3 n^2 - b\gamma - \frac{1}{36}n^4\gamma\right.\\
&\qquad \left. + \frac{1}{6}a\gamma n^2 - \frac{1}{56}\gamma^2 c + \gamma a^2 + \frac{1}{336}c n^2 - \frac{1}{120}n^4\gamma^3\right)x^3\\
&\quad + \left(\frac{1}{3}a\gamma^3 - \frac{1}{6}a\gamma n^2\right)x^4 + \left(\frac{1}{120}n^4\gamma + \frac{2}{15}\gamma^5 - \frac{1}{12}\gamma^3 n^2\right.\\
&\qquad + \frac{3}{1120}n^2\gamma^2 c + \frac{3}{20}a\gamma^3 n^2\\
&\qquad + \frac{1}{20}\gamma^5 n^2 - \frac{1}{40}n^4\gamma^3\\
&\qquad \left. - \frac{3}{400}n^4\gamma^5\right)x^5 - \frac{1}{20}a\gamma^3 n^2 x^6\\
&\quad + \left(\frac{11}{840}n^4\gamma^3 - \frac{11}{420}\gamma^5 n^2\right)x^7 + \frac{1}{56}c x^8\\
&\quad + \left(\frac{1}{56}ac + \frac{1}{480}n^4\gamma^5\right)x^9 + \left(-\frac{11}{2520}\gamma^2 c - \frac{1}{5040}c n^2\right)x^{10}\\
&\quad - \frac{1}{2464}n^2\gamma^2 c x^{12}.
\end{aligned} \tag{41}$$

Substituting (41) into (39), and calculating the limit when $p \to 1$, we obtain the second order approximated solution of (36)

$$u_2(x) = \lim_{p\to 1} v = (v_0 + v_1 + v_2)\left(1 + ax + bx^2\right). \tag{42}$$

Finally, from (35) and (42), the proposed solution of Troesch's problem is

$$y(x) = \frac{4}{n}\tanh^{-1}(u_2(x)), \quad 0 \le x \le 1. \tag{43}$$

If we consider $n = 0.5$, then we choose the adjustment parameter as: $a = 0.4144179560E\text{-}5$, $b = 1.7343105421E\text{-}8$, and $c = 0.2046343553E\text{-}2$, by using the procedure explained for the Van der Pol Oscillator problem.

7. Numerical Simulation and Discussion

A comparison of the exact solution (26) of the Van der Pol Oscillator against the approximated solutions obtained by LPFTH (33), HPM [6], and VIM [6] is shown in Table 1 and Figure 1. In the comparison can be seen that the LPFTH method exhibited the higher accuracy for a large period of time reproducing successfully the oscillatory behaviour of the exact solution (26), therefore, being a suitable alternative to solve nonlinear oscillators.

The Troesch's BVP problem is a benchmark equation for numerical [74, 75] and semi-analytical methods [51–57] which has been solved by FTH to obtain the approximated solution (43). Table 2 shows a comparison of the results obtained with other semi-analytical methods as: homotopy perturbation method (HPM) [52, 54, 55], decomposition method approximation (DMA) [53, 54], homotopy analysis method (HAM) [56], and Laplace transform decomposition method (LTDM) [57]. The comparison shows that the average absolute relative error (A.A.R.E.) of (43) is lower than most of the reported results and similar to LDTM [57].

For both cases of study, polynomial functions were used for fixed and auxiliary terms as they generate the best results. Nonetheless, functions as exponential, trigonometric or hyperbolic may enlarge the domain of convergence. Thereby, a methodology that allows the selection of fixed and auxiliary terms like these to obtain more accurate solutions is a window of opportunity to be exploited.

Since FTH does not rely in the linearization of the input equations, a perturbation parameter nor assumption of weak nonlinearity, the solution generated may be general and more realistic than the method of simplifying the physical problems.

8. Conclusions

In this work, the Fixed Term Homotopy and the Laplace-Padé Fixed Term Homotopy methods are presented as novel tools to solve nonlinear ordinary differential equations. The proposed methods were tested using two nonlinear problems: a second order nonlinear oscillator and a high nonlinear boundary value problem. From a comparison against several semi-analytical methods from the literature, FTH and LPFTH probed to be power tools that generate highly accurate easy manageable expressions. Furthermore, the homotopy given in (9) may be replaced or modified by a formulation inspired by some other homotopy method proposed in literature [76–96], which can lead to improvement of convergence of proposed methods. Additionally, being FTH a modified version of HPM can be assumed that the differential equations solved by HPM should be also solvable by FTH. In that fashion, further research can be focused on applying FTH method in the solution of nonlinear partial differential equations, nonlinear fractional differential equations, among others.

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