Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 576237, 8 pages
http://dx.doi.org/10.1155/2013/576237

Research Article

On Fuzzy Modular Spaces

Yonghong Shen[1,2] and Wei Chen[3]

[1] *School of Mathematics, Beijing Institute of Technology, Beijing 100081, China*
[2] *School of Mathematics and Statistics, Tianshui Normal University, Tianshui 741001, China*
[3] *School of Information, Capital University of Economics and Business, Beijing 100070, China*

Correspondence should be addressed to Yonghong Shen; shenyonghong2008@hotmail.com

Received 12 November 2012; Revised 30 January 2013; Accepted 18 February 2013

Academic Editor: Luis Javier Herrera

The concept of fuzzy modular space is first proposed in this paper. Afterwards, a Hausdorff topology induced by a β-homogeneous fuzzy modular is defined and some related topological properties are also examined. And then, several theorems on μ-completeness of the fuzzy modular space are given. Finally, the well-known Baire's theorem and uniform limit theorem are extended to fuzzy modular spaces.

1. Introduction and Preliminaries

In the 1960s, the concept of modular space was introduced by Nakano [1]. Soon after, Musielak and Orlicz [2] redefined and generalized the notion of modular space. A real function ρ on an arbitrary vector space X is said to be a *modular* if it satisfies the following conditions:

(M-1) $\rho(x) = 0$ if and only if $x = \theta$ (i.e., x is the null vector θ),

(M-2) $\rho(x) = \rho(-x)$,

(M-3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for all $x, y \in X$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

A *modular space* X_ρ is defined by a corresponding modular ρ, that is, $X_\rho = \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}$.

Based on definition of the modular space, Kozłowski [3, 4] introduced the notion of modular function space. In the sequel, Kozłowski and Lewicki [5] considered the problem of analytic extension of measurable functions in modular function spaces and discussed some extension properties by means of polynomial approximation. Afterwards, Kilmer and Kozłowski [6] studied the existence of best approximations in modular function spaces by elements of sublattices. In 1990, Khamsi et al. [7] initiated the study of fixed point theory for nonexpansive mappings defined on some subsets of modular function spaces. More researches on fixed point theory in modular function spaces can be found in [8–13].

In 2007, Nourouzi [14] proposed probabilistic modular spaces based on the theory of modular spaces and some researches on the Menger's probabilistic metric spaces. A pair (X, ρ) is called a *probabilistic modular space* if X is a real vector space, ρ is a mapping from X into the set of all distribution functions (for $x \in X$, the distribution function $\rho(x)$ is denoted by ρ_x, and $\rho_x(t)$ is the value ρ_x at $t \in \mathbb{R}$) satisfying the following conditions:

(PM-1) $\rho_x(0) = 0$,

(PM-2) $\rho_x(t) = 1$ for all $t > 0$ if and only if $x = \theta$,

(PM-3) $\rho_{-x}(t) = \rho_x(t)$,

(PM-4) $\rho_{\alpha x+\beta y}(s + t) \geq \rho_x(s) \wedge \rho_y(t)$ for all $x, y \in X$ and $\alpha, \beta, s, t \in \mathbb{R}_0^+$, $\alpha + \beta = 1$.

Especially, for every $x \in X$, $t > 0$ and $\alpha \in \mathbb{R} \setminus \{0\}$, if

$$\rho_{\alpha x}(t) = \rho_x\left(\frac{t}{|\alpha|^\beta}\right), \quad \text{where } \beta \in (0, 1], \tag{1}$$

then we say that (X, ρ) is *β-homogeneous*.

Recently, further studies have been made on the probabilistic modular spaces. Nourouzi [15] extended the well-known Baire's theorem to probabilistic modular spaces by using a special condition. Fallahi and Nourouzi [16] investigated the continuity and boundedness of linear operators defined between probabilistic modular spaces in the probabilistic sense.

In this paper, following the idea of probabilistic modular space and the definition of fuzzy metric space in the sense of George and Veeramani [17], we apply fuzzy concept to the classical notions of modular and modular spaces and propose a novel concept named fuzzy modular spaces.

2. Fuzzy Modular Spaces

In this section, following the idea of probabilistic modular space, we will introduce the concept of fuzzy modular space by using continuous t-norm and present some related notions.

Definition 1 (Schweizer and Sklar [18]). A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a *continuous t-norm* if it satisfies the following conditions:

(TN-1) $*$ is commutative and associative;

(TN-2) $*$ is continuous;

(TN-3) $a * 1 = a$ for every $a \in [0,1]$;

(TN-4) $a * b \le c * d$ whenever $a \le c$, $b \le d$ and $a, b, c, d \in [0,1]$.

Three common examples of the continuous t-norm are (1) $a *_M b = \min\{a, b\}$; (2) $a *_P b = a \cdot b$; (3) $a *_L b = \max\{a + b - 1, 0\}$. For more examples, the reader can be referred to [19].

Definition 2 (George and Veeramani [17]). A *fuzzy metric space* is an ordered triple $(X, M, *)$ such that X is a nonempty set, $*$ is a continuous t-norm, and M is a fuzzy set on $X \times X \times (0,\infty)$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

(F-1) $M(x, y, t) > 0$,

(F-2) $M(x, y, t) = 1$ if and only if $x = y$,

(F-3) $M(x, y, t) = M(y, x, t)$,

(F-4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,

(F-5) $M(x, y, \cdot) : (0,\infty) \to (0, 1]$ is continuous.

Based on the notion of probabilistic modular space and Definition 2, we will propose a novel concept named fuzzy modular spaces.

Definition 3. The triple $(X, \mu, *)$ is said to be a *fuzzy modular space* (shortly, $\mathcal{F}$-modular space) if X is a real or complex vector space, $*$ is a continuous t-norm, and μ is a fuzzy set on $X \times (0,\infty)$ satisfying the following conditions, for all $x, y \in X$, $s, t > 0$ and $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$:

(FM-1) $\mu(x, t) > 0$,

(FM-2) $\mu(x, t) = 1$ for all $t > 0$ if and only if $x = \theta$,

(FM-3) $\mu(x, t) = \mu(-x, t)$,

(FM-4) $\mu(\alpha x + \beta y, s + t) \ge \mu(x, s) * \mu(y, t)$,

(FM-5) $\mu(x, \cdot) : (0,\infty) \to (0, 1]$ is continuous.

Generally, if $(X, \mu, *)$ is a fuzzy modular space, we say that $(\mu, *)$ is a *fuzzy modular* on X. Moreover, the triple $(X, \mu, *)$ is called *β-homogeneous* if for every $x \in X$, $t > 0$ and $\lambda \in \mathbb{R} \setminus \{0\}$,

$$\mu(\lambda x, t) = \mu\left(x, \frac{t}{|\lambda|^{\beta}}\right), \quad \text{where } \beta \in (0,1]. \tag{2}$$

Example 4. Let X be a real or complex vector space and let ρ be a modular on X. Take t-norm $a * b = a *_M b$. For every $t \in (0,\infty)$, define $\mu(x, t) = t/(t + \rho(x))$ for all $x \in X$. Then $(X, \mu, *)$ is a $\mathcal{F}$-modular space.

Remark 5. Note that the above conclusion still holds even if the t-norm is replaced by $a * b = a *_P b$ and $a * b = a *_L b$, respectively.

Example 6. Let $X = R$. ρ is a modular on X, which is defined by $\rho(x) = |x|^{\beta}$, where $\beta \in (0, 1]$. Take t-norm $a * b = a *_P b$. For every $t \in (0,\infty)$, we define

$$\mu(x, t) = \frac{1}{e^{\rho(x)/t}} \tag{3}$$

for all $x \in X$. Then $(X, \mu, *)$ is a β-homogeneous $\mathcal{F}$-modular space.

Proof. We just need to prove the condition (FM-4) of Definition 3 and formula (2), because other conditions hold obviously. In the following, we first verify $\mu(\alpha x + \beta y, s + t) \ge \mu(x, s) * \mu(y, t)$, as $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$.

Since ρ is a modular on X, for all $x, y \in X$, we have

$$\rho(\alpha x + \beta y) \le \rho(x) + \rho(y). \tag{4}$$

Then, we can obtain

$$\rho(\alpha x + \beta y) \le \frac{t+s}{t}\rho(x) + \frac{t+s}{s}\rho(y), \tag{5}$$

that is,

$$\frac{1}{t+s}\rho(\alpha x + \beta y) \le \frac{1}{t}\rho(x) + \frac{1}{s}\rho(y). \tag{6}$$

Therefore

$$e^{\rho(\alpha x+\beta y)/(t+s)} \le e^{\rho(x)/t} \cdot e^{\rho(y)/s} = e^{\rho(x)/t} *_P e^{\rho(y)/s}. \tag{7}$$

Thus, we have $\mu(\alpha x + \beta y, s + t) \ge \mu(x, s) * \mu(y, t)$.

On the other hand, for all $\lambda \in \mathbb{R} \setminus \{0\}$, since $\rho(\lambda x) = |\lambda x|^{\beta} = |\lambda|^{\beta} \cdot |x|^{\beta} = |\lambda|^{\beta}\rho(x)$, it follows that

$$\mu(\lambda x, t) = \mu\left(x, \frac{t}{|\lambda|^{\beta}}\right). \tag{8}$$

Hence, we know that $(X, \mu, *)$ is a β-homogeneous $\mathcal{F}$-modular space. □

Theorem 7. *If $(X, \mu, *)$ is a $\mathcal{F}$-modular space, then $\mu(x, \cdot)$ is nondecreasing for all $x \in X$.*

Proof. Suppose that $\mu(x,t) < \mu(x,s)$ for some $t > s > 0$. Without loss of generality, we can take $\alpha = 1$, $\beta = 0$, and $y = \theta$ is the null vector in X. By Definition 3, we can obtain

$$\begin{aligned}\mu(x,s) * \mu(\theta,t-s) = \mu(x,s) * \mu(y,t-s) \leq \mu(\alpha x + \beta y, t) \\ = \mu(x,t) < \mu(x,s).\end{aligned} \tag{9}$$

Since $\mu(\theta, t-s) = 1$, we have $\mu(x,s) < \mu(x,s)$. Obviously, this leads to a contradiction. □

It should be noted that, in general, a fuzzy modular and a fuzzy metric (in the sense of George and Veeramani [17]) do not necessarily induce mutually when the triangular norm is the same one. In essence, the fuzzy modular and fuzzy metric can be viewed as two different characterizations for the same set. The former is regarded as a kind of fuzzy quantization on the classical vector modular, while the latter is regarded as a fuzzy measure on the distance between two points. Next, we construct two examples to show that there does not exist direct relationship between a fuzzy modular and a fuzzy metric.

Example 8. Let $X = \mathbb{R}$. Take t-norm $a * b = a *_M b$. For every $t \in (0,\infty)$, we define

$$\mu(x,t) = \frac{k}{k+|x|}, \tag{10}$$

where $k > 0$ is a constant.

Here, we only show that $\mu(x,t)$ satisfies the condition (FM-4) of Definition 3, since other conditions can be easily verified.

For every $x, y \in \mathbb{R}$, and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Without loss of generality, we assume that $|x| \leq |y|$. Since $|\alpha x + \beta y| \leq |y|$, we then obtain

$$\begin{aligned}\mu(\alpha x + \beta y, t+s) &= \frac{k}{k+|\alpha x + \beta y|} \geq \frac{k}{k+|y|} \\ &= \min\left\{\frac{k}{k+|x|}, \frac{k}{k+|y|}\right\} \\ &= \mu(x,t) *_M \mu(y,s).\end{aligned} \tag{11}$$

Hence $(\mu, *_M)$ is a fuzzy modular on X. However, if we set

$$M(x,y,t) = \mu(x-y,t) = \frac{k}{k+|x-y|}, \tag{12}$$

it is easy to verify that $(M, *_M)$ is not a fuzzy metric on X.

Example 9. Let $X = \mathbb{R}$. Take t-norm $a * b = a *_M b$. For every $x, y \in X$ and $t \in (0,\infty)$, we define

$$M(x,y,t) = \begin{cases} 1, & x = y, \\ \frac{1}{2}, & x \neq y,\ x, y \in \mathbb{Z}, \\ \frac{1}{4}, & x \in \mathbb{Z}, y \in \mathbb{R}\setminus\mathbb{Z} \text{ or } x \in \mathbb{R}\setminus\mathbb{Z},\ y \in \mathbb{Z}, \\ \frac{1}{4}, & x \neq y,\ x, y \in \mathbb{R}\setminus\mathbb{Z}. \end{cases} \tag{13}$$

It can easily be shown that $(M, *_M)$ is a fuzzy metric on X. Set

$$\mu(x,t) = M(x,\theta,t) = \begin{cases} 1, & x = 0, \\ \frac{1}{2}, & x \in \mathbb{Z}\setminus\{0\}, \\ \frac{1}{4}, & x \in \mathbb{R}\setminus\mathbb{Z}. \end{cases} \tag{14}$$

If we take $\alpha = \sqrt{2}/2$, $\beta = 1 - \sqrt{2}/2$, $x \neq y$, and $x, y \in \mathbb{Z}$, then we know that $\alpha x + \beta y \in \mathbb{R}\setminus\mathbb{Z}$. Thus, for all $t, s > 0$, we have $\mu(\alpha x + \beta y, t+s) = 1/4$. But $\mu(x,t) *_M \mu(y,s) = \min\{\mu(x,t), \mu(y,s)\} = 1/2$. Obviously, $(\mu, *_M)$ is not a fuzzy modular on X.

3. Topology Induced by a β-Homogeneous Fuzzy Modular

In this section, we will define a topology induced by a β-homogeneous fuzzy modular and examine some topological properties. Let $\mathbb{N}$ denote the set of all positive integers.

Definition 10. Let $(X, \mu, *)$ be a $\mathcal{F}$-modular space. The *μ-ball* $B(x,r,t)$ with center $x \in X$ and radius r, $0 < r < 1$, $t > 0$ is defined as

$$B(x,r,t) = \{y \in X : \mu(x-y,t) > 1-r\}. \tag{15}$$

An element $x \in E$ is called a *μ-interior point* of E if there exist $r \in (0,1)$ and $t > 0$ such that $B(x,r,t) \subseteq E$. Meantime, we say that E is a *μ-open set* in X if and only if every element of E is a μ-interior point.

Lemma 11 (George and Veeramani [17]). *If the t-norm $*$ is continuous, then*

(L1) *for every $r_1, r_2 \in (0,1)$ with $r_1 > r_2$, there exists $r_3 \in (0,1)$ such that $r_1 * r_3 \geq r_2$,*

(L2) *for every $r_4 \in (0,1)$, there exists $r_5 \in (0,1)$ such that $r_5 * r_5 \geq r_4$.*

Theorem 12. *If $(X, \mu, *)$ is a β-homogeneous $\mathcal{F}$-modular space, then $B(x,r,t/2^{\beta+1}) \subset B(x,r,t/2)$.*

Proof. By Theorem 7, for every $r \in (0,1)$ and $t > 0$, since $\mu(x-y, t/2) \geq \mu(x-y, t/2^{\beta+1})$, it is obvious that $\{y \in X : \mu(x-y, t/2^{\beta+1} > 1-r)\} \subset \{y \in X : \mu(x-y, t/2) > 1-r\}$. □

Theorem 13. *Let $(X, \mu, *)$ be a β-homogeneous $\mathcal{F}$-modular space. Every μ-ball $B(x,r,t)$ in $(X,\mu,*)$ is a μ-open set.*

Proof. By Definition 10, for every $y \in B(x,r,t)$, we have $\mu(x-y, t) > 1-r$. Without loss of generality, we may assume that $t = 2t_1$. Since $\mu(x-y, \cdot)$ is continuous, there exists an $\epsilon_y > 0$ such that $\mu(x-y, (t_1-\epsilon)/2^{\beta-1}) > 1-r$ for some $\epsilon > 0$ with $(t_1-\epsilon)/2^{\beta-1} > 0$ and $\epsilon/2^{\beta-1} \in (0, \epsilon_y)$. Set $r_0 = \mu(x-y, (t_1-\epsilon)/2^{\beta-1})$. Since $r_0 > 1-r$, there exists an $s \in (0,1)$ such that $r_0 > 1-s > 1-r$. According to Lemma 11, we can find an $r_1 \in (0,1)$ such that $r_0 * r_1 \geq 1-s$.

Next, we show that $B(y, 1-r_1, \epsilon/2^{\beta-1}) \subset B(x, r, 2t_1)$. For every $z \in B(y, 1-r_1, \epsilon/2^{\beta-1})$, we have $\mu(y-z, \epsilon/2^{\beta-1}) > r_1$. Therefore,

$$\begin{aligned}\mu(x-z,t) &= \mu(x-z,2t_1) \geq \mu(2(x-y), 2(t_1-\epsilon)) \\ &\quad * \mu(2(y-z), 2\epsilon) \\ &= \mu\left(x-y, \frac{t_1-\epsilon}{2^{\beta-1}}\right) * \mu\left(y-z, \frac{\epsilon}{2^{\beta-1}}\right) \\ &\geq r_0 * r_1 \geq 1-s > 1-r.\end{aligned} \tag{16}$$

Thus $z \in B(x,r,t)$ and hence $B(y, 1-r_1, \epsilon/2^{\beta-1}) \subset B(x,r,t)$. □

Theorem 14. *Let $(X,\mu,*)$ be a β-homogeneous $\mathcal{F}$-modular space. Define*

$$\begin{aligned}\mathcal{T}_\mu = \{A \subset X : x \in A \text{ if and only if there exist } t > 0 \text{ and} \\ r \in (0,1) \text{ such that } B(x,r,t) \subset A\}.\end{aligned} \tag{17}$$

Then $\mathcal{T}_\mu$ is a topology on X.

Proof. The proof will be divided into three parts.

(i) Obviously, $\emptyset, X \in \mathcal{T}_\mu$.

(ii) Suppose that $A, B \in \mathcal{T}_\mu$. If $x \in A \cap B$, then $x \in A$ and $x \in B$.

Therefore, there exist $0 < r_1, r_2 < 1$ and $t_1, t_2 > 0$ such that $B(x, r_1, t_1) \subset A$ and $B(x, r_2, t_2) \subset B$. Set $r = \min\{r_1, r_2\}$, $t = \min\{t_1, t_2\}$. Now, we claim that $B(x,r,t) \subset B(x, r_1, t_1)$.

If $y \in B(x,r,t)$, then we know that $\mu(x-y,t) > 1-r$. According to Theorem 7, we can obtain

$$\mu(x-y, t_1) \geq \mu(x-y, t) > 1-r \geq 1-r_1. \tag{18}$$

Thus, $y \in B(x, r_1, t_1)$, that is, $B(x,r,t) \subset B(x, r_1, t_1)$.

Similarly, $B(x,r,t) \subset B(x, r_2, t_2)$.

Hence, $B(x,r,t) \subset B(x,r_1,t_1) \cap B(x,r_2,t_2) \subset A \cap B$. That is to say, $A \cap B \in \mathcal{T}_\mu$.

(iii) Suppose that $\mathcal{T}'_\mu \subset \mathcal{T}_\mu$. If $x \in \bigcup_{A \in \mathcal{T}'_\mu} A$, then there exists $U \in \mathcal{T}'_\mu$ such that $x \in U$. Since $U \in \mathcal{T}_\mu$, there exist $0 < r < 1$ and $t > 0$ such that $B(x,r,t) \subset U \subset \bigcup_{A \in \mathcal{T}'_\mu} A$. Hence, $\bigcup_{A \in \mathcal{T}'_\mu} A \in \mathcal{T}_\mu$. □

Obviously, if we take $r = t = (1/n)$ $(n = 1,2,3,\ldots)$, then the family of μ-ball $B(x, 1/n, 1/n)$, $(n = 1,2,3,\ldots)$ constitutes a countable local base at x. Therefore, we can obtain Theorem 15.

Theorem 15. *The topology $\mathcal{T}_\mu$ induced by a β-homogeneous $\mathcal{F}$-modular space is first countable.*

Theorem 16. *Every β-homogeneous $\mathcal{F}$-modular space is Hausdorff.*

Proof. For the β-homogeneous $\mathcal{F}$-modular space $(X,\mu,*)$, let x and y be two distinct points in X. By Definition 3, we can easily obtain $0 < \mu(x-y,t) < 1$ for all $t > 0$. Set $r = \mu(x-y,t)$. According to Lemma 11, for every $r_0 \in (r,1)$, there exists $r_1 \in (0,1)$ such that $r_1 * r_1 \geq r_0$.

Next, we consider the μ-balls $B(x, 1-r_1, t/2^{\beta+1})$ and $B(y, 1-r_1, t/2^{\beta+1})$ and then show that $B(x, 1-r_1, t/2^{\beta+1}) \cap B(y, 1-r_1, t/2^{\beta+1}) = \emptyset$ using reduction to absurdity. If there exists $z \in B(x, 1-r_1, t/2^{\beta+1}) \cap B(y, 1-r_1, t/2^{\beta+1})$, then

$$\begin{aligned}r = \mu(x-y,t) &\geq \mu\left(2(x-z), \frac{t}{2}\right) * \mu\left(2(z-y), \frac{t}{2}\right) \\ &= \mu\left(x-z, \frac{t}{2^{\beta+1}}\right) * \mu\left(z-y, \frac{t}{2^{\beta+1}}\right) \\ &\geq r_1 * r_1 \geq r_0,\end{aligned} \tag{19}$$

which is a contradiction. Hence $(X,\mu,*)$ is Hausdorff. □

In order to obtain some further properties, several basic notions derived from general topology are introduced in the $\mathcal{F}$-modular space.

Definition 17. Let $(X,\mu,*)$ be a $\mathcal{F}$-modular space.

(i) A sequence $\{x_n\}$ in X is said to be *μ-convergent* to a point $x \in X$, denoted by $x_n \xrightarrow{\mu} x$, if for every $r \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in B(x,r,t)$ for all $n \geq n_0$.

(ii) A subset $A \subset X$ is called *μ-bounded* if and only if there exist $t > 0$ and $r \in (0,1)$ such that $\mu(x,t) > 1-r$ for all $x \in A$.

(iii) A subset $B \subset X$ is called *μ-compact* if and only if every μ-open cover of B has a finite subcover (or equivalently, every sequence in B has a μ-convergent subsequence in B).

(iv) A subset $C \subset X$ is called a *μ-closed* if and only if for every sequence $\{x_n\} \subset C$, $x_n \xrightarrow{\mu} x$ implies $x \in C$.

Theorem 18. *Every μ-compact subset A of a β-homogeneous $\mathcal{F}$-modular space $(X,\mu,*)$ is μ-bounded.*

Proof. Suppose that A is a μ-compact subset of the given β-homogeneous $\mathcal{F}$-modular space $(X, \mu, *)$. Fix $t > 0$ and $r \in (0,1)$, it is easy to see that the family of μ-ball $\{B(x, r, t/2^{\beta+1}) : x \in A\}$ is a μ-open cover of A. Since A is μ-compact, there exist $x_1, x_2, \ldots, x_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n} B(x_i, r, t/2^{\beta+1})$. For every $x \in A$, there exists i such that $x \in B(x_i, r, t/2^{\beta+1})$. Therefore, we have $\mu(x - x_i, t/2^{\beta+1}) > 1 - r$. Set $\alpha = \min\{\mu(x_i, t/2^{\beta+1}) : 1 \le i \le n\}$. Clearly, we know that $\alpha > 0$. Thus, we have

$$\begin{aligned}\mu(x,t) &= \mu((x - x_i) + x_i, t) \ge \mu\left(2(x - x_i), \frac{t}{2}\right) \\ &\quad * \mu\left(2x_i, \frac{t}{2}\right) \\ &= \mu\left(x - x_i, \frac{t}{2^{\beta+1}}\right) * \mu\left(x_i, \frac{t}{2^{\beta+1}}\right) \\ &\ge (1 - r) * \alpha > 1 - s\end{aligned} \tag{20}$$

for some $s \in (0,1)$. This shows that A is μ-bounded. □

Theorem 19. *Let $(X, \mu, *)$ be a β-homogeneous $\mathcal{F}$-modular space, and let $\mathcal{T}_\mu$ be the topology induced by the β-homogeneous modular. Then for a sequence $\{x_n\}$ in X, $x_n \xrightarrow{\mu} x$ if and only if $\mu(x - x_n, t) \to 1$ as $n \to \infty$.*

Proof. Fix $t > 0$. Suppose that $x_n \xrightarrow{\mu} x$. Then for every $r \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in B(x, r, t)$ for all $n \ge n_0$. Namely, $\mu(x_n - x, t) > 1 - r$ for all $n \ge n_0$. Thus, we have $1 - \mu(x_n - x, t) < r$ for all $n \ge n_0$. Because r is arbitrary, we can verify that $\mu(x_n - x, t) \to 1$ as $n \to \infty$.

On the other hand, if for every $t > 0$, $\mu(x - x_n, t) \to 1$ as $n \to \infty$, then for every $r \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $1 - \mu(x - x_n, t) < r$ for all $n \ge n_0$. Therefore, we know that $\mu(x - x_n, t) > 1 - r$ for all $n \ge n_0$. Thus $x_n \in B(x, r, t)$ for all $n \ge n_0$, and hence $x_n \xrightarrow{\mu} x$ as $n \to \infty$. □

4. μ-Completeness of a Fuzzy Modular Space

In this section, we will establish some related theorems of μ-completeness of a fuzzy modular space.

Definition 20. Let $(X, \mu, *)$ be a $\mathcal{F}$-modular space.

(i) A sequence $\{x_n\}$ in X is a μ-*Cauchy sequence* if and only if for every $\epsilon \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu(x_m - x_n, t) > 1 - \epsilon$ for all $m, n \ge n_0$.

(ii) The $\mathcal{F}$-modular space $(X, \mu, *)$ is called μ-*complete* if every μ-Cauchy sequence is μ-convergent.

In [16], Fallahi and Nourouzi proved that every μ-convergent sequence is a μ-Cauchy sequence in the β-homogeneous $\mathcal{F}$-modular space. Here we will propose a similar result in $\mathcal{F}$-modular space. Noticing that the following theorem shows that a μ-convergent sequence is not necessarily a μ-Cauchy sequence in a general $\mathcal{F}$-modular space.

Theorem 21. *Let $(X, \mu, *_M)$ be a β-homogeneous $\mathcal{F}$-modular space. Then every μ-convergent sequence $\{x_n\}$ in X is a μ-Cauchy sequence.*

Proof. Suppose that the sequence $\{x_n\}$ μ-converges to $x \in X$. Therefore, for every $\epsilon \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu(x_n - x, t/2^{\beta+1}) > 1 - \epsilon$ for all $n \ge n_0$. For all $m, n \ge n_0$, we have

$$\begin{aligned}\mu(x_m - x_n, t) &\ge \mu\left(2(x_m - x), \frac{t}{2}\right) *_M \mu\left(2(x_n - x), \frac{t}{2}\right) \\ &\ge \mu\left(x_m - x, \frac{t}{2^{\beta+1}}\right) *_M \mu\left(x_n - x, \frac{t}{2^{\beta+1}}\right) \\ &> (1 - \epsilon) *_M (1 - \epsilon) = 1 - \epsilon.\end{aligned} \tag{21}$$

Hence $\{x_n\}$ is a μ-Cauchy sequence in X. □

Remark 22. The proof of Theorem 21 shows that, in the $\mathcal{F}$-modular space, a μ-convergent sequence is not necessarily a μ-Cauchy sequence. However, the β-homogeneity and the choice of triangular norms are essential to guarantee the establishment of theorem.

Theorem 23. *Every μ-closed subspace of μ-complete $\mathcal{F}$-modular space is μ-complete.*

Proof. From Definition 20, it is evident to see that the theorem holds. □

Theorem 24. *Let $(X, \mu, *)$ be a β-homogeneous $\mathcal{F}$-modular space, and let Y be a subset of X. If every μ-Cauchy sequence of Y is μ-convergent in X, then every μ-Cauchy sequence of $\overline{Y}$ is also μ-convergent in X, where $\overline{Y}$ denotes the μ-closure of Y.*

Proof. Suppose that the sequence $\{x_n\}$ is a μ-Cauchy sequence of $\overline{Y}$. Therefore, for every $n \in \mathbb{N}$ and $t > 0$, there exists $y_n \in Y$ such that $\mu(x_n - y_n, t/4^{\beta+1}) > 1 - 1/(n+1)$. According to Theorem 7, we have $\mu(x_n - y_n, t/2^{\beta+1}) > 1 - (1/(n+1))$. In addition, for every $r \in (0,1)$ and $t > 0$, there exists an $n_0 \in \mathbb{N}$ such that $\mu(x_n - x_m, t/4^{\beta+1}) > 1 - r$ for all $m, n \ge n_0$. That is to say, $\mu(x_n - x_m, t/4^{\beta+1}) \to 1$ as $m, n \to \infty$. Next, we will show that the sequence $\{y_n\}$ is a μ-Cauchy sequence of Y. For every $m, n \ge n_0$, we have

$$\begin{aligned}\mu(y_n - y_m, t) &\ge \mu\left(2(y_n - x_n), \frac{t}{2}\right) * \mu\left(2(x_n - y_m), \frac{t}{2}\right) \\ &\ge \mu\left(2(y_n - x_n), \frac{t}{2}\right) * \mu\left(4(x_n - x_m), \frac{t}{4}\right) \\ &\quad * \mu\left(4(x_m - y_m), \frac{t}{4}\right)\end{aligned}$$

$$= \mu\left(y_n - x_n, \frac{t}{2^{\beta+1}}\right) * \mu\left(x_n - x_m, \frac{t}{4^{\beta+1}}\right) * \mu\left(x_m - y_m, \frac{t}{4^{\beta+1}}\right) > \left(1 - \frac{1}{n+1}\right) * (1-r) * \left(1 - \frac{1}{m+1}\right). \quad (22)$$

Since t-norm $*$ is continuous, it follows that $\mu(y_n - y_m, t) \to 1$ as $m, n \to \infty$. Now, we assume that the sequence $\{y_n\}$ μ-converges to $x \in X$. Thus, for every $\epsilon \in (0,1)$ and $t > 0$, there exists an $n_1 \in \mathbb{N}$ such that $\mu(x - y_n, t/2^{\beta+1}) > 1 - \epsilon$ for all $n \geq n_1$. Therefore, for all $n \geq n_1$, we can obtain

$$\mu(x_n - x, t) \geq \mu\left(2(x_n - y_n), \frac{t}{2}\right) * \mu\left(2(y_n - x_n), \frac{t}{2}\right) = \mu\left(x_n - y_n, \frac{t}{2^{\beta+1}}\right) * \mu\left(y_n - x_n, \frac{t}{2^{\beta+1}}\right) > (1-\epsilon) * \left(1 - \frac{1}{n+1}\right). \quad (23)$$

According to the arbitrary of ϵ and by letting $n \to \infty$, it follows that $\lim_{n\to\infty}\mu(x_n - x, t) = 1$. That is, an arbitrary μ-Cauchy sequence $\{x_n\}$ of $\overline{Y}$ μ-converges to $x \in X$. The proof of the theorem is now completed. □

Theorem 25. *Let $(X, \mu, *)$ be a β-homogeneous $\mathcal{F}$-modular space, and let Y be a dense subset of X. If every μ-Cauchy sequence of Y is μ-convergent in X, then the β-homogeneous $\mathcal{F}$-modular space $(X, \mu, *)$ is μ-complete.*

Proof. It follows from Theorem 24. □

5. Baire's Theorem and Uniform Limit Theorem

In [15], Nourouzi extended the well-know Baire's theorem to probabilistic modular spaces. In this section, we will extend the Baire's theorem to fuzzy modular spaces in an analogous way. Moreover, the uniform limit theorem also can be extended to this type of spaces.

Theorem 26 (Baire's theorem). *Let U_n $(n = 1, 2, \ldots)$ be a countable number of μ-dense and μ-open sets in the μ-complete β-homogeneous $\mathcal{F}$-modular space $(X, \mu, *_M)$. Then $\bigcap_{n=1}^{\infty} U_n$ is μ-dense in X.*

Proof. First of all, if $B(x, r, 2t)$ is a μ-ball in X and y is an arbitrary element of it, then we know that $\mu(x - y, 2t) > 1 - r$. Since $\mu(x - y, \cdot)$ is continuous, there exists an $\epsilon_y > 0$ such that $\mu(x - y, (t-\epsilon)/2^{\beta-1}) > 1 - r$ for some $\epsilon > 0$ with $(t-\epsilon)/2^{\beta-1} > 0$ and $\epsilon/2^{\beta-1} \in (0, \epsilon_y)$. Choose $r' \in (0, r)$, $\epsilon/2^{\beta-1} \in (0, \epsilon_y)$ and $z \in \overline{B(y, r', \epsilon/4^{\beta})}$, there exists a sequence $\{z_n\}$ in $\overline{B(y, r', \epsilon/4^{\beta})}$ such that $z_n \xrightarrow{\mu} z$ and hence we have

$$\mu\left(z - y, \frac{\epsilon}{2^{\beta-1}}\right) \geq \mu\left(2(z - z_n), \frac{\epsilon}{2^{\beta}}\right) *_M \mu\left(2(z_n - y), \frac{\epsilon}{2^{\beta}}\right) = \mu\left(z - z_n, \frac{\epsilon}{4^{\beta}}\right) *_M \mu\left(z_n - y, \frac{\epsilon}{4^{\beta}}\right) > 1 - r \quad (24)$$

for some $n \in \mathbb{N}$. Therefore, we can obtain

$$\mu(x - z, 2t) = \mu(2(z - y), 2\epsilon) *_M \mu(2(x - y), 2(t - \epsilon)) = \mu\left(z - y, \frac{\epsilon}{2^{\beta-1}}\right) *_M \mu\left(x - y, \frac{t-\epsilon}{2^{\beta-1}}\right) > (1-r) *_M (1-r) = 1 - r. \quad (25)$$

This shows that $\overline{B(y, r', \epsilon/4^{\beta})} \subseteq B(x, r, 2t)$. It means that if A is a nonempty μ-open set of X, then $A \cap U_1$ is nonempty and μ-open. Now, let $x_1 \in A \cap U_1$, there exist $r_1 \in (0,1)$ and $t_1 > 0$ such that $B(x_1, r_1, t_1/2^{\beta-1}) \subseteq A \cap U_1$. Choose $r_1' < r_1$ and $t_1' = \min\{t_1, 1\}$ such that $\overline{B(x_1, r_1', t_1'/2^{\beta-1})} \subseteq A \cap U_1$. Since U_2 is μ-dense in X, we can obtain $B(x_1, r_1', t_1'/2^{\beta-1}) \cap U_2 \neq \emptyset$. Let $x_2 \in B(x_1, r_1', t_1'/2^{\beta-1}) \cap U_2$, there exist $r_2 \in (0, 1/2)$ and $t_2 > 0$ such that $B(x_2, r_2, t_2/2^{\beta-1}) \subseteq B(x_1, r_1', t_1'/2^{\beta-1}) \cap U_2$. Choose $r_2' < r_2$ and $t_2' = \min\{t_2, 1/2\}$ such that $\overline{B(x_2, r_2', t_2'/2^{\beta-1})} \subseteq A \cap U_2$. By induction, we can obtain a sequence $\{x_n\}$ in X and two sequence $\{r_n'\}$, $\{t_n'\}$ such that $0 < r_n' < 1/n$, $0 < t_n' < 1/n$ and $\overline{B(x_n, r_n', t_n'/2^{\beta-1})} \subseteq A \cap U_n$.

Next, we show that $\{x_n\}$ is a μ-Cauchy sequence. For given $t > 0$ and $r \in (0,1)$, we can choose $k \in \mathbb{N}$ such that $2t_k' < t$ and $r_k' < r$. Then for $m, n \geq k$, since $x_m, x_n \in B(x_k, r_k', t_k'/2^{\beta-1})$, we have

$$\mu(x_m - x_n, 2t) \geq \mu(x_m - x_n, 4t_k') \geq \mu(2(x_m - x_k), 2t_k') *_M \mu(2(x_n - x_k), 2t_k') = \mu\left(x_m - x_k, \frac{t_k'}{2^{\beta-1}}\right) *_M \mu\left(x_n - x_k, \frac{t_k'}{2^{\beta-1}}\right) \geq 1 - r_k > 1 - r. \quad (26)$$

According to the arbitrary of t, it follows that $\{x_n\}$ is a μ-Cauchy sequence. Since X is μ-complete, there exists $x \in X$ such that $x_n \xrightarrow{\mu} x$. But $x_n \in B(x_k, r_k', t_k'/2^{\beta-1})$ for all $n \geq k$, and therefore $x \in \overline{B(x_k, r_k', t_k'/2^{\beta-1})} \subseteq A \cap U_k$ for all k. Thus $A \cap (\bigcap_{n=1}^{\infty} U_n) \neq \emptyset$. Hence $\bigcap_{n=1}^{\infty} U_n$ is μ-dense in X. □

Definition 27. Let X be any nonempty set and let $(Y, \mu, *)$ be a $\mathcal{F}$-modular space. A sequence $\{f_n\}$ of functions from X to

Y is said to *μ-converge uniformly* to a function f from X to Y if given $t > 0$ and $r \in (0, 1)$; there exists $n_0 \in \mathbb{N}$ such that $\mu(f_n(x) - f(x), t) > 1 - r$ for all $n \geq n_0$ and for every $x \in X$.

Theorem 28 (Uniform limit theorem). *Let $f_n : X \rightarrow Y$ be a sequence of continuous functions from a topological space X to a β-homogeneous $\mathcal{F}$-modular space $(Y, \mu, *)$. If $\{f_n\}$ μ-converges uniformly to $f : X \rightarrow Y$, then f is continuous.*

Proof. Let V be a μ-open set of Y and $x_0 \in f^{-1}(V)$. Since V is μ-open, there exist $r \in (0, 1)$ and $t > 0$ such that $B(f(x_0), r, t) \subset V$. Owing to $r \in (0, 1)$, we can choose $s \in (0, 1)$ such that $(1 - s) * (1 - s) * (1 - s) > 1 - r$. Since $\{f_n\}$ μ-converges uniformly to f, given $s \in (0, 1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu(f_n(x) - f(x), t/4^{\beta+1}) > 1 - s$ for all $n \geq n_0$ and for every $x \in X$. Moreover, f_n is continuous for every $n \in \mathbb{N}$, there exists a neighborhood U of x_0 such that $f_n(U) \subset B(f_n(x_0), s, t/4^{\beta+1})$. Therefore, we know that $\mu(f_n(x) - f_n(x_0), t/4^{\beta+1}) > 1 - s$ for every $x \in U$. Thus, we have

$$\begin{aligned}
\mu(f(x) - f(x_0), t) &\geq \mu\left(2(f(x) - f_n(x)), \frac{t}{2}\right) \\
&\quad * \mu\left(2(f_n(x) - f(x_0)), \frac{t}{2}\right) \\
&= \mu\left(f(x) - f_n(x), \frac{t}{2^{\beta+1}}\right) \\
&\quad * \mu\left(f_n(x) - f(x_0), \frac{t}{2^{\beta+1}}\right) \\
&\geq \mu\left(f(x) - f_n(x), \frac{t}{2^{\beta+1}}\right) \\
&\quad * \mu\left(2(f_n(x) - f_n(x_0)), \frac{t}{2^{\beta+2}}\right) \\
&\quad * \mu\left(2(f_n(x_0) - f(x_0)), \frac{t}{2^{\beta+2}}\right) \\
&= \mu\left(f(x) - f_n(x), \frac{t}{2^{\beta+1}}\right) \\
&\quad * \mu\left(f_n(x) - f_n(x_0), \frac{t}{4^{\beta+1}}\right) \\
&\quad * \mu\left(f_n(x_0) - f(x_0), \frac{t}{4^{\beta+1}}\right) \\
&\geq (1 - s) * (1 - s) * (1 - s) \\
&> 1 - r.
\end{aligned} \tag{27}$$

This shows that $f(x) \in B(f(x_0), r, t) \subset V$. Hence $f(U) \subset V$; that is, f is continuous. □

Remark 29. All the results in this paper are still valid if the condition (FM-5) in Definition 3 is replaced by left continuity.

6. Conclusions

In this paper, we have proposed the concept of fuzzy modular space based on the (probabilistic) modular space and continuous t-norm, which can be regarded as a generalization of (probabilistic) modular space in the fuzzy sense. Meantime, two examples are given to show that a fuzzy modular and a fuzzy metric do not necessarily induce mutually when the triangular norm is the same one. In the sequel, we have defined a Hausdorff topology induced by a β-homogeneous fuzzy modular and examined some related topological properties. It should be pointed out that the β-homogeneity is essential to ensure the establishment of most important conclusions, and some properties also depend on the choice of triangular norms. Finally, we have extended the well-known Baire's theorem and uniform limit theorem to β-homogeneous fuzzy modular spaces.

Further research will focus on the following problems. (1) We first address the problem whether there is a relationship between a fuzzy modular and a fuzzy metric. If the aforementioned relationship exists, then the following issue should be simultaneously considered. (2) It has important theoretical values to explore what conditions a fuzzy modular and a fuzzy metric can induce mutually. (3) Similar to the fixed point theory in probabilistic or fuzzy metric spaces, it is an interesting and valuable research direction to construct fixed point theorems in fuzzy modular spaces. (4) Inspired by [3, 4, 20–22], a problem worthy to be considered is extending the modular sequence (function) space and the Orlicz sequence space to fuzzy setting by the method used in this paper.

Acknowledgments

This work was supported by "Qing Lan" Talent Engineering Funds by Tianshui Normal University. The second author acknowledge the support of the Beijing Municipal Education Commission Foundation of China (no. KM201210038001), the National Natural Science Foundation (no. 71240002) and the Funding Project for Academic Human Resources Development in Institutions of Higher Learning Under the Jurisdiction of Beijing Municipality (no. PHR201108333).

References

[1] H. Nakano, "Modular Semi-Ordered Spaces," Tokoyo, Japan, 1959.

[2] J. Musielak and W. Orlicz, "On modular spaces," *Studia Mathematica*, vol. 18, pp. 49–65, 1959.

[3] W. M. Kozłowski, "Notes on modular function spaces—I," *Commentationes Mathematicae*, vol. 28, no. 1, pp. 87–100, 1988.

[4] W. M. Kozłowski, "Notes on modular function spaces—II," *Commentationes Mathematicae*, vol. 28, no. 1, pp. 101–116, 1988.

[5] W. M. Kozłowski and G. Lewicki, "Analyticity and polynomial approximation in modular function spaces," *Journal of Approximation Theory*, vol. 58, no. 1, pp. 15–35, 1989.

[6] S. J. Kilmer, W. M. Kozłowski, and G. Lewicki, "Best approximants in modular function spaces," *Journal of Approximation Theory*, vol. 63, no. 3, pp. 338–367, 1990.

[7] M. A. Khamsi, W. M. Kozłowski, and S. Reich, "Fixed point theory in modular function spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 14, no. 11, pp. 935–953, 1990.

[8] T. Dominguez Benavides, M. A. Khamsi, and S. Samadi, "Uniformly Lipschitzian mappings in modular function spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 46, pp. 267–278, 2001.

[9] N. Hussain, M. A. Khamsi, and A. Latif, "Banach operator pairs and common fixed points in modular function spaces," *Fixed Point Theory and Applications*, vol. 2011, article 75, 2011.

[10] M. A. Japón, "Some geometric properties in modular spaces and application to fixed point theory," *Journal of Mathematical Analysis and Applications*, vol. 295, no. 2, pp. 576–594, 2004.

[11] M. A. Khamsi and W. M. Kozlowski, "On asymptotic pointwise contractions in modular function spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 73, no. 9, pp. 2957–2967, 2010.

[12] M. A. Khamsi and W. M. Kozlowski, "On asymptotic pointwise nonexpansive mappings in modular function spaces," *Journal of Mathematical Analysis and Applications*, vol. 380, no. 2, pp. 697–708, 2011.

[13] C. Mongkolkeha and P. Kumam, "Fixed point theorems for generalized asymptotic pointwise ρ-contraction mappings involving orbits in modular function spaces," *Applied Mathematics Letters*, vol. 25, no. 10, pp. 1285–1290, 2012.

[14] K. Nourouzi, "Probabilistic modular spaces," in *Proceedings of the 6th International ISAAC Congress*, Ankara, Turkey, 2007.

[15] K. Nourouzi, "Baire's theorem in probabilistic modular spaces," in *Proceedings of the World Congress on Engineering (WCE '08)*, vol. 2, pp. 916–917, 2008.

[16] K. Fallahi and K. Nourouzi, "Probabilistic modular spaces and linear operators," *Acta Applicandae Mathematicae*, vol. 105, no. 2, pp. 123–140, 2009.

[17] A. George and P. Veeramani, "On some results in fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 64, no. 3, pp. 395–399, 1994.

[18] B. Schweizer and A. Sklar, "Statistical metric spaces," *Pacific Journal of Mathematics*, vol. 10, pp. 313–334, 1960.

[19] P. Klement and R. Mesiar, "Triangular norms," *Tatra Mountains Mathematical Publications*, vol. 13, pp. 169–193, 1997.

[20] H. Dutta, I. H. Jebril, B. S. Reddy, and S. Ravikumar, "A generalization of modular sequence spaces by Cesaro mean of order one," *Revista Notas De Matematica*, vol. 7, no. 1, pp. 1–13, 2011.

[21] H. Dutta and F. Başar, "A generalization of Orlicz sequence spaces by Cesàro mean of order one," *Acta Mathematica Universitatis Comenianae*, vol. 80, no. 2, pp. 185–200, 2011.

[22] V. Karakaya and H. Dutta, "On some vector valued generalized difference modular sequence spaces," *Filomat*, vol. 25, no. 3, pp. 15–27, 2011.


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