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Research Article

The General Solution of Impulsive Systems with Caputo-Hadamard Fractional Derivative of Order $q \in \mathbb{C}$ $(\Re(q) \in (1,2))$

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Motivated by some preliminary works about general solution of impulsive system with fractional derivative, the generalized impulsive differential equations with Caputo-Hadamard fractional derivative of $q \in \mathbb{C}$ $(\Re(q) \in (1,2))$ are further studied by analyzing the limit case (as impulses approach zero) in this paper. The formulas of general solution are found for the impulsive systems.

1. Introduction

Hadamard fractional calculus is a key part of the theory of fractional calculus. The authors in [1–6] made an important development of the fractional calculus within the frame of Hadamard fractional derivative. For the general theory of Hadamard fractional calculus, one can see the monograph of Kilbas et al. [7].

Recently, Jarad et al. made a progress on Hadamard fractional derivative to present the definition of Caputo-Hadamard fractional derivative in [8] and developed the fundamental theorem of this fractional derivative in [8, 9].

Furthermore, impulsive differential equations are utilized as a valuable tool to describe the dynamics of processes in which sudden, discontinuous jumps occur, and impulsive differential equations with Caputo fractional derivative were widely researched in [10–26]. Next, the general solutions of several kinds of impulsive fractional differential equations have been found in [27–30], respectively.

Motivated by the above-mentioned works, we will further seek the general solution of generalized impulsive system with Caputo-Hadamard fractional derivative:

$$\begin{aligned}
&{}_{\text{C-H}}D^{q}_{a^+}x(t) = f(t, x(t)), \\
&t \in (a,T],\ t \neq t_k\ (k = 1,2,\dots,m),\ t \neq \bar{t}_l\ (l = 1,2,\dots,p), \\
&\qquad \Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)) \in \mathbb{C}, \\
&\qquad\qquad k = 1,2,\dots,m, \\
&\qquad \Delta x'\big|_{t=\bar{t}_l} = x'(\bar{t}_l^+) - x'(\bar{t}_l^-) = \bar{I}_l(x(\bar{t}_l^-)) \in \mathbb{C}, \\
&\qquad\qquad l = 1,2,\dots,p, \\
&\qquad x(a) = x_a \in \mathbb{C}, \\
&\qquad x'(a) = \bar{x}_a \in \mathbb{C}.
\end{aligned} \tag{1}$$

Here $q \in \mathbb{C}$ and $\Re(q) \in (1,2)$, ${}_{\text{C-H}}D^{q}_{a^+}$ denotes left-sided Caputo-Hadamard fractional derivative of order q and $a > 0$,

$f : [a, T] \times \mathbb{C} \to \mathbb{C}$ is an appropriate continuous function, $a = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, and $a = \bar{t}_0 < \bar{t}_1 < \cdots < \bar{t}_p < \bar{t}_{p+1} = T$. Here $x(t_k^+) = \lim_{\varepsilon \to 0^+} x(t_k + \varepsilon)$ and $x(t_k^-) = \lim_{\varepsilon \to 0^-} x(t_k + \varepsilon)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively, and $x'(\bar{t}_l^+)$ and $x'(\bar{t}_l^-)$ have similar meaning. Let us queue $a, t_1, \ldots, t_m, \bar{t}_1, \ldots, \bar{t}_p, T$ to $a = t'_0 < t'_1 < t'_2 < \cdots < t'_\Pi < t'_{\Pi+1} = T$ such that

$$\text{set}\left\{t_1, \ldots, t_m, \bar{t}_1, \ldots, \bar{t}_p\right\} = \text{set}\left\{t'_\Pi, t'_2, \ldots, t'_\Pi\right\}. \tag{2}$$

For each $[a, t'_k]$ $(k = 0, 1, \ldots, \Pi)$, suppose $[a, t_{k_1}] \subseteq [a, t'_k] \subset [a, t_{k_1+1}]$ (here $k_1 \in \{1, 2, \ldots, m\}$) and $[a, \bar{t}_{k_2}] \subseteq [a, t'_k] \subset [a, \bar{t}_{k_2+1}]$ (here $k_2 \in \{1, 2, \ldots, p\}$), respectively.

In order to get the solution of (1), we will first consider the following system:

$$\begin{aligned} &{}_{\text{C-H}}D_{a^+}^{q} x(t) = f(t, x(t)), \\ &t \in (a, T], \ t \neq t_k \ (k = 1, \ldots, m), \ t \neq \bar{t}_l \ (l = 1, \ldots, p), \end{aligned} \tag{3a}$$

$$\begin{aligned} \Delta x|_{t=t_k} &= x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)) \in \mathbb{C}, \\ & k = 1, 2, \ldots, m, \end{aligned} \tag{3b}$$

$$\begin{aligned} \Delta\delta x|_{t=\bar{t}_l} &= \delta x(\bar{t}_l^+) - \delta x(\bar{t}_l^-) = J_l(x(\bar{t}_l^-)) \in \mathbb{C}, \\ & l = 1, 2, \ldots, p, \end{aligned} \tag{3c}$$

$$\begin{aligned} &x(a) = x_a \in \mathbb{C}, \\ &\delta x(a) = \widehat{x}_a \in \mathbb{C}, \end{aligned} \tag{3d}$$

where differential operator $\delta = t(d/dt)$, $\delta^0 x(t) = x(t)$.

Next, some definitions and conclusions are introduced in Section 2, and the formulas of general solution will be given for some impulsive differential equations with Caputo-Hadamard fractional derivative in Section 3.

2. Preliminaries

Definition 1 (see [7, p. 110]). Let $0 \le a \le b \le \infty$ be finite or infinite interval of the half-axis $\mathbb{R}^+$. The left-sided Hadamard fractional integral of order $\alpha \in \mathbb{C}$ of function $\varphi(x)$ is defined by

$$\left({}_{\text{H}}\mathcal{J}_{a^+}^{\alpha}\varphi\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln\frac{x}{s}\right)^{\alpha-1} \varphi(s)\frac{ds}{s}, \quad (a < x < b), \tag{4}$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2 (see [7, p. 110]). The left-sided Hadamard fractional derivative of order $\alpha \in \mathbb{C}$ $(\mathfrak{R}(\alpha) \ge 0)$ on (a, b) is defined by

$$\begin{aligned} \left({}_{\text{H}}D_{a^+}^{\alpha}\varphi\right)(x) &= \delta^n\left({}_{\text{H}}\mathcal{J}_{a^+}^{n-\alpha}\varphi\right)(x) \\ &= \left(x\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(\ln\frac{x}{s}\right)^{n-\alpha-1} \varphi(s)\frac{ds}{s}, \\ &\quad (a < x < b), \end{aligned} \tag{5}$$

where $n = [\mathfrak{R}(\alpha)] + 1$ and differential operator $\delta = x(d/dx)$ and $\delta^0 y(x) = y(x)$.

Lemma 3 (see [7, p. 114–116]). *Let $\alpha, \beta \in \mathbb{C}$ such that $\mathfrak{R}(\alpha) > \mathfrak{R}(\beta) > 0$. For $0 < a < b < \infty$, if $\varphi \in L^p(a, b)$ $(1 \le p < \infty)$, then ${}_{H}D_{a^+}^{\beta}\, {}_{H}\mathcal{J}_{a^+}^{\alpha}\varphi = {}_{H}\mathcal{J}_{a^+}^{\alpha-\beta}\varphi$ and ${}_{H}\mathcal{J}_{a^+}^{\alpha}\, {}_{H}\mathcal{J}_{a^+}^{\beta}\varphi = {}_{H}\mathcal{J}_{a^+}^{\alpha+\beta}\varphi$.*

The left-sided Caputo-Hadamard fractional derivative was defined in [8] by

$$\begin{aligned} &{}_{\text{C-H}}D_{a^+}^{\alpha}\varphi(x) \\ &\quad = {}_{\text{H}}D_{a^+}^{\alpha}\left[\varphi(x) - \sum_{k=0}^{n-1}\frac{\delta^k\varphi(a)}{k!}\left(\ln\frac{t}{a}\right)^k\right](x); \end{aligned} \tag{6}$$

here $\mathfrak{R}(\alpha) \ge 0$, $n = [\mathfrak{R}(\alpha)] + 1$, $0 < a < b < \infty$, differential operator $\delta = x(d/dx)$, $\delta^0 y(x) = y(x)$, and

$$\begin{aligned} \varphi(x) \in AC_\delta^n[a, b] = \Big\{&\varphi : [a, b] \longrightarrow \mathbb{C} : \delta^{(n-1)}\varphi(x) \\ &\in AC[a, b], \ \delta = x\frac{d}{dx}\Big\}. \end{aligned} \tag{7}$$

Theorem 4 (see [8, p. 4]). *Let $\mathfrak{R}(\alpha) \ge 0$, $n = [\mathfrak{R}(\alpha)] + 1$, and $\varphi \in AC_\delta^n[a, b]$, $0 < a < b < \infty$. Then, ${}_{C\text{-}H}D_{a^+}^{\alpha}\varphi(x)$ exist everywhere on $[a, b]$ and*

(a) *if $\alpha \notin \mathbb{N}_0$,*

$$\begin{aligned} {}_{C\text{-}H}D_{a^+}^{\alpha}\varphi(x) &= \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(\ln\frac{x}{s}\right)^{n-\alpha-1} \delta^n\varphi(s)\frac{ds}{s} \\ &= {}_{H}\mathcal{J}_{a^+}^{n-\alpha}\delta^n\varphi(x), \end{aligned} \tag{8}$$

(b) *if $\alpha = n \in \mathbb{N}_0$,*

$${}_{C\text{-}H}D_{a^+}^{\alpha}\varphi(x) = \delta^n\varphi(x). \tag{9}$$

In particular,

$${}_{C\text{-}H}D_{a^+}^{0}\varphi(x) = \varphi(x). \tag{10}$$

Lemma 5 (see [8, p. 5]). *Let $\mathfrak{R}(\alpha) > 0$, $n = [\mathfrak{R}(\alpha)] + 1$ and $\varphi \in C[a, b]$. If $\mathfrak{R}(\alpha) \neq 0$ or $\alpha \in \mathbb{N}$, then*

$${}_{C\text{-}H}D_{a^+}^{\alpha}\left({}_{H}\mathcal{J}_{a^+}^{\alpha}\varphi\right)(x) = \varphi(x). \tag{11}$$

Lemma 6 (see [8, p. 6]). *Let $\varphi \in AC_\delta^n[a, b]$ or let $C_\delta^n[a, b]$ and $\alpha \in \mathbb{C}$; then,*

$${}_{H}\mathcal{J}_{a^+}^{\alpha}\left({}_{C\text{-}H}D_{a^+}^{\alpha}\varphi\right)(x) = \varphi(x) - \sum_{k=0}^{n-1}\frac{\delta^k\varphi(a)}{k!}\left(\ln\frac{x}{a}\right)^k. \tag{12}$$

Lemma 7 (see [29, p. 4]). *Let $w \in \mathbb{C}$ and $\mathfrak{R}(w) \in (0,1)$, and let ξ be a constant. A function $u(t) : [a,T] \rightarrow \mathbb{C}$ is general solution of system*

$$
\begin{aligned}
&{}_{C\text{-}H}D^{w}_{a^+}u(t) = g(t,u(t)), \\
&\qquad t \in (a,T],\ t \neq t_k\ \ (k = 1,2,\dots,m), \\
&\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-) = \Delta_k(u(t_k^-)) \in \mathbb{C}, \\
&\qquad k = 1,2,\dots,m, \\
&u(a) = u_a, \quad u_a \in \mathbb{C},
\end{aligned} \tag{13}
$$

if and only if $u(t)$ satisfies the integral equation

$$
u(t) = \begin{cases}
u_a + \dfrac{1}{\Gamma(w)}\displaystyle\int_a^t \left(\ln\frac{t}{s}\right)^{w-1} g(s,u(s))\frac{ds}{s} & \text{for } t \in (a,t_1], \\[2ex]
u_a + \displaystyle\sum_{i=1}^{k}\Delta_i(u(t_i^-)) + \dfrac{1}{\Gamma(w)}\displaystyle\int_a^t \left(\ln\frac{t}{s}\right)^{w-1} g(s,u(s))\frac{ds}{s} & \\
\quad + \xi\displaystyle\sum_{i=1}^{k}\frac{\Delta_i(u(t_i^-))}{\Gamma(w)}\left[\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{w-1} g(s,u(s))\frac{ds}{s} + \int_{t_i}^{t}\left(\ln\frac{t}{s}\right)^{w-1} g(s,u(s))\frac{ds}{s}\right. & \\
\quad \left. - \displaystyle\int_a^t \left(\ln\frac{t}{s}\right)^{w-1} g(s,u(s))\frac{ds}{s}\right] & \text{for } t \in (t_k,t_{k+1}],\ k = 1,2,\dots,m
\end{cases} \tag{14}
$$

provided that the integral in (14) exists.

3. Main Results

Firstly, let us consider some limit cases in system (3a), (3b), (3c), and (3d):

$$
\lim_{\substack{I_k(x(t_k^-))\to 0\ \forall k\in\{1,2,\dots,m\},\\ J_l(x(\bar{t}_l^-))\to 0\ \forall l\in\{1,2,\dots,p\}}} \{\text{system (3a), (3b), (3c), and (3d)}\} \longrightarrow \begin{cases}
{}_{C\text{-}H}D^{q}_{a^+}x(t) = f(t,x(t)), \quad t \in (a,T], \\
x(a) = x_a \in \mathbb{C}, \\
\delta x(a) = \widehat{x}_a \in \mathbb{C},
\end{cases} \tag{15}
$$

$$
\begin{aligned}
&\lim_{\substack{J_l(x(\bar{t}_l^-))\to 0,\\ \forall l\in\{1,2,\dots,p\}}} \{\text{system (3a), (3b), (3c), and (3d)}\} \\
&\longrightarrow \begin{cases}
{}_{C\text{-}H}D^{q}_{a^+}x(t) = f(t,x(t)), & t \in (a,T],\ t \neq t_k\ \ (k = 1,\dots,m), \\
\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)) \in \mathbb{C}, & k = 1,2,\dots,m, \\
x(a) = x_a \in \mathbb{C}, & \\
\delta x(a) = \widehat{x}_a \in \mathbb{C},
\end{cases}
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
&\lim_{\substack{I_k(x(t_k^-))\to 0,\\ \forall k\in\{1,2,\dots,m\}}} \{\text{system (3a), (3b), (3c), and (3d)}\} \\
&\longrightarrow \begin{cases}
{}_{C\text{-}H}D^{q}_{a^+}x(t) = f(t,x(t)), & t \in (a,T],\ t \neq \bar{t}_l\ \ (l = 1,\dots,p), \\
\Delta\delta x|_{t=\bar{t}_l} = \delta x(\bar{t}_l^+) - \delta x(\bar{t}_l^-) = J_l(x(\bar{t}_l^-)) \in \mathbb{C}, & l = 1,2,\dots,p, \\
x(a) = x_a \in \mathbb{C}, & \\
\delta x(a) = \widehat{x}_a \in \mathbb{C}.
\end{cases}
\end{aligned} \tag{17}
$$

Thus,

$$\text{(i)} \lim_{\substack{I_k(x(t_k^-))\to 0\,\forall k\in\{1,2,\ldots,m\},\\ J_l(x(\bar{t}_l^-))\to 0\,\forall l\in\{1,2,\ldots,p\}}} \{\text{the solution of system (3a)}, \text{(3b)}, \text{(3c)}, \text{and (3d)}\} = \{\text{the solution of system (15)}\},$$

$$\text{(ii)} \lim_{\substack{J_l(x(\bar{t}_l^-))\to 0,\\ \forall l\in\{1,2,\ldots,p\}}} \{\text{the solution of system (3a)}, \text{(3b)}, \text{(3c)}, \text{and (3d)}\} = \{\text{the solution of system (16)}\}, \quad (18)$$

$$\text{(iii)} \lim_{\substack{I_k(x(t_k^-))\to 0,\\ \forall k\in\{1,2,\ldots,m\}}} \{\text{the solution of system (3a)}, \text{(3b)}, \text{(3c)}, \text{and (3d)}\} = \{\text{the solution of system (17)}\}.$$

Thus, the definition of solution of system (3a), (3b), (3c), and (3d) is presented as follows.

Definition 8. A function $z(t) : [a,T] \to \mathbb{C}$ is said to be a solution of (3a), (3b), (3c), and (3d) if $z(a) = x_a$ and $\delta z(a) = \widehat{x}_a$, the equation condition ${}_{\text{C-H}}D^q_{a^+} z(t) = f(t, z(t))$ for each $t \in [a,T]/\{t'_1, t'_2, \ldots, t'_\Pi\}$ is verified, the impulsive conditions $\Delta z|_{t=t_k} = I_k(z(t_k^-))$ (here $k = 1,2,\ldots,m$) and $\Delta\delta z|_{t=\bar{t}_l} = J_l(z(\bar{t}_l^-))$ (here $l = 1,2,\ldots,p$) are satisfied, the restriction of $z(\cdot)$ to the interval $(t'_k, t'_{k+1}]$ (here $k = 0,1,2,\ldots,\Pi$) is continuous, and the conditions (i)–(iii) hold.

Next, define a function by

$$\widetilde{x}(t) = x(t_k^+) + \delta x(t_k^+)\ln\frac{t}{t_k} + \frac{1}{\Gamma(q)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s}, \quad \text{for } t \in (t_k, t_{k+1}] \ (\text{here } k = 0,1,2,\ldots,m). \quad (19)$$

By Theorem 4, we have

$$\begin{aligned}\left[{}_{\text{C-H}}D^q_{a^+}\widetilde{x}(t)\right]_{t\in(t_k,t_{k+1}]} &= \left\{\frac{1}{\Gamma(2-q)}\int_a^t\left(\ln\frac{t}{s}\right)^{2-q-1} \cdot \delta^2\left[x(t_k^+) + \delta x(t_k^+)\ln\frac{s}{t_k} + \frac{1}{\Gamma(q)} \cdot \int_{t_k}^{s}\left(\ln\frac{s}{\eta}\right)^{q-1} f(\eta, x(\eta))\frac{d\eta}{\eta}\right]\frac{ds}{s}\right\}_{t\in(t_k,t_{k+1}]} \\ &= \left\{\frac{1}{\Gamma(2-q)\Gamma(q)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{2-q-1} \cdot \delta^2\left[\int_{t_k}^{s}\left(\ln\frac{s}{\eta}\right)^{q-1} f(\eta, x(\eta))\frac{d\eta}{\eta}\right]\frac{ds}{s}\right\}_{t\in(t_k,t_{k+1}]} \\ &= f(t, x(t))|_{t\in(t_k,t_{k+1}]}.\end{aligned} \quad (20)$$

This means that $\widetilde{x}(t)$ satisfies (3a), and $\widetilde{x}(t)$ satisfies (3b)–(3d). However, $\widetilde{x}(t)$ does not satisfy the conditions (i)–(iii), and it is not a solution of system (3a), (3b), (3c), and (3d). Therefore, $\widetilde{x}(t)$ is considered as an approximate solution to seek the exact solutions of (3a), (3b), (3c), and (3d). Next, let us prove some useful conclusions.

Lemma 9. *Let $q \in \mathbb{C}$, $\mathfrak{R}(q) \in (1,2)$, and ξ is a constant. System (16) is equivalent to the integral equation*

$$x(t) = \begin{cases} x_a + \widehat{x}_a\ln\frac{t}{a} + \frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s} & \text{for } t \in (a, t_1], \\ x_a + \widehat{x}_a\ln\frac{t}{a} + \sum_{i=1}^{k} I_i(x(t_i^-)) + \frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s} \\ \quad + \xi\sum_{i=1}^{k} I_i(x(t_i^-))\left\{\frac{1}{\Gamma(q)}\left[\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s} + \int_{t_i}^{t}\left(\ln\frac{t}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s}\right.\right. \\ \quad \left.\left. - \int_a^t\left(\ln\frac{t}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s}\right] + \frac{\ln(t/t_i)}{\Gamma(q-1)}\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-2} f(s, x(s))\frac{ds}{s}\right\} & \text{for } t \in (t_k, t_{k+1}],\ 1 \le k \le m \end{cases} \quad (21)$$

provided that the integral in (21) exists.

Proof.

Necessity. Letting $I_k(x(t_k^-)) \to 0$ $(k = 1, 2, \dots, m)$ in (16), we have

$$\lim_{\substack{I_k(x(t_k^-))\to 0,\\ \forall k\in\{1,2,\dots,m\}}} \{\text{system (16)}\} \longrightarrow \begin{cases} {}_{\text{C-H}}D_{a^+}^{q} x(t) = f(t, x(t)), & t \in (a, T], \\ x(a) = x_a \in \mathbb{C}, \\ \delta x(a) = \widehat{x}_a \in \mathbb{C}. \end{cases} \tag{22}$$

That is,

$$\lim_{\substack{I_k(x(t_k^-))\to 0,\\ \forall k\in\{1,2,\dots,m\}}} \{\text{the solution of system (16)}\} = \{\text{the solution of system (15)}\}. \tag{23}$$

In fact, we can verify that (21) satisfies the condition (23).

Next, taking fractional derivative to (21) for $t \in (t_k, t_{k+1}]$ (here $k = 0, 1, 2, \dots, m$), we get

$$\begin{aligned}
&\left[{}_{\text{C-H}}D_{a^+}^{q} x(t)\right]_{t\in(t_k,t_{k+1}]} = \Bigg(\frac{1}{\Gamma(2-q)} \int_a^t \left(\ln\frac{t}{s}\right)^{2-q-1} \delta^2 \Bigg\{ x_a + \widehat{x}_a \ln\frac{s}{a} + \sum_{i=1}^{k} I_i(x(t_i^-)) + \frac{1}{\Gamma(q)} \int_a^s \left(\ln\frac{s}{\eta}\right)^{q-1} f(\eta, x(\eta)) \frac{d\eta}{\eta} + \xi \sum_{i=1}^{k} I_i(x(t_i^-)) \\
&\quad \cdot \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{t_i} \left(\ln\frac{t_i}{\eta}\right)^{q-1} f(\eta, x(\eta)) \frac{d\eta}{\eta} + \int_{t_i}^{s} \left(\ln\frac{s}{\eta}\right)^{q-1} f(\eta, x(\eta)) \frac{d\eta}{\eta} - \int_a^{s} \left(\ln\frac{s}{\eta}\right)^{q-1} f(\eta, x(\eta)) \frac{d\eta}{\eta} \right] + \frac{\ln(s/t_i)}{\Gamma(q-1)} \int_a^{t_i} \left(\ln\frac{t_i}{\eta}\right)^{q-2} f(\eta, x(\eta)) \frac{d\eta}{\eta} \right\} \Bigg\} \frac{ds}{s} \Bigg)_{t\in(t_k,t_{k+1}]} \\
&= \frac{1}{\Gamma(2-q)\Gamma(q)} \Bigg\{ \int_a^t \left(\ln\frac{t}{s}\right)^{2-q-1} \delta^2 \left(\int_a^s \left(\ln\frac{s}{\eta}\right)^{q-1} f(\eta, x(\eta)) \frac{d\eta}{\eta} \right) \frac{ds}{s} + \xi \sum_{i=1}^{k} I_i(x(t_i^-)) \Bigg[\int_{t_i}^{t} \left(\ln\frac{t}{s}\right)^{2-q-1} \delta^2 \left(\int_{t_i}^{s} \left(\ln\frac{s}{\eta}\right)^{q-1} f(\eta, x(\eta)) \frac{d\eta}{\eta} \right) \frac{ds}{s} - \int_a^t \left(\ln\frac{t}{s}\right)^{2-q-1} \\
&\quad \cdot \delta^2 \left(\int_a^s \left(\ln\frac{s}{\eta}\right)^{q-1} f(\eta, x(\eta)) \frac{d\eta}{\eta} \right) \frac{ds}{s} \Bigg] \Bigg\}_{t\in(t_k,t_{k+1}]} = \left\{ f(t, x(t))\big|_{t\ge a} + \sum_{i=1}^{k} \xi I_i(x(t_i^-)) \left[f(t, x(t))\big|_{t\ge t_i} - f(t, x(t))\big|_{t\ge a} \right] \right\}_{t\in(t_k,t_{k+1}]} = f(t, x(t))\big|_{t\in(t_k,t_{k+1}]}.
\end{aligned} \tag{24}$$

So, (21) satisfies the condition of fractional derivative in system (16).

Finally, using (21) for each t_k (here $k \in \{1, 2, \dots, m\}$), we have

$$\begin{aligned}
x(t_k^+) - x(t_k^-) &= \lim_{t\to t_k^+} x(t) - x(t_k) = x_a + \widehat{x}_a \ln\frac{t_k}{a} \\
&\quad + \sum_{i=1}^{k} I_i(x(t_i^-)) + \frac{1}{\Gamma(q)} \int_a^{t_k} \left(\ln\frac{t_k}{s}\right)^{q-1} \\
&\quad \cdot f(s, x(s)) \frac{ds}{s} + \xi \sum_{i=1}^{k} I_i(x(t_i^-)) \\
&\quad \cdot \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{t_i} \left(\ln\frac{t_i}{s}\right)^{q-1} f(s, x(s)) \frac{ds}{s} \right.\right. \\
&\quad + \int_{t_i}^{t_k} \left(\ln\frac{t_k}{s}\right)^{q-1} f(s, x(s)) \frac{ds}{s} \\
&\quad \left. - \int_a^{t_k} \left(\ln\frac{t_k}{s}\right)^{q-1} f(s, x(s)) \frac{ds}{s} \right] + \frac{\ln(t_k/t_i)}{\Gamma(q-1)} \\
&\quad \left. \cdot \int_a^{t_i} \left(\ln\frac{t_i}{s}\right)^{q-2} f(s, x(s)) \frac{ds}{s} \right\} - x_a - \widehat{x}_a \ln\frac{t_k}{a} \\
&\quad - \sum_{i=1}^{k-1} I_i(x(t_i^-)) - \frac{1}{\Gamma(q)} \int_a^{t_k} \left(\ln\frac{t_k}{s}\right)^{q-1} \\
&\quad \cdot f(s, x(s)) \frac{ds}{s} - \xi \sum_{i=1}^{k-1} I_i(x(t_i^-)) \\
&\quad \cdot \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{t_i} \left(\ln\frac{t_i}{s}\right)^{q-1} f(s, x(s)) \frac{ds}{s} \right.\right. \\
&\quad + \int_{t_i}^{t_k} \left(\ln\frac{t_k}{s}\right)^{q-1} f(s, x(s)) \frac{ds}{s} \\
&\quad \left. - \int_a^{t_k} \left(\ln\frac{t_k}{s}\right)^{q-1} f(s, x(s)) \frac{ds}{s} \right] + \frac{\ln(t_k/t_i)}{\Gamma(q-1)} \\
&\quad \left. \cdot \int_a^{t_i} \left(\ln\frac{t_i}{s}\right)^{q-2} f(s, x(s)) \frac{ds}{s} \right\} = I_k(x(t_k^-)).
\end{aligned} \tag{25}$$

It means that (21) satisfies the impulsive condition of (16). Hence, (21) satisfies all conditions of system (16).

Sufficiency (by mathematical induction). By Lemma 6, the solution of (16) satisfies

$$\begin{aligned}
x(t) &= x_a + \widehat{x}_a \ln\frac{t}{a} \\
&\quad + \frac{1}{\Gamma(q)} \int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f(s, x(s)) \frac{ds}{s} \\
&\qquad \text{for } t \in (a, t_1].
\end{aligned} \tag{26}$$

Using (26), we obtain

$$x(t_1^+) = x(t_1^-) + I_1(x(t_1^-)) = x_a + \widehat{x}_a \ln\frac{t_1}{a} + I_1(x(t_1^-)) + \frac{1}{\Gamma(q)}\int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s},$$
$$\delta x(t_1^+) = \delta x(t_1^-) = \widehat{x}_a + \frac{1}{\Gamma(q-1)}\int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}. \quad (27)$$

Thus, the approximate solution $\widetilde{x}(t)$ is given by

$$\widetilde{x}(t) = x(t_1^+) + \delta x(t_1^+)\ln\frac{t}{t_1} + \frac{1}{\Gamma(q)} \cdot \int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} = x_a + \widehat{x}_a\ln\frac{t}{a} + I_1(x(t_1^-)) + \frac{1}{\Gamma(q)}\left[\int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] + \frac{\ln(t/t_1)}{\Gamma(q-1)} \cdot \int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s} \quad \text{for } t\in(t_1,t_2]. \quad (28)$$

Let $e_1(t) = x(t) - \widetilde{x}(t)$, for $t \in (t_1, t_2]$. By (26), the exact solution $x(t)$ of system (16) satisfies

$$\lim_{I_1(x(t_1^-))\to 0} x(t) = x_a + \widehat{x}_a\ln\frac{t}{a} + \frac{1}{\Gamma(q)}\int_a^{t}\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}, \quad \text{for } t\in(t_1,t_2]. \quad (29)$$

Then,

$$\lim_{I_1(x(t_1^-))\to 0} e_1(t) = \lim_{I_1(x(t_1^-))\to 0}\{x(t) - \widetilde{x}(t)\} = \frac{-1}{\Gamma(q)}\left[\int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} - \int_a^{t}\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] - \frac{\ln(t/t_1)}{\Gamma(q-1)} \cdot \int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}. \quad (30)$$

This shows that $e_1(t)$ is connected with $I_1(x(t_1^-))$ and $\lim_{I_1(x(t_1^-))}e_1(t)$. Thus, we assume

$$e_1(t) = \chi(I_1(x(t_1^-)))\lim_{I_1(x(t_1^-))\to 0} e_1(t) = -\chi(I_1(x(t_1^-))) \cdot \left\{\frac{1}{\Gamma(q)}\left[\int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} - \int_a^{t}\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] + \frac{\ln(t/t_1)}{\Gamma(q-1)} \cdot \int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right\} \quad \text{for } t\in(t_1,t_2], \quad (31)$$

where function χ is an undetermined function with $\chi(0) = 1$. So,

$$x(t) = \widetilde{x}(t) + e_1(t) = x_a + \widehat{x}_a\ln\frac{t}{a} + I_1(x(t_1^-)) + \frac{1}{\Gamma(q)}\int_a^{t}\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \left[1 - \chi(I_1(x(t_1^-)))\right] \cdot \left\{\frac{1}{\Gamma(q)}\left[\int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} - \int_a^{t}\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] + \frac{\ln(t/t_1)}{\Gamma(q-1)} \cdot \int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right\} \quad \text{for } t\in(t_1,t_2]. \quad (32)$$

Letting $\gamma(I_1(x(t_1^-))) = 1 - \chi(I_1(x(t_1^-)))$, we get

$$\begin{aligned} x(t) &= x_a + \hat{x}_a \ln\frac{t}{a} + I_1(x(t_1^-)) + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} \\ &\quad \cdot f(s,x(s))\frac{ds}{s} + \gamma(I_1(x(t_1^-))) \\ &\quad \cdot \left\{\frac{1}{\Gamma(q)}\left[\int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right.\right. \\ &\quad + \int_{t_1}^t \left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} \\ &\quad \left. - \int_a^t\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] + \frac{\ln(t/t_1)}{\Gamma(q-1)} \\ &\quad \left.\cdot \int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right\} \\ &\quad \text{for } t\in(t_1,t_2]. \end{aligned} \tag{33}$$

Using (33), we obtain

$$\begin{aligned} x(t_2^+) &= x(t_2^-) + I_2(x(t_2^-)) = x_a + \hat{x}_a \ln\frac{t_2}{a} \\ &\quad + I_1(x(t_1^-)) + I_2(x(t_2^-)) + \frac{1}{\Gamma(q)}\int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-1} \\ &\quad \cdot f(s,x(s))\frac{ds}{s} + \gamma(I_1(x(t_1^-))) \\ &\quad \cdot \left\{\frac{1}{\Gamma(q)}\left[\int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right.\right. \\ &\quad + \int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} \\ &\quad \left. - \int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] + \frac{\ln(t_2/t_1)}{\Gamma(q-1)} \\ &\quad \left.\cdot \int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right\}, \\ \delta x(t_2^+) &= \delta x(t_2^-) = \hat{x}_a + \frac{1}{\Gamma(q-1)}\int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-2} \\ &\quad \cdot f(s,x(s))\frac{ds}{s} \\ &\quad + \frac{\gamma(I_1(x(t_1^-)))}{\Gamma(q-1)}\left[\int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right. \\ &\quad + \int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s} \\ &\quad \left. - \int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right]. \end{aligned} \tag{34}$$

Therefore, the approximate solution $\tilde{x}(t)$ is provided by

$$\begin{aligned} \tilde{x}(t) &= x(t_2^+) + \delta x(t_2^+)\ln\frac{t}{t_2} + \frac{1}{\Gamma(q)}\int_{t_2}^t\left(\ln\frac{t}{s}\right)^{q-1} \\ &\quad \cdot f(s,x(s))\frac{ds}{s} = x_a + \hat{x}_a\ln\frac{t}{a} \\ &\quad + \sum_{i=1,2} I_i(x(t_i^-)) \\ &\quad + \frac{1}{\Gamma(q)}\left[\int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right. \\ &\quad \left. + \int_{t_2}^t\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] + \frac{\ln(t/t_2)}{\Gamma(q-1)} \\ &\quad \cdot \int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s} + \gamma(I_1(x(t_1^-))) \\ &\quad \cdot \left\{\frac{1}{\Gamma(q)}\left[\int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right.\right. \\ &\quad + \int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} \\ &\quad \left. - \int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] + \frac{\ln(t_2/t_1)}{\Gamma(q-1)} \\ &\quad \left.\cdot \int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right\} \\ &\quad + \frac{\gamma(I_1(x(t_1^-)))}{\Gamma(q-1)}\ln\frac{t}{t_2} \\ &\quad \cdot \left[\int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right. \\ &\quad + \int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s} \\ &\quad \left. - \int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right] \quad \text{for } t\in(t_2,t_3]. \end{aligned} \tag{35}$$

Let $e_2(t) = x(t) - \tilde{x}(t)$ for $t \in (t_2, t_3]$. Moreover, by (33), the exact solution $x(t)$ of system (16) satisfies

$$\begin{aligned} \lim_{I_1(x(t_1^-))\to 0} x(t) &= x_a + \hat{x}_a\ln\frac{t}{a} + I_2(x(t_2^-)) + \frac{1}{\Gamma(q)} \\ &\quad \cdot \int_a^t\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \gamma(I_2(x(t_2^-))) \\ &\quad \cdot \left\{\frac{1}{\Gamma(q)}\left[\int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right.\right. \\ &\quad + \int_{t_2}^t\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} \end{aligned}$$

$$- \int_a^t \left(\ln \frac{t}{s}\right)^{q-1} f(s, x(s)) \frac{ds}{s} \Bigg] + \frac{\ln (t/t_2)}{\Gamma (q-1)} \cdot \int_a^{t_2} \left(\ln \frac{t_2}{s}\right)^{q-2} f(s, x(s)) \frac{ds}{s} \Bigg\} \quad \text{for } t \in (t_2, t_3],$$

$$\lim_{I_2(x(t_2^-))\to 0} x(t) = x_a + \widehat{x}_a \ln \frac{t}{a} + I_1(x(t_1^-)) + \frac{1}{\Gamma(q)} \cdot \int_a^t \left(\ln \frac{t}{s}\right)^{q-1} f(s, x(s)) \frac{ds}{s} + \gamma(I_1(x(t_1^-))) \cdot \Bigg\{ \frac{1}{\Gamma(q)} \Bigg[\int_a^{t_1} \left(\ln \frac{t_1}{s}\right)^{q-1} f(s, x(s)) \frac{ds}{s} + \int_{t_1}^t \left(\ln \frac{t}{s}\right)^{q-1} f(s, x(s)) \frac{ds}{s} - \int_a^t \left(\ln \frac{t}{s}\right)^{q-1} f(s, x(s)) \frac{ds}{s} \Bigg] + \frac{\ln (t/t_1)}{\Gamma (q-1)} \cdot \int_a^{t_1} \left(\ln \frac{t_1}{s}\right)^{q-2} f(s, x(s)) \frac{ds}{s} \Bigg\} \quad \text{for } t \in (t_2, t_3],$$

$$\lim_{\substack{I_1(x(t_1^-))\to 0,\\ I_2(x(t_2^-))\to 0}} x(t) = x_a + \widehat{x}_a \ln \frac{t}{a} + \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s}\right)^{q-1} \cdot f(s, x(s)) \frac{ds}{s} \quad \text{for } t \in (t_2, t_3]. \tag{36}$$

Then,

$$\lim_{I_1(x(t_1^-))\to 0} e_2(t) = \lim_{I_1(x(t_1^-))\to 0} \{x(t) - \widetilde{x}(t)\} = [-1 + \gamma(I_2(x(t_2^-)))] \cdot \Bigg\{ \frac{1}{\Gamma(q)} \Bigg[\int_a^{t_2} \left(\ln \frac{t_2}{s}\right)^{q-1} f(s, x(s)) \frac{ds}{s} + \int_{t_2}^t \left(\ln \frac{t}{s}\right)^{q-1} f(s, x(s)) \frac{ds}{s} - \int_a^t \left(\ln \frac{t}{s}\right)^{q-1} f(s, x(s)) \frac{ds}{s} \Bigg] + \frac{\ln (t/t_2)}{\Gamma (q-1)} \cdot \int_a^{t_2} \left(\ln \frac{t_2}{s}\right)^{q-2} f(s, x(s)) \frac{ds}{s} \Bigg\} \quad \text{for } t \in (t_2, t_3],$$

$$\lim_{I_2(x(t_2^-))\to 0} e_2(t) = \lim_{I_2(x(t_2^-))\to 0} \{x(t) - \widetilde{x}(t)\} = [-1 + \gamma(I_1(x(t_1^-)))] \cdot \Bigg\{ \frac{1}{\Gamma(q)} \Bigg[\int_a^{t_2} \left(\ln \frac{t_2}{s}\right)^{q-1} f(s, x(s)) \frac{ds}{s} + \int_{t_2}^t \left(\ln \frac{t}{s}\right)^{q-1} f(s, x(s)) \frac{ds}{s} - \int_a^t \left(\ln \frac{t}{s}\right)^{q-1} f(s, x(s)) \frac{ds}{s} \Bigg] + \frac{\ln (t/t_2)}{\Gamma (q-1)} \cdot \int_a^{t_2} \left(\ln \frac{t_2}{s}\right)^{q-2} f(s, x(s)) \frac{ds}{s} \Bigg\} - \gamma(I_1(x(t_1^-))) \cdot \Bigg\{ \frac{1}{\Gamma(q)} \Bigg[\int_{t_1}^{t_2} \left(\ln \frac{t_2}{s}\right)^{q-1} f(s, x(s)) \frac{ds}{s} + \int_{t_2}^t \left(\ln \frac{t}{s}\right)^{q-1} f(s, x(s)) \frac{ds}{s} - \int_{t_1}^t \left(\ln \frac{t}{s}\right)^{q-1} f(s, x(s)) \frac{ds}{s} \Bigg] + \frac{\ln (t/t_2)}{\Gamma (q-1)} \cdot \int_{t_1}^{t_2} \left(\ln \frac{t_2}{s}\right)^{q-2} f(s, x(s)) \frac{ds}{s} \Bigg\} \quad \text{for } t \in (t_2, t_3],$$

$$\lim_{\substack{I_1(x(t_1^-))\to 0,\\ I_2(x(t_2^-))\to 0}} e_2(t) = \lim_{\substack{I_1(x(t_1^-))\to 0,\\ I_2(x(t_2^-))\to 0}} \{x(t) - \widetilde{x}(t)\} = \frac{-1}{\Gamma(q)} \Bigg[\int_a^{t_2} \left(\ln \frac{t_2}{s}\right)^{q-1} f(s, x(s)) \frac{ds}{s} + \int_{t_2}^t \left(\ln \frac{t}{s}\right)^{q-1} f(s, x(s)) \frac{ds}{s} - \int_a^t \left(\ln \frac{t}{s}\right)^{q-1} f(s, x(s)) \frac{ds}{s} \Bigg] - \frac{\ln (t/t_2)}{\Gamma (q-1)} \cdot \int_a^{t_2} \left(\ln \frac{t_2}{s}\right)^{q-2} f(s, x(s)) \frac{ds}{s} \quad \text{for } t \in (t_2, t_3]. \tag{37}$$

By (37), we get

$$e_2(t) = \left[\gamma\left(I_1\left(x\left(t_1^-\right)\right)\right) + \gamma\left(I_2\left(x\left(t_2^-\right)\right)\right) - 1\right] \cdot \left\{\frac{1}{\Gamma(q)}\left[\int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \int_{t_2}^{t}\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} - \int_a^{t}\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] + \frac{\ln(t/t_2)}{\Gamma(q-1)} \cdot \int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right\} - \gamma\left(I_1\left(x\left(t_1^-\right)\right)\right) \cdot \left\{\frac{1}{\Gamma(q)}\left[\int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \int_{t_2}^{t}\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} - \int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] + \frac{\ln(t/t_2)}{\Gamma(q-1)} \cdot \int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right\} \quad \text{for } t \in (t_2, t_3]. \tag{38}$$

Then,

$$x(t) = \tilde{x}(t) + e_2(t) = x_a + \hat{x}_a \ln\frac{t}{a} + \sum_{i=1,2} I_i\left(x\left(t_i^-\right)\right) + \frac{1}{\Gamma(q)}\int_a^{t}\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \gamma\left(I_1\left(x\left(t_1^-\right)\right)\right) \cdot \left\{\frac{1}{\Gamma(q)}\left[\int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} - \int_a^{t}\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] + \frac{\ln(t/t_1)}{\Gamma(q-1)} \cdot \int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right\} + \gamma\left(I_2\left(x\left(t_2^-\right)\right)\right) \cdot \left\{\frac{1}{\Gamma(q)}\left[\int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \int_{t_2}^{t}\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} - \int_a^{t}\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] + \frac{\ln(t/t_2)}{\Gamma(q-1)} \cdot \int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right\} \quad \text{for } t \in (t_2, t_3]. \tag{39}$$

Consider the following limit case

$$\lim_{t_2 \to t_1} \begin{cases} {}_{\text{C-H}}D_{a^+}^{q} x(t) = f(t, x(t)), & t \in (a, t_3],\ t \neq t_1,\ t \neq t_2, \\ \Delta x|_{t=t_k} = x\left(t_k^+\right) - x\left(t_k^-\right) = I_k\left(x\left(t_k^-\right)\right) \in \mathbb{C}, & k = 1, 2, \\ x(a) = x_a \in \mathbb{C}, \\ \delta x(a) = \hat{x}_a \in \mathbb{C}. \end{cases} \tag{40}$$

$$\longrightarrow \begin{cases} {}_{\text{C-H}}D_{a^+}^{q} x(t) = f(t, x(t)), & t \in (a, t_3],\ t \neq t_1 \\ \Delta x|_{t=t_1} = I_1\left(x\left(t_1^-\right)\right) + I_2\left(x\left(t_2^-\right)\right) \\ x(a) = x_a \in \mathbb{C}, \\ \delta x(a) = \hat{x}_a \in \mathbb{C}. \end{cases} \tag{41}$$

Using (33) and (39) for (41) and (40), respectively, we have

$$\chi\left(I_1\left(x\left(t_1^-\right)\right)+I_2\left(x\left(t_2^-\right)\right)\right) = \chi\left(I_1\left(x\left(t_1^-\right)\right)\right)+\chi\left(I_2\left(x\left(t_2^-\right)\right)\right) \quad \text{for } \forall I_1\left(x\left(t_1^-\right)\right), I_2\left(x\left(t_2^-\right)\right) \in \mathbb{C}. \tag{42}$$

Therefore, $\chi(z) = \xi z\ \forall z \in \mathbb{C}$; here ξ is a constant. Thus,

$$\begin{aligned} x(t) &= x_a + \widehat{x}_a \ln\frac{t}{a} + I_1\left(x\left(t_1^-\right)\right) + \frac{1}{\Gamma(q)} \cdot \int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \xi I_1\left(x\left(t_1^-\right)\right) \\ &\cdot \left\{\frac{1}{\Gamma(q)}\left[\int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \int_{t_1}^t \left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} - \int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] + \frac{\ln(t/t_1)}{\Gamma(q-1)} \cdot \int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right\} \\ &\quad \text{for } t \in (t_1, t_2], \\ x(t) &= x_a + \widehat{x}_a \ln\frac{t}{a} + I_1\left(x\left(t_1^-\right)\right) + I_2\left(x\left(t_2^-\right)\right) + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \xi I_1\left(x\left(t_1^-\right)\right) \\ &\cdot \left\{\frac{1}{\Gamma(q)}\left[\int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \int_{t_1}^t \left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} - \int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] + \frac{\ln(t/t_1)}{\Gamma(q-1)} \cdot \int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right\} + \xi I_2\left(x\left(t_2^-\right)\right) \\ &\cdot \left\{\frac{1}{\Gamma(q)}\left[\int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \int_{t_2}^t \left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} - \int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] + \frac{\ln(t/t_2)}{\Gamma(q-1)} \cdot \int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right\} \\ &\quad \text{for } t \in (t_2, t_3]. \end{aligned} \tag{43}$$

Next, suppose

$$\begin{aligned} x(t) &= x_a + \widehat{x}_a \ln\frac{t}{a} + \sum_{i=1}^k I_i\left(x\left(t_i^-\right)\right) + \frac{1}{\Gamma(q)} \cdot \int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \xi\sum_{i=1}^k I_i\left(x\left(t_i^-\right)\right) \\ &\cdot \left\{\frac{1}{\Gamma(q)}\left[\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \int_{t_i}^t \left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} - \int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] + \frac{\ln(t/t_i)}{\Gamma(q-1)} \cdot \int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right\} \\ &\quad \text{for } t \in (t_k, t_{k+1}]. \end{aligned} \tag{44}$$

Using (44), we obtain

$$\begin{aligned} x\left(t_{k+1}^+\right) &= x\left(t_{k+1}^-\right) + I_{k+1}\left(x\left(t_{k+1}^-\right)\right) = x_a + \widehat{x}_a \ln\frac{t_{k+1}}{a} + \sum_{i=1}^{k+1} I_i\left(x\left(t_i^-\right)\right) + \frac{1}{\Gamma(q)}\int_a^{t_{k+1}}\left(\ln\frac{t_{k+1}}{s}\right)^{q-1} \\ &\cdot f(s,x(s))\frac{ds}{s} + \xi\sum_{i=1}^k I_i\left(x\left(t_i^-\right)\right) \\ &\cdot \left\{\frac{1}{\Gamma(q)}\left[\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \int_{t_i}^{t_{k+1}} \left(\ln\frac{t_{k+1}}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} - \int_a^{t_{k+1}} \left(\ln\frac{t_{k+1}}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] + \frac{\ln(t_{k+1}/t_i)}{\Gamma(q-1)} \cdot \int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right\}, \end{aligned}$$

$$\delta x\left(t_{k+1}^{+}\right)=\delta x\left(t_{k+1}^{-}\right)=\widehat{x}_a+\frac{1}{\Gamma(q-1)}\cdot\int_a^{t_{k+1}}\left(\ln\frac{t_{k+1}}{s}\right)^{q-2}f(s,x(s))\frac{ds}{s}+\xi\sum_{i=1}^{k}\frac{I_i\left(x\left(t_i^{-}\right)\right)}{\Gamma(q-1)}\left[\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-2}f(s,x(s))\frac{ds}{s}+\int_{t_i}^{t_{k+1}}\left(\ln\frac{t_{k+1}}{s}\right)^{q-2}f(s,x(s))\frac{ds}{s}-\int_a^{t_{k+1}}\left(\ln\frac{t_{k+1}}{s}\right)^{q-2}f(s,x(s))\frac{ds}{s}\right]. \quad (45)$$

Therefore, the approximate solution $\widetilde{x}(t)$ is presented by

$$\widetilde{x}(t)=x\left(t_{k+1}^{+}\right)+\delta x\left(t_{k+1}^{+}\right)\ln\frac{t}{t_{k+1}}+\frac{1}{\Gamma(q)}\cdot\int_{t_{k+1}}^{t}\left(\ln\frac{t}{s}\right)^{q-1}f(s,x(s))\frac{ds}{s}=x_a+\widehat{x}_a\ln\frac{t}{a}+\sum_{i=1}^{k+1}I_i\left(x\left(t_i^{-}\right)\right)+\frac{1}{\Gamma(q)}\left[\int_a^{t_{k+1}}\left(\ln\frac{t_{k+1}}{s}\right)^{q-1}f(s,x(s))\frac{ds}{s}+\int_{t_{k+1}}^{t}\left(\ln\frac{t}{s}\right)^{q-1}f(s,x(s))\frac{ds}{s}\right]+\frac{\ln\left(t/t_{k+1}\right)}{\Gamma(q-1)}\cdot\int_a^{t_{k+1}}\left(\ln\frac{t_{k+1}}{s}\right)^{q-2}f(s,x(s))\frac{ds}{s}+\xi\sum_{i=1}^{k}I_i\left(x\left(t_i^{-}\right)\right)\cdot\left\{\frac{1}{\Gamma(q)}\left[\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-1}f(s,x(s))\frac{ds}{s}+\int_{t_i}^{t_{k+1}}\left(\ln\frac{t_{k+1}}{s}\right)^{q-1}f(s,x(s))\frac{ds}{s}-\int_a^{t_{k+1}}\left(\ln\frac{t_{k+1}}{s}\right)^{q-1}f(s,x(s))\frac{ds}{s}\right]+\frac{\ln\left(t_{k+1}/t_i\right)}{\Gamma(q-1)}\cdot\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-2}f(s,x(s))\frac{ds}{s}\right\}+\xi\sum_{i=1}^{k}\frac{I_i\left(x\left(t_i^{-}\right)\right)\ln\left(t/t_{k+1}\right)}{\Gamma(q-1)}\cdot\left[\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-2}f(s,x(s))\frac{ds}{s}+\int_{t_i}^{t_{k+1}}\left(\ln\frac{t_{k+1}}{s}\right)^{q-2}f(s,x(s))\frac{ds}{s}-\int_a^{t_{k+1}}\left(\ln\frac{t_{k+1}}{s}\right)^{q-2}f(s,x(s))\frac{ds}{s}\right]\quad\text{for } t\in\left(t_{k+1},t_{k+2}\right]. \quad (46)$$

Let $e_{k+1}(t)=x(t)-\widetilde{x}(t)$ for $t\in(t_{k+1},t_{k+2}]$. In addition, by (44), the exact solution $x(t)$ of system (16) satisfies

$$\lim_{\substack{I_i(x(t_i^{-}))\to 0,\\ \forall i\in\{1,2,\ldots,k+1\}}}x(t)=x_a+\widehat{x}_a\ln\frac{t}{a}+\frac{1}{\Gamma(q)}\int_a^{t}\left(\ln\frac{t}{s}\right)^{q-1}\cdot f(s,x(s))\frac{ds}{s}\quad\text{for } t\in\left(t_k,t_{k+1}\right],$$

$$\lim_{I_j(x(t_j^{-}))\to 0}x(t)=x_a+\widehat{x}_a\ln\frac{t}{a}+\sum_{\substack{1\le i\le k+1,\\ i\ne j}}I_i\left(x\left(t_i^{-}\right)\right)+\frac{1}{\Gamma(q)}\int_a^{t}\left(\ln\frac{t}{s}\right)^{q-1}f(s,x(s))\frac{ds}{s}+\xi\sum_{\substack{1\le i\le k+1,\\ i\ne j}}I_i\left(x\left(t_i^{-}\right)\right)\cdot\left\{\frac{1}{\Gamma(q)}\left[\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-1}f(s,x(s))\frac{ds}{s}+\int_{t_i}^{t}\left(\ln\frac{t}{s}\right)^{q-1}f(s,x(s))\frac{ds}{s}-\int_a^{t}\left(\ln\frac{t}{s}\right)^{q-1}f(s,x(s))\frac{ds}{s}\right]+\frac{\ln\left(t/t_i\right)}{\Gamma(q-1)}\cdot\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-2}f(s,x(s))\frac{ds}{s}\right\}\quad\text{for } t\in\left(t_k,t_{k+1}\right],\ j\in\{1,2,\ldots,k+1\}. \quad (47)$$

Then,

$$\lim_{\substack{I_i(x(t_i^-))\to 0,\\ \forall i\in\{1,2,\ldots,k+1\}}} e_{k+1}(t) = \lim_{\substack{I_i(x(t_i^-))\to 0,\\ \forall i\in\{1,2,\ldots,k+1\}}} x(t)-\widetilde{x}(t) = \frac{-1}{\Gamma(q)}\left[\int_a^{t_{k+1}}\left(\ln\frac{t_{k+1}}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \int_{t_{k+1}}^{t}\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} - \int_a^t\left(\ln\frac{t}{s}\right)^{q-1} \cdot f(s,x(s))\frac{ds}{s}\right] - \frac{\ln(t/t_{k+1})}{\Gamma(q-1)} \cdot \int_a^{t_{k+1}}\left(\ln\frac{t_{k+1}}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s} \quad \text{for } t\in(t_k,t_{k+1}],$$

$$\lim_{I_j(x(t_j^-))\to 0} e_{k+1}(t) = \lim_{I_j(x(t_j^-))\to 0} x(t)-\widetilde{x}(t) = \left\{-1 + \xi\sum_{\substack{1\le i\le k+1,\\ i\ne j}} I_i(x(t_i^-))\right\} \cdot\left\{\frac{1}{\Gamma(q)}\left[\int_a^{t_{k+1}}\left(\ln\frac{t_{k+1}}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \int_{t_{k+1}}^{t}\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} - \int_a^t\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] + \frac{\ln(t/t_{k+1})}{\Gamma(q-1)} \cdot\int_a^{t_{k+1}}\left(\ln\frac{t_{k+1}}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right\} - \xi\sum_{\substack{1\le i\le k+1,\\ i\ne j}} I_i(x(t_i^-)) \cdot\left\{\frac{1}{\Gamma(q)}\left[\int_{t_i}^{t_{k+1}}\left(\ln\frac{t_{k+1}}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \int_{t_{k+1}}^{t}\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} - \int_{t_i}^t\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] - \frac{\ln(t/t_{k+1})}{\Gamma(q-1)} \cdot\int_{t_i}^{t_{k+1}}\left(\ln\frac{t_{k+1}}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right\} \quad \text{for } t\in(t_k,t_{k+1}],\ j\in\{1,2,\ldots,k+1\}. \tag{48}$$

By (48), we obtain

$$e_{k+1}(t) = \left\{-1 + \xi\sum_{i=1}^{k+1} I_i(x(t_i^-))\right\} \cdot\left\{\frac{1}{\Gamma(q)}\left[\int_a^{t_{k+1}}\left(\ln\frac{t_{k+1}}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \int_{t_{k+1}}^{t}\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} - \int_a^t\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] + \frac{\ln(t/t_{k+1})}{\Gamma(q-1)} \cdot\int_a^{t_{k+1}}\left(\ln\frac{t_{k+1}}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right\} - \xi\sum_{i=1}^{k+1} I_i(x(t_i^-)) \cdot\left\{\frac{1}{\Gamma(q)}\left[\int_{t_i}^{t_{k+1}}\left(\ln\frac{t_{k+1}}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \int_{t_{k+1}}^{t}\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} - \int_{t_i}^t\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] - \frac{\ln(t/t_{k+1})}{\Gamma(q-1)} \cdot\int_{t_i}^{t_{k+1}}\left(\ln\frac{t_{k+1}}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right\} \quad \text{for } t\in(t_k,t_{k+1}],\ j\in\{1,2,\ldots,k+1\}. \tag{49}$$

Thus, we have

$$x(t) = \widetilde{x}(t) + e_{k+1}(t) = x_a + \widehat{x}_a \ln\frac{t}{a} + \sum_{i=1}^{k+1} I_i(x(t_i^-)) + \frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \xi\sum_{i=1}^{k+1} I_i(x(t_i^-)) \cdot\left\{\frac{1}{\Gamma(q)}\left[\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \int_{t_i}^{t}\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} - \int_a^t\left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] + \frac{\ln(t/t_i)}{\Gamma(q-1)} \cdot\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right\} \quad \text{for } t\in(t_{k+1},t_{k+2}]. \tag{50}$$

Then, the solution of system (16) satisfies (21).

By the proof of Sufficiency and Necessity, system (16) is equivalent to (21). The proof is completed. □

Lemma 10. *Let $q \in \mathbb{C}$, $\Re(q) \in (1,2)$, and ζ is a constant. System (17) is equivalent to the integral equation*

$$x(t) = \begin{cases} x_a + \hat{x}_a \ln\frac{t}{a} + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}, & \text{for } t\in(a,\bar{t}_1], \\ x_a + \hat{x}_a \ln\frac{t}{a} + \sum_{j=1}^{l} J_j\left(x\left(\bar{t}_j^-\right)\right)\ln\frac{t}{\bar{t}_j} + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} & \\ \quad + \zeta\sum_{j=1}^{l} J_j\left(x\left(\bar{t}_j^-\right)\right)\left\{\frac{1}{\Gamma(q)}\left[\int_a^{\bar{t}_j}\left(\ln\frac{\bar{t}_j}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \int_{\bar{t}_j}^t \left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right.\right. & \\ \quad \left.\left. - \int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] + \frac{\ln\left(t/\bar{t}_j\right)}{\Gamma(q-1)}\int_a^{\bar{t}_j}\left(\ln\frac{\bar{t}_j}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right\}, & \text{for } t\in(\bar{t}_l,\bar{t}_{l+1}],\ 1\le l\le p \end{cases} \tag{51}$$

provided that the integral in (51) exists.

Remark 11. For (17), we have

$$\lim_{J_1(x(\bar{t}_1^-))\to 0,\ldots,J_p(x(\bar{t}_p^-))\to 0} \{\text{system (17)}\} \longrightarrow \{\text{system (15)}\}. \tag{52}$$

Then,

$$\lim_{J_1(x(\bar{t}_1^-))\to 0,\ldots,J_p(x(\bar{t}_p^-))\to 0} \{\text{the solution of system (17)}\} = \{\text{the solution of system (15)}\}. \tag{53}$$

In fact, we can verify that (51) satisfies the condition (53). Moreover, the approximate solution $\tilde{x}(t)$ of system (17) is defined by

$$\tilde{x}(t) = x\left(\bar{t}_l^+\right) + \delta x\left(\bar{t}_l^+\right)\ln\frac{t}{\bar{t}_l} + \frac{1}{\Gamma(q)}\int_{\bar{t}_l}^t \left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} \quad \text{for } t\in(\bar{t}_l,\bar{t}_{l+1}]; \tag{54}$$

here $x(\bar{t}_l^+) = x(\bar{t}_l^-)$ and $\delta x(\bar{t}_l^+) = \delta x(\bar{t}_l^-) + J_l(x(\bar{t}_l^-))$, $l = 1, 2, \ldots, p$.

Due to similarity with Lemma 9, the proof is omitted.

Corollary 12. *Let $q \in \mathbb{C}$, $\Re(q) \in (1,2)$, and ξ is a constant. A function $x(t) : [a,T] \to \mathbb{C}$ is general solution of the system (16); then,*

$$\delta x(t) = \begin{cases} \hat{x}_a + \frac{1}{\Gamma(q-1)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s} & \text{for } t\in(a,t_1], \\ \hat{x}_a + \frac{1}{\Gamma(q-1)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s} & \\ \quad + \xi\sum_{i=1}^{k}\frac{I_i\left(x\left(t_i^-\right)\right)}{\Gamma(q-1)}\left[\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s} + \int_{t_i}^t \left(\ln\frac{t}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right. & \\ \quad \left. - \int_a^t \left(\ln\frac{t}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right] & \text{for } t\in(t_k,t_{k+1}],\ 1\le k\le m \end{cases} \tag{55}$$

provided that the integral in (55) exists.

Corollary 13. *Let $q \in \mathbb{C}$, $\Re(q) \in (1,2)$, and ζ is a constant. A function $x(t) : [a,T] \to \mathbb{C}$ is general solution of the system (17); then,*

$$\delta x(t) = \begin{cases} \hat{x}_a + \dfrac{1}{\Gamma(q-1)} \displaystyle\int_a^t \left(\ln\frac{t}{s}\right)^{q-2} f(s,x(s)) \frac{ds}{s} & \text{for } t \in (a,\bar{t}_1], \\ \hat{x}_a + \displaystyle\sum_{j=1}^{l} J_j\left(x\left(\bar{t}_j^-\right)\right) + \dfrac{1}{\Gamma(q-1)} \displaystyle\int_a^t \left(\ln\frac{t}{s}\right)^{q-2} f(s,x(s)) \frac{ds}{s} & \\ +\zeta \displaystyle\sum_{j=1}^{l} \dfrac{J_j\left(x\left(\bar{t}_j^-\right)\right)}{\Gamma(q-1)} \left[\displaystyle\int_a^{\bar{t}_j} \left(\ln\frac{\bar{t}_j}{s}\right)^{q-2} f(s,x(s)) \frac{ds}{s} + \displaystyle\int_{\bar{t}_j}^t \left(\ln\frac{t}{s}\right)^{q-2} f(s,x(s)) \frac{ds}{s} \right. & \\ \left. - \displaystyle\int_a^t \left(\ln\frac{t}{s}\right)^{q-2} f(s,x(s)) \frac{ds}{s} \right] & \text{for } t \in (\bar{t}_l,\bar{t}_{l+1}],\ 1 \le l \le p \end{cases} \tag{56}$$

provided that the integral in (56) exists.

Remark 14. By Corollaries 12 and 13, it is shown that two kinds of impulses $\Delta x|_{t=t_k}$ $(k = 1,2,\dots,m)$ and $\Delta\delta x|_{t=\bar{t}_l}$ $(l = 1,2,\dots,p)$ have similar effect on $\delta x(t)$ of system (15).

Lemma 15. *Let $q \in \mathbb{C}$, $\Re(q) \in (1,2)$, and ξ and ζ are two constants. A function $x(t) : [a,T] \to \mathbb{C}$ is general solution of the system (3a), (3b), (3c), and (3d); then,*

$$\delta x(t) = \begin{cases} \hat{x}_a + \dfrac{1}{\Gamma(q-1)} \displaystyle\int_a^t \left(\ln\frac{t}{s}\right)^{q-2} f(s,x(s)) \frac{ds}{s} & \text{for } t \in \left(a,t'_1\right], \\ \hat{x}_a + \displaystyle\sum_{j=1}^{k_2} J_j\left(x\left(\bar{t}_j^-\right)\right) + \dfrac{1}{\Gamma(q-1)} \displaystyle\int_a^t \left(\ln\frac{t}{s}\right)^{q-2} f(s,x(s)) \frac{ds}{s} & \\ +\xi \displaystyle\sum_{i=1}^{k_1} \dfrac{I_i\left(x\left(t_i^-\right)\right)}{\Gamma(q-1)} \left[\displaystyle\int_a^{t_i} \left(\ln\frac{t_i}{s}\right)^{q-2} f(s,x(s)) \frac{ds}{s} + \displaystyle\int_{t_i}^t \left(\ln\frac{t}{s}\right)^{q-2} f(s,x(s)) \frac{ds}{s} \right. & \\ \left. - \displaystyle\int_a^t \left(\ln\frac{t}{s}\right)^{q-2} f(s,x(s)) \frac{ds}{s} \right] & \\ +\zeta \displaystyle\sum_{j=1}^{k_2} \dfrac{J_j\left(x\left(\bar{t}_j^-\right)\right)}{\Gamma(q-1)} \left[\displaystyle\int_a^{\bar{t}_j} \left(\ln\frac{\bar{t}_j}{s}\right)^{q-2} f(s,x(s)) \frac{ds}{s} + \displaystyle\int_{\bar{t}_j}^t \left(\ln\frac{t}{s}\right)^{q-2} f(s,x(s)) \frac{ds}{s} \right. & \\ \left. - \displaystyle\int_a^t \left(\ln\frac{t}{s}\right)^{q-2} f(s,x(s)) \frac{ds}{s} \right] & \text{for } t \in \left(t'_k,t'_{k+1}\right],\ 1 \le k \le \Pi \end{cases} \tag{57}$$

provided that the integral in (57) exists.

Proof. According to Corollaries 12 and 13, the solutions of system (3a), (3b), (3c), and (3d) satisfy

$$\delta x(t) = \hat{x}_a + \frac{1}{\Gamma(q-1)} \int_a^t \left(\ln\frac{t}{s}\right)^{q-2} f(s,x(s)) \frac{ds}{s}, \quad \text{for } t \in \left(a,t'_1\right]. \tag{58}$$

By the definition of Caputo-Hadamard fractional derivative, system (3a), (3b), (3c), and (3d) satisfies

$$\{\text{system (3a), (3b), (3c), and (3d)}\} \Longleftrightarrow \begin{cases} {}_{\text{C-H}}D_{a^+}^{q-1}(\delta x(t)) = f(t, x(t)), & t \in J = (a, T],\ t \neq t_k\ (k = 1, \ldots, m),\ t \neq \bar{t}_l\ (l = 1, \ldots, p), \\ \Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)) \in \mathbb{C}, & k = 1, 2, \ldots, m, \\ \Delta \delta x|_{t=\bar{t}_l} = \delta x(\bar{t}_l^+) - \delta x(\bar{t}_l^-) = J_l(x(\bar{t}_l^-)) \in \mathbb{C}, & l = 1, 2, \ldots, p, \\ x(a) = x_a \in \mathbb{C}, \\ \delta x(a) = \widehat{x}_a \in \mathbb{C}. \end{cases} \tag{59}$$

Moreover, it is reasonable that impulses $\Delta x|_{t=t_k}$ ($k = 1, 2, \ldots, m$) are considered as special impulses $\Delta\delta x|_{t=\bar{t}_l}$ ($l = 1, 2, \ldots, p$) in system (59) by Remark 14. Therefore, using Lemma 7 for system (59) (as $t \in (t'_k, t'_{k+1}]$, here $k = 1, 2, \ldots, \Pi$), we have

$$\begin{aligned} \delta x(t) = {} & \widehat{x}_a + \sum_{j=1}^{k_2} J_j(x(\bar{t}_j^-)) + \frac{1}{\Gamma(q-1)} \int_a^t \left(\ln\frac{t}{s}\right)^{q-2} \cdot f(s, x(s)) \frac{ds}{s} \\ & + \sum_{i=1}^{k_1} \frac{\xi_i I_i(x(t_i^-))}{\Gamma(q-1)} \left[\int_a^{t_i} \left(\ln\frac{t_i}{s}\right)^{q-2} f(s, x(s)) \frac{ds}{s} + \int_{t_i}^t \left(\ln\frac{t}{s}\right)^{q-2} f(s, x(s)) \frac{ds}{s} - \int_a^t \left(\ln\frac{t}{s}\right)^{q-2} \cdot f(s, x(s)) \frac{ds}{s} \right] \\ & + \sum_{j=1}^{k_2} \frac{\zeta_j J_j(x(\bar{t}_j^-))}{\Gamma(q-1)} \left[\int_a^{\bar{t}_j} \left(\ln\frac{\bar{t}_j}{s}\right)^{q-2} f(s, x(s)) \frac{ds}{s} + \int_{\bar{t}_j}^t \left(\ln\frac{t}{s}\right)^{q-2} f(s, x(s)) \frac{ds}{s} - \int_a^t \left(\ln\frac{t}{s}\right)^{q-2} \cdot f(s, x(s)) \frac{ds}{s} \right], \end{aligned} \tag{60}$$

where ξ_i ($i = 1, 2, \ldots, k_1$) and ζ_j ($j = 1, 2, \ldots, k_2$) are undetermined constants. Letting $J_j(x(\bar{t}_j^-)) = 0$ (for all $j \in \{1, 2, \ldots, k_2\}$) and $I_i(x(t_i^-)) = 0$ (for all $i \in \{1, 2, \ldots, k_1\}$), respectively, we get $\xi_i = \xi$ (for all $i \in \{1, 2, \ldots, k_1\}$) and $\zeta_j = \zeta$ (for all $j \in \{1, 2, \ldots, k_2\}$) by Corollaries 12 and 13. Thus,

$$\begin{aligned} \delta x(t) = {} & \widehat{x}_a + \sum_{j=1}^{k_2} J_j(x(\bar{t}_j^-)) + \frac{1}{\Gamma(q-1)} \int_a^t \left(\ln\frac{t}{s}\right)^{q-2} \cdot f(s, x(s)) \frac{ds}{s} \\ & + \xi \sum_{i=1}^{k_1} \frac{I_i(x(t_i^-))}{\Gamma(q-1)} \left[\int_a^{t_i} \left(\ln\frac{t_i}{s}\right)^{q-2} f(s, x(s)) \frac{ds}{s} + \int_{t_i}^t \left(\ln\frac{t}{s}\right)^{q-2} f(s, x(s)) \frac{ds}{s} - \int_a^t \left(\ln\frac{t}{s}\right)^{q-2} f(s, x(s)) \frac{ds}{s} \right] \\ & + \zeta \sum_{j=1}^{k_2} \frac{J_j(x(\bar{t}_j^-))}{\Gamma(q-1)} \left[\int_a^{\bar{t}_j} \left(\ln\frac{\bar{t}_j}{s}\right)^{q-2} f(s, x(s)) \frac{ds}{s} + \int_{\bar{t}_j}^t \left(\ln\frac{t}{s}\right)^{q-2} f(s, x(s)) \frac{ds}{s} - \int_a^t \left(\ln\frac{t}{s}\right)^{q-2} f(s, x(s)) \frac{ds}{s} \right], \\ & \text{for } t \in (t'_k, t'_{k+1}],\ 1 \le k \le \Pi. \end{aligned} \tag{61}$$

This proof is completed. □

Theorem 16. *Let $q \in \mathbb{C}$, $\Re(q) \in (1,2)$, and ξ and ζ are two constants. System (3a), (3b), (3c), and (3d) is equivalent to the integral equation*

$$x(t) = \begin{cases} x_a + \widehat{x}_a \ln\frac{t}{a} + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} & \text{for } t \in \left(a, t_1'\right], \\ x_a + \widehat{x}_a \ln\frac{t}{a} + \sum_{i=1}^{k_1} I_i\left(x\left(t_i^-\right)\right) + \sum_{j=1}^{k_2} J_j\left(x\left(\bar{t}_j^-\right)\right)\ln\frac{t}{\bar{t}_j} + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} & \\ + \xi\sum_{i=1}^{k_1} I_i\left(x\left(t_i^-\right)\right)\left\{\frac{1}{\Gamma(q)}\left[\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \int_{t_i}^t \left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right.\right. & \\ \left.\left. - \int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] + \frac{\ln(t/t_i)}{\Gamma(q-1)}\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right\} & \\ + \zeta\sum_{j=1}^{k_2} J_j\left(x\left(\bar{t}_j^-\right)\right)\left\{\frac{1}{\Gamma(q)}\left[\int_a^{\bar{t}_j}\left(\ln\frac{\bar{t}_j}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \int_{\bar{t}_j}^t \left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right.\right. & \\ \left.\left. - \int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] + \frac{\ln\left(t/\bar{t}_j\right)}{\Gamma(q-1)}\int_a^{\bar{t}_j}\left(\ln\frac{\bar{t}_j}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right\} & \text{for } t \in \left(t_k', t_{k+1}'\right],\ 1 \le k \le \Pi \end{cases} \tag{62}$$

provided that the integral in (62) exists.

Proof.

Necessity. We can verify that (62) satisfies conditions (i)–(iii) by Lemmas 9, 10, and 6.

Next, taking the fractional derivative to (62) for $t \in (t_k', t_{k+1}']$ (here $k = 0, 1, 2, \ldots, \Pi$), we get

$$\begin{aligned} \left[{}_{\text{C-H}}D_{a^+}^q x(t)\right]_{t\in(t_k',t_{k+1}']} &= \left\{\frac{1}{\Gamma(2-q)}\int_a^t\left(\ln\frac{t}{\eta}\right)^{2-q-1}\delta^2\left(x_a + \widehat{x}_a\ln\frac{\eta}{a} + \sum_{i=1}^{k_1} I_i\left(x\left(t_i^-\right)\right) + \sum_{j=1}^{k_2} J_j\left(x\left(\bar{t}_j^-\right)\right)\ln\frac{\eta}{\bar{t}_j} + \frac{1}{\Gamma(q)}\int_a^\eta\left(\ln\frac{\eta}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \xi\sum_{i=1}^{k_1} I_i\left(x\left(t_i^-\right)\right)\right.\right. \\ &\cdot\left\{\frac{1}{\Gamma(q)}\left[\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \int_{t_i}^\eta\left(\ln\frac{\eta}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} - \int_a^\eta\left(\ln\frac{\eta}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] + \frac{\ln(\eta/t_i)}{\Gamma(q-1)}\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right\} + \zeta\sum_{j=1}^{k_2} J_j\left(x\left(\bar{t}_j^-\right)\right) \\ &\cdot\left.\left.\left\{\frac{1}{\Gamma(q)}\left[\int_a^{\bar{t}_j}\left(\ln\frac{\bar{t}_j}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \int_{\bar{t}_j}^\eta\left(\ln\frac{\eta}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} - \int_a^\eta\left(\ln\frac{\eta}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] + \frac{\ln\left(\eta/\bar{t}_j\right)}{\Gamma(q-1)}\int_a^{\bar{t}_j}\left(\ln\frac{\bar{t}_j}{s}\right)^{q-2} f(s,x(s))\frac{ds}{s}\right\}\right)\frac{d\eta}{\eta}\right\}_{t\in(t_k',t_{k+1}']} \\ &= \frac{1}{\Gamma(2-q)\Gamma(q)}\left\{\int_a^t\left(\ln\frac{t}{\eta}\right)^{2-q-1}\delta^2\left\{\int_a^\eta\left(\ln\frac{\eta}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} + \xi\sum_{i=1}^{k_1} I_i\left(x\left(t_i^-\right)\right)\left[\int_{t_i}^\eta\left(\ln\frac{\eta}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} - \int_a^\eta\left(\ln\frac{\eta}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right] + \zeta\sum_{j=1}^{k_2} J_j\left(x\left(\bar{t}_j^-\right)\right)\right.\right. \\ &\cdot\left.\left.\left[\int_{\bar{t}_j}^\eta\left(\ln\frac{\eta}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s} - \int_a^\eta\left(\ln\frac{\eta}{s}\right)^{q-1} f(s,x(s))\frac{ds}{s}\right]\right\}\frac{d\eta}{\eta}\right\}_{t\in(t_k',t_{k+1}']} = \left\{ f(t,x(t))\big|_{t\ge a} + \xi\sum_{i=1}^{k_1} I_i\left(x\left(t_i^-\right)\right)\left[f(t,x(t))\big|_{t\ge t_i} - f(t,x(t))\big|_{t\ge a}\right]\right. \\ &\left. + \zeta\sum_{j=1}^{k_2} J_j\left(x\left(\bar{t}_j^-\right)\right)\left[f(t,x(t))\big|_{t\ge\bar{t}_j} - f(t,x(t))\big|_{t\ge a}\right]\right\}_{t\in(t_k',t_{k+1}']} = f(t,x(t))\big|_{t\in(t_k',t_{k+1}']}. \end{aligned} \tag{63}$$

So, (62) satisfies (3a).

Finally, it is straightforward to verify that (62) satisfies (3b) and (3c). So, (62) satisfies all conditions of system (3a), (3b), (3c), and (3d).

Sufficiency. According to Lemmas 9 and 10, the solutions of system (3a), (3b), (3c), and (3d) satisfy

$$x(t) = x_a + \hat{x}_a \ln\frac{t}{a} + \frac{1}{\Gamma(q)}\int_a^t \left(\ln\frac{t}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s}, \quad \text{for } t \in \left(a, t_1'\right]. \tag{64}$$

Next, by Lemma 15, the solutions of system (3a), (3b), (3c), and (3d) satisfy

$$\begin{aligned}\delta x(t) &= \hat{x}_a + \sum_{j=1}^{k_2} J_j\left(x\left(\bar{t}_j^-\right)\right) + \frac{1}{\Gamma(q-1)} \cdot \int_a^t \left(\ln\frac{t}{s}\right)^{q-2} f(s, x(s))\frac{ds}{s} + \xi\sum_{i=1}^{k_1}\frac{I_i\left(x\left(t_i^-\right)\right)}{\Gamma(q-1)} \\ &\quad\cdot\left[\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-2} f(s, x(s))\frac{ds}{s} + \int_{t_i}^t \left(\ln\frac{t}{s}\right)^{q-2} f(s, x(s))\frac{ds}{s} - \int_a^t\left(\ln\frac{t}{s}\right)^{q-2} f(s, x(s))\frac{ds}{s}\right] + \zeta\sum_{j=1}^{k_2}\frac{J_j\left(x\left(\bar{t}_j^-\right)\right)}{\Gamma(q-1)} \\ &\quad\cdot\left[\int_a^{\bar{t}_j}\left(\ln\frac{\bar{t}_j}{s}\right)^{q-2} f(s, x(s))\frac{ds}{s} + \int_{\bar{t}_j}^t\left(\ln\frac{t}{s}\right)^{q-2} f(s, x(s))\frac{ds}{s} - \int_a^t\left(\ln\frac{t}{s}\right)^{q-2} f(s, x(s))\frac{ds}{s}\right], \\ &\quad \text{for } t \in \left(t_k', t_{k+1}'\right],\ 1 \le k \le \Pi.\end{aligned} \tag{65}$$

Using (65), we have

$$\begin{aligned}x(t) &= C + \int\left\{\frac{\hat{x}_a}{\eta} + \sum_{j=1}^{k_2}\frac{J_j\left(x\left(\bar{t}_j^-\right)\right)}{\eta} + \frac{1}{\Gamma(q-1)} \cdot \int_a^\eta\left(\ln\frac{\eta}{s}\right)^{q-2}\frac{1}{\eta} f(s, x(s))\frac{ds}{s}\right. \\ &\quad + \xi\sum_{i=1}^{k_1}\frac{I_i\left(x\left(t_i^-\right)\right)}{\Gamma(q-1)}\left[\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-2}\frac{1}{\eta} f(s, x(s))\frac{ds}{s} + \int_{t_i}^\eta\left(\ln\frac{t}{s}\right)^{q-2}\frac{1}{\eta} f(s, x(s))\frac{ds}{s} - \int_a^\eta\left(\ln\frac{\eta}{s}\right)^{q-2}\frac{1}{\eta} f(s, x(s))\frac{ds}{s}\right] \\ &\quad + \zeta\sum_{j=1}^{k_2}\frac{J_j\left(x\left(\bar{t}_j^-\right)\right)}{\Gamma(q-1)} \cdot\left[\int_a^{\bar{t}_j}\left(\ln\frac{\bar{t}_j}{s}\right)^{q-2}\frac{1}{\eta} f(s, x(s))\frac{ds}{s} + \int_{\bar{t}_j}^\eta\left(\ln\frac{\eta}{s}\right)^{q-2}\frac{1}{\eta} f(s, x(s))\frac{ds}{s}\right. \\ &\quad \left.\left. - \int_a^\eta\left(\ln\frac{\eta}{s}\right)^{q-2}\frac{1}{\eta} f(s, x(s))\frac{ds}{s}\right]\right\} d\eta = C \\ &\quad + \hat{x}_a \ln t + \sum_{j=1}^{k_2} J_j\left(x\left(\bar{t}_j^-\right)\right)\ln t + \frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1} \cdot f(s, x(s))\frac{ds}{s} + \sum_{i=1}^{k_1}\frac{\xi I_i\left(x\left(t_i^-\right)\right)\ln t}{\Gamma(q-1)} \\ &\quad \cdot\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-2} f(s, x(s))\frac{ds}{s} + \sum_{j=1}^{k_2}\frac{\zeta J_j\left(x\left(\bar{t}_j^-\right)\right)\ln t}{\Gamma(q-1)}\int_a^{\bar{t}_j}\left(\ln\frac{\bar{t}_j}{s}\right)^{q-2} f(s, x(s))\frac{ds}{s} \\ &\quad + \sum_{i=1}^{k_1}\frac{\xi I_i\left(x\left(t_i^-\right)\right)}{\Gamma(q)}\left[\int_{t_i}^t\left(\ln\frac{t}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s} - \int_a^t\left(\ln\frac{\eta}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s}\right] \\ &\quad + \sum_{j=1}^{k_2}\frac{\zeta J_j\left(x\left(\bar{t}_j^-\right)\right)}{\Gamma(q)}\left[\int_{\bar{t}_j}^t\left(\ln\frac{t}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s} - \int_a^t\left(\ln\frac{t}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s}\right].\end{aligned} \tag{66}$$

Letting $J_j(x(\bar{t}_j^-)) = 0$ (for all $j \in \{1, 2, \ldots, p\}$) and $I_i(x(t_i^-)) = 0$ (for all $i \in \{1, 2, \ldots, m\}$) in (66), respectively, by Lemmas 9 and 10, we obtain

$$C = x_a - \widehat{x}_a \ln a + \sum_{i=1}^{k_1} I_i\left(x\left(t_i^-\right)\right) - \sum_{j=1}^{k_2} J_j\left(x\left(\bar{t}_j^-\right)\right)\bar{t}_j + \sum_{i=1}^{k_1} \xi I_i\left(x\left(t_i^-\right)\right) \cdot \left\{\frac{1}{\Gamma(q)}\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s} - \frac{\ln t_i}{\Gamma(q-1)}\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-2} f(s, x(s))\frac{ds}{s}\right\} + \sum_{j=1}^{k_2} \zeta J_j\left(x\left(\bar{t}_j^-\right)\right) \cdot \left\{\frac{1}{\Gamma(q)}\int_a^{\bar{t}_j}\left(\ln\frac{\bar{t}_j}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s} - \frac{\ln \bar{t}_j}{\Gamma(q-1)}\int_a^{\bar{t}_j}\left(\ln\frac{\bar{t}_j}{s}\right)^{q-2} f(s, x(s))\frac{ds}{s}\right\}. \tag{67}$$

Thus,

$$x(t) = x_a + \widehat{x}_a \ln\frac{t}{a} + \sum_{i=1}^{k_1} I_i\left(x\left(t_i^-\right)\right) + \sum_{j=1}^{k_2} J_j\left(x\left(\bar{t}_j^-\right)\right) \cdot \ln\frac{t}{\bar{t}_j} + \frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s} + \xi\sum_{i=1}^{k_1} I_i\left(x\left(t_i^-\right)\right) \cdot \left\{\frac{1}{\Gamma(q)}\left[\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s} + \int_{t_i}^t\left(\ln\frac{t}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s} - \int_a^t\left(\ln\frac{t}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s}\right] + \frac{\ln(t/t_i)}{\Gamma(q-1)} \cdot \int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-2} f(s, x(s))\frac{ds}{s}\right\} + \zeta\sum_{j=1}^{k_2} J_j\left(x\left(\bar{t}_j^-\right)\right) \cdot \left\{\frac{1}{\Gamma(q)}\left[\int_a^{\bar{t}_j}\left(\ln\frac{\bar{t}_j}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s} + \int_{\bar{t}_j}^t\left(\ln\frac{t}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s} - \int_a^t\left(\ln\frac{t}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s}\right] + \frac{\ln\left(t/\bar{t}_j\right)}{\Gamma(q-1)} \cdot \int_a^{\bar{t}_j}\left(\ln\frac{\bar{t}_j}{s}\right)^{q-2} f(s, x(s))\frac{ds}{s}\right\} \quad \text{for } t \in \left(t'_k, t'_{k+1}\right],\ 1 \le k \le \Pi. \tag{68}$$

So, the solutions of system (3a), (3b), (3c), and (3d) satisfy (62). This proof is completed. □

Corollary 17. *Let $q \in \mathbb{C}$, $\mathfrak{R}(q) \in (1, 2)$, and ξ and ζ are two constants. System (1) is equivalent to the integral equation*

$$x(t) = \begin{cases} x_a + a\bar{x}_a \ln\frac{t}{a} + \frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s} & \textit{for } t \in \left(a, t'_1\right], \\ x_a + a\bar{x}_a \ln\frac{t}{a} + \sum_{i=1}^{k_1} I_i\left(x\left(t_i^-\right)\right) + \sum_{j=1}^{k_2} \bar{t}_j\bar{I}_j\left(x\left(\bar{t}_j^-\right)\right)\ln\frac{t}{\bar{t}_j} + \frac{1}{\Gamma(q)}\int_a^t\left(\ln\frac{t}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s} \\ \quad + \xi\sum_{i=1}^{k_1} I_i\left(y\left(t_i^-\right)\right)\left\{\frac{1}{\Gamma(q)}\left[\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s} + \int_{t_i}^t\left(\ln\frac{t}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s}\right.\right. \\ \quad \left.\left. - \int_a^t\left(\ln\frac{t}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s}\right] + \frac{\ln(t/t_i)}{\Gamma(q-1)}\int_a^{t_i}\left(\ln\frac{t_i}{s}\right)^{q-2} f(s, x(s))\frac{ds}{s}\right\} \\ \quad + \zeta\sum_{j=1}^{k_2} \bar{t}_j\bar{I}_j\left(x\left(\bar{t}_j^-\right)\right)\left\{\frac{1}{\Gamma(q)}\left[\int_a^{\bar{t}_j}\left(\ln\frac{\bar{t}_j}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s} + \int_{\bar{t}_j}^t\left(\ln\frac{t}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s}\right.\right. \\ \quad \left.\left. - \int_a^t\left(\ln\frac{t}{s}\right)^{q-1} f(s, x(s))\frac{ds}{s}\right] + \frac{\ln\left(t/\bar{t}_j\right)}{\Gamma(q-1)}\int_a^{\bar{t}_j}\left(\ln\frac{\bar{t}_j}{s}\right)^{q-2} f(s, x(s))\frac{ds}{s}\right\} & \textit{for } t \in \left(t'_k, t'_{k+1}\right],\ 1 \le k \le \Pi \end{cases} \tag{69}$$

provided that the integral in (69) exists.

Remark 18. Substituting $\widehat{x}_a = a\overline{x}_a$ and $J_j(x(\overline{t}_j^-)) = \overline{t}_j \overline{I}_j(x(\overline{t}_j^-))$ into (62), (69) can be obtained. Next, let us analyze the limited case of system (1):

$$\lim_{q\to 2^-}\{\text{system (1)}\} \longrightarrow \begin{cases} \delta^2(x(t)) = f(t,x(t)), & t\in J=(a,T],\ t\neq t_k\ (k=1,\dots,m),\ t\neq \overline{t}_l\ (l=1,\dots,p), \\ \Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)) \in \mathbb{C}, & k=1,2,\dots,m, \\ \Delta x'\big|_{t=\overline{t}_l} = x'(\overline{t}_l^+) - x'(\overline{t}_l^-) = \overline{I}_l(x(\overline{t}_l^-)) \in \mathbb{C}, & l=1,2,\dots,p, \\ x(a) = x_a \in \mathbb{C}, \\ x'(a) = \overline{x}_a \in \mathbb{C}. \end{cases} \quad (70)$$

On the other hand, by (69), we have

$$\lim_{q\to 2^-} x(t) = \begin{cases} x_a + a\overline{x}_a \ln\dfrac{t}{a} + \dfrac{1}{\Gamma(q)}\displaystyle\int_a^t \ln\frac{t}{s} f(s,x(s))\frac{ds}{s}, & \text{for } t\in(a,t_1'], \\ x_a + a\overline{x}_a \ln\dfrac{t}{a} + \displaystyle\sum_{i=1}^{k_1} I_i(x(t_i^-)) + \sum_{j=1}^{k_2}\overline{I}_j(x(\overline{t}_j^-))\overline{t}_j \ln\frac{t}{\overline{t}_j} + \int_a^t \ln\frac{t}{s} f(s,x(s))\frac{ds}{s} & \text{for } t\in(t_k',t_{k+1}'],\ 1\le k\le \Pi. \end{cases} \quad (71)$$

It can be verified that (71) is the solution of (70), which indirectly supports our conclusion.

Competing Interests

The authors declare that they have no competing interests.

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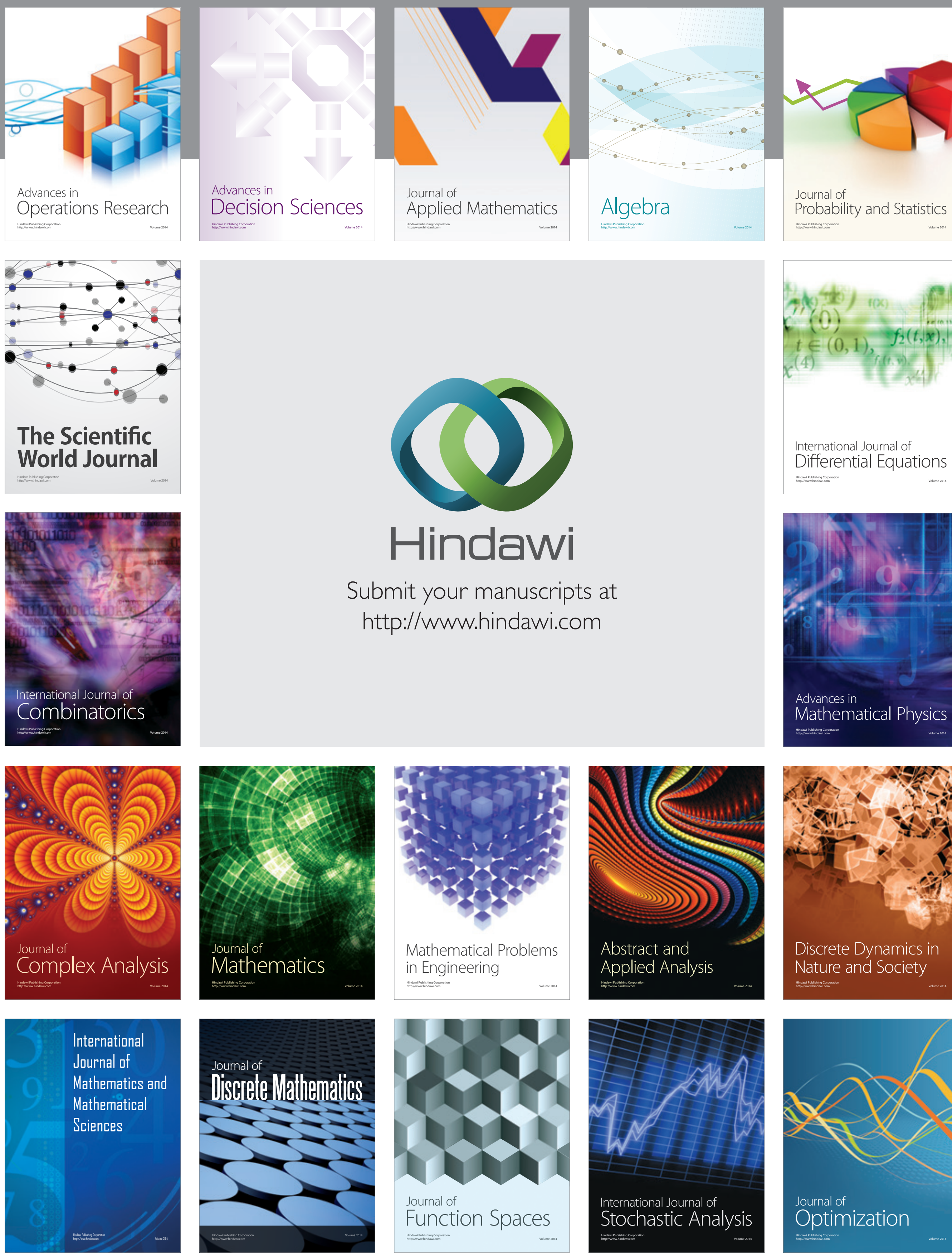

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