Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 879657, 14 pages
doi:10.1155/2012/879657

Research Article

Asymptotic Stability Results for Nonlinear Fractional Difference Equations

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Received 1 August 2011; Revised 27 December 2011; Accepted 2 January 2012

Academic Editor: Michela Redivo-Zaglia

We present some results for the asymptotic stability of solutions for nonlinear fractional difference equations involvingRiemann-Liouville-like difference operator. The results are obtained by using Krasnoselskii's fixed point theorem and discrete Arzela-Ascoli's theorem. Three examples are also provided to illustrate our main results.

1. Introduction

In this paper we consider the asymptotic stability of solutions for nonlinear fractional difference equations:

$$\Delta^{\alpha}x(t) = f(t+\alpha, x(t+\alpha)), \quad t \in N_0,\ 0 < \alpha \le 1,$$
$$\Delta^{\alpha-1}x(t)|_{t=0} = x_0, \tag{1.1}$$

where Δ^{α} is aRiemann-Liouville-like discrete fractional difference, $f : [0,+\infty) \times R \to R$ is continuous with respect to t and x, $N_a = \{a, a+1, a+2, \ldots\}$.

Fractional differential equations have received increasing attention during recent years since these equations have been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Most of the present works were focused on fractional differential equations, see [1–12] and the references therein. However, very little progress has been made to develop the theory of the analogous fractional finite difference equation [13–19].

Due to the lack of geometry interpretation of the fractional derivatives, it is difficult to find a valid tool to analyze the stability of fractional difference equations. In the case that it is difficult to employ Liapunov's direct method, fixed point theorems are usually considered in stability [20–25]. Motivated by this idea, in this paper, we discuss asymptotic stability of nonlinear fractional difference equations by using Krasnoselskii's fixed point theorem and discrete Arzela-Ascoli's theorem. Different from our previous work [18], in this paper, the sufficient conditions of attractivity are irrelevant to the initial value x_0.

2. Preliminaries

In this section, we introduce preliminary facts of discrete fractional calculus. For more details, see [14].

Definition 2.1 (see [14]). Let $\nu > 0$. The ν-th fractional sum x is defined by

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} f(s), \tag{2.1}$$

where f is defined for $s = a \bmod (1)$ and $\Delta^{-\nu} f$ is defined for $t = (a+\nu) \bmod (1)$, and $t^{(\nu)} = \Gamma(t+1)/\Gamma(t-\nu+1)$. The fractional sum $\Delta^{-\nu}$ maps functions defined on N_a to functions defined on $N_{a+\nu}$.

Definition 2.2 (see [14]). Let $\mu > 0$ and $m-1 < \mu < m$, where m denotes a positive integer, $m = \lceil \mu \rceil$, $\lceil \cdot \rceil$ ceiling of number. Set $\nu = m - \mu$. The μ-th fractional difference is defined as

$$\Delta^{\mu} f(t) = \Delta^{m-\nu} f(t) = \Delta^{m} \left(\Delta^{-\nu} f(t)\right). \tag{2.2}$$

Theorem 2.3 (see [15]). *Let f be a real-value function defined on N_a and $\mu, \nu > 0$, then the following equalities hold:*

(i) $\Delta^{-\nu}[\Delta^{-\mu} f(t)] = \Delta^{-(\mu+\nu)} f(t) = \Delta^{-\mu}[\Delta^{-\nu} f(t)]$;

(ii) $\Delta^{-\nu} \Delta f(t) = \Delta \Delta^{-\nu} f(t) - \dfrac{(t-a)^{(\nu-1)}}{\Gamma(\nu)} f(a)$.

Lemma 2.4 (see [15]). *Let $\mu \neq 1$ and assume $\mu + \nu + 1$ is not a nonpositive integer, then*

$$\Delta^{-\nu} t^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} t^{(\mu+\nu)}. \tag{2.3}$$

Lemma 2.5 (see [15]). *Assume that the following factorial functions are well defined:*

(i) *If $0 < \alpha < 1$, then $t^{(\alpha\gamma)} \geq (t^{(\gamma)})^{\alpha}$;*

(ii) $t^{(\beta+\gamma)} = (t-\gamma)^{(\beta)} t^{(\gamma)}$.

Lemma 2.6 (see [13]). *Let $\mu > 0$ be noninteger, $m = \lceil \mu \rceil$, $\lceil \cdot \rceil$, $\nu = m - \mu$, thus one has*

$$\sum_{s=a+\nu}^{t-\mu} (t-s-1)^{(\mu-1)} = \frac{(t-a-\nu)^{(\mu)}}{\mu}. \tag{2.4}$$

Lemma 2.7. *The equivalent fractional Taylor's difference formula of* (1.1) *is*

$$x(t) = \frac{x_0}{\Gamma(\alpha)} t^{(\alpha-1)} + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s+\alpha, x(s+\alpha)), \quad t \in N_\alpha. \tag{2.5}$$

Proof. Apply the $\Delta^{-\alpha}$ operator to each side of the first formula of (1.1) to obtain

$$\Delta^{-\alpha}\Delta^{\alpha} x(t) = \Delta^{-\alpha} f(t+\alpha, x(t+\alpha)), \quad t \in N_\alpha. \tag{2.6}$$

Apply Theorem 2.3 to the left-hand side of (2.6) to obtain

$$\begin{aligned} \Delta^{-\alpha}\Delta^{\alpha} x(t) &= \Delta^{-\alpha}\Delta\Delta^{-(1-\alpha)} x(t) = \Delta\Delta^{-\alpha}\Delta^{-(1-\alpha)} x(t) - \frac{t^{(\alpha-1)}}{\Gamma(\alpha)} x(\alpha-1) \\ &= x(t) - \frac{x_0}{\Gamma(\alpha)} t^{(\alpha-1)}. \end{aligned} \tag{2.7}$$

So, applying Definition 2.1 to the right-hand side of (2.6), for $t \in N_\alpha$ we obtain (2.5). The recursive iteration to this Taylor's difference formula implies that (2.5) represents the unique solution of the IVP (1.1). This completes the proof. □

Lemma 2.8 (see [4, (1.5.15)]). *The quotient expansion of two gamma functions at infinityis*

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left[1 + O\left(\frac{1}{z}\right)\right], \quad \left(\left|\arg(z+a)\right| < \pi, \ |z| \longrightarrow \infty\right). \tag{2.8}$$

Corollary 2.9. *One has*

$$t^{(-\beta)} > (t+\alpha)^{(-\beta)} \quad \text{for } \alpha, \beta, t > 0. \tag{2.9}$$

Proof. According to Lemma 2.8,

$$\begin{aligned} \frac{t^{(-\beta)}}{(t+\alpha)^{(-\beta)}} &= \frac{\Gamma(t+1)}{\Gamma(t+\beta+1)} \cdot \frac{\Gamma(t+\alpha+\beta+1)}{\Gamma(t+\alpha+1)} \\ &= \frac{\Gamma(t+1)}{\Gamma(t+\alpha+1)} \cdot \frac{\Gamma(t+\alpha+\beta+1)}{\Gamma(t+\beta+1)} \\ &= t^{-\alpha}\left[1 + O\left(\frac{1}{t}\right)\right] \cdot (t+\beta)^{\alpha}\left[1 + O\left(\frac{1}{t+\beta}\right)\right] \\ &= \left(1 + \frac{\beta}{t}\right)^{\alpha}\left[1 + O\left(\frac{1}{t}\right)\right]\left[1 + O\left(\frac{1}{t+\beta}\right)\right] \\ &> 1. \end{aligned} \tag{2.10}$$

Then, $t^{(-\beta)} > (t+\alpha)^{(-\beta)}$ for $\alpha, \beta, t > 0$. This completes the proof. □

Definition 2.10. The solution $x = \varphi(t)$ of the IVP (1.1) is said to be

(i) stable if for any $\varepsilon > 0$ and $t_0 \in R^+$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that

$$\left| x(t, x_0, t_0) - \varphi(t) \right| < \varepsilon \tag{2.11}$$

for $|x_0 - \varphi(t_0)| \leq \delta(t_0, \varepsilon)$ and all $t \geq t_0$;

(ii) attractive if there exists $\eta(t_0) > 0$ such that $\|x_0\| \leq \eta$ implies

$$\lim_{t \to \infty} x(t, x_0, t_0) = 0; \tag{2.12}$$

(iii) asymptotically stable if it is stable and attractive.

The space $l_{n_0}^{\infty}$ is the set of real sequences defined on the set of positive integers where any individual sequence is bounded with respect to the usual supremum norm. It is well known that under the supremum norm $l_{n_0}^{\infty}$ is a Banach space [26].

Definition 2.11 (see [27]). A set Ω of sequences in $l_{n_0}^{\infty}$ is uniformly Cauchy (or equi-Cauchy), if for every $\varepsilon > 0$, there exists an integer N such that $|x(i) - x(j)| < \varepsilon$, whenever $i, j > N$ for any $x = \{x(n)\}$ in Ω.

Theorem 2.12 (see [27, (discrete Arzela-Ascoli's theorem)]). *A bounded, uniformly Cauchy subset Ω of $l_{n_0}^{\infty}$ is relatively compact.*

Theorem 2.13 (see [20, (Krasnoselskii's fixed point theorem)]). *Let S be a nonempty, closed, convex, and bounded subset of the Banach space X and let $A : X \to X$ and $B : S \to X$ be two operators such that*

(a) *A is a contraction with constant $L < 1$,*

(b) *B is continuous, BS resides in a compact subset of X,*

(c) $[x = Ax + By,\ y \in S] \Rightarrow x \in S$.

Then the operator equation $Ax + Bx = x$ has a solution in S.

3. Main Results

Let l_{α}^{∞} be the set ofall real sequences $x = \{x(t)\}_{t=\alpha}^{\infty}$ with norm $\|x\| = \sup_{t \in N_{\alpha}} |x(t)|$, then l_{α}^{∞} is a Banach space.

Define the operator

$$\begin{aligned} Px(t) &= \frac{x_0}{\Gamma(\alpha)} t^{(\alpha-1)} + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s+\alpha, x(s+\alpha)), \\ Ax(t) &= \frac{x_0}{\Gamma(\alpha)} t^{(\alpha-1)}, \\ Bx(t) &= \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s+\alpha, x(s+\alpha)), \quad t \in N_{\alpha}. \end{aligned} \tag{3.1}$$

Obviously, $Px = Ax + Bx$, the operator A is a contraction with the constant 0, which implies that condition (a) of Theorem 2.13 holds, and $x(t)$ is a solution of (1.1) if it is a fixed point of the operator P.

Lemma 3.1. *Assume that the following condition is satisfied:*

(H_1) *there exist constants $\beta_1 \in (\alpha, 1)$ and $L_1 \geq 0$ such that*

$$\left| f(t, x(t)) \right| \leq L_1 t^{(-\beta_1)} \quad \text{for } t \in N_\alpha. \tag{3.2}$$

Then the operator B is continuous and BS_1 is a compact subset of R for $t \in N_{\alpha+n_1}$, where

$$S_1 = \left\{ x(t) : |x(t)| \leq t^{(-\gamma_1)} \text{ for } t \in N_{\alpha+n_1} \right\}, \tag{3.3}$$

$\gamma_1 = (-1/2)(\alpha - \beta_1)$, *and $n_1 \in N$ satisfies that*

$$\frac{|x_0|}{\Gamma(\alpha)} (\alpha + n_1 + \gamma_1)^{((1/2)(\alpha+\beta_1)-1)} + \frac{L_1 \Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)} (\alpha + n_1 + \gamma_1)^{(-\gamma_1)} \leq 1. \tag{3.4}$$

Proof. For $t \in N_\alpha$, apply Lemma 2.8 and $\gamma_1 > 0$,

$$t^{(-\gamma_1)} = \frac{\Gamma(t+1)}{\Gamma(t+\gamma_1+1)} = t^{-\gamma_1} \left[1 + O\left(\frac{1}{t} \right) \right], \tag{3.5}$$

and we have that $t^{(-\gamma_1)} \to 0$ as $t \to \infty$, then there exists a $n_1 \in N$ such that inequality (3.4) holds, which implies that the set S_1 exists.

We firstly show that B maps S_1 in S_1.

It is easy to know that S_1 is a closed, bounded, and convex subset of R.

Apply condition (H_1), Lemma 2.5, Corollary 2.9 and (3.4), for $t \in N_{\alpha+n_1}$, we have

$$\begin{aligned}
|Bx(t)| &\leq \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{(\alpha-1)} \left| f(s+\alpha, x(s+\alpha)) \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{(\alpha-1)} L_1 (s+\alpha)^{(-\beta_1)} \\
&= L_1 \Delta^{-\alpha} (t+\alpha)^{(-\beta_1)} \\
&= \frac{L_1 \Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)} (t+\alpha)^{(\alpha-\beta_1)} \\
&< \frac{L_1 \Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)} t^{(\alpha-\beta_1)} \\
&= \frac{L_1 \Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)} (t+\gamma_1)^{(-\gamma_1)} t^{(-\gamma_1)}
\end{aligned}$$

$$\leq \frac{L_1\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}(\alpha+n_1+\gamma_1)^{(-\gamma_1)}t^{(-\gamma_1)}$$
$$\leq t^{(-\gamma_1)}, \tag{3.6}$$

which implies that $BS_1 \subset S_1$ for $t \in N_{\alpha+n_1}$.

Nextly, we show that B is continuous on S_1.

Let $\varepsilon > 0$ be given then there exist $T_1 \in N$ and $T_1 \geq n_1$ such that $t \in N_{\alpha+T_1}$ implies that

$$\frac{L_1\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}t^{(\alpha-\beta_1)} < \frac{\varepsilon}{2}. \tag{3.7}$$

Let $\{x_n\}$ be a sequence such that $x_n \longrightarrow x$. For $t \in \{\alpha+n_1, \alpha+n_1+1, \ldots, \alpha+T_1-1\}$, applying the continuity of f and Lemma 2.6, we have

$$\begin{aligned}
|Bx_n(t)-Bx(t)| &\leq \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{(\alpha-1)}\left|f(s+\alpha,x_n(s+\alpha))-f(s+\alpha,x(s+\alpha))\right| \\
&\leq \max_{s\in\{0,1,\ldots,T_1-1\}}\left|f(s+\alpha,x_n(s+\alpha))-f(s+\alpha,x(s+\alpha))\right| \\
&\quad\times \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{(\alpha-1)} \\
&= \frac{t^{(\alpha)}}{\Gamma(\alpha+1)}\max_{s\in\{0,1,\ldots,T_1-1\}}\left|f(s+\alpha,x_n(s+\alpha))-f(s+\alpha,x(s+\alpha))\right| \\
&\leq \frac{(\alpha+T_1-1)^{(\alpha)}}{\Gamma(\alpha+1)}\max_{s\in\{0,1,\ldots,T_1-1\}}\left|f(s+\alpha,x_n(s+\alpha))-f(s+\alpha,x(s+\alpha))\right| \\
&\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.8}$$

For $t \in N_{\alpha+T_1}$,

$$\begin{aligned}
|Bx_n(t)-Bx(t)| &\leq \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{(\alpha-1)}\left[\left|f(s+\alpha,x_n(s+\alpha))\right|+\left|f(s+\alpha,x(s+\alpha))\right|\right] \\
&\leq \frac{2L_1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{(\alpha-1)}(s+\alpha)^{(-\beta_1)} \\
&= 2L_1\Delta^{-\alpha}(t+\alpha)^{(-\beta_1)} \\
&= \frac{2L_1\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}(t+\alpha)^{(\alpha-\beta_1)} \\
&< \frac{2L_1\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}t^{(\alpha-\beta_1)} \\
&< \varepsilon.
\end{aligned} \tag{3.9}$$

Thus, for all $t \in N_{\alpha+n_1}$, we have

$$|Bx_n(t) - Bx(t)| \longrightarrow 0 \quad \text{as } n \to \infty. \tag{3.10}$$

which implies that B is continuous.

Lastly, we show that BS_1 is relatively compact.

Let $t_1, t_2 \in N_{\alpha+T_1}$ and $t_2 > t_1$, thus we have

$$\begin{aligned}
|Bx(t_2) - Bx(t_1)| &= \left| \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t_2-\alpha} (t_2 - s - 1)^{(\alpha-1)} f(s+\alpha, x(s+\alpha)) \right. \\
&\qquad \left. - \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t_1-\alpha} (t_1 - s - 1)^{(\alpha-1)} f(s+\alpha, x(s+\alpha)) \right| \\
&\le \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t_2-\alpha} (t_2 - s - 1)^{(\alpha-1)} \left| f(s+\alpha, x(s+\alpha)) \right| \\
&\quad + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t_1-\alpha} (t_1 - s - 1)^{(\alpha-1)} \left| f(s+\alpha, x(s+\alpha)) \right| \\
&\le \frac{L_1 \Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)} (t_2+\alpha)^{(\alpha-\beta_1)} + \frac{L_1 \Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)} (t_1+\alpha)^{(\alpha-\beta_1)} \\
&< \frac{L_1 \Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)} t_2^{(\alpha-\beta_1)} + \frac{L_1 \Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)} t_1^{(\alpha-\beta_1)} \\
&< \varepsilon.
\end{aligned} \tag{3.11}$$

Thus, $\{Bx : x \in S_1\}$ is a bounded and uniformly Cauchy subset by Definition 2.11, and BS_1 is relatively compact by means of Theorem 2.12. This completes the proof. □

Lemma 3.2. *Assume that condition* (H_1) *holds, then a solution of* (1.1) *is in* S_1 *for* $t \in N_{\alpha+n_1}$.

Proof. Notice if that $x(t)$ is a fixed point of P, then it is a solution of (1.1). To prove this, it remains to show that, for fixed $y \in S_1$, $x = Ax + By \Rightarrow x \in S_1$ holds.

If $x = Ax + By$, applying condition (H_1) and (3.4), for $t \in N_{\alpha+n_1}$, we have

$$\begin{aligned}
|x(t)| &\le |Ax(t)| + |By(t)| \\
&\le \frac{|x_0|}{\Gamma(\alpha)} t^{(\alpha-1)} + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{(\alpha-1)} \left| f(s+\alpha, y(s+\alpha)) \right| \\
&\le \frac{|x_0|}{\Gamma(\alpha)} t^{(\alpha-1)} + \frac{L_1 \Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)} (t+\alpha)^{(\alpha-\beta_1)} \\
&< \frac{|x_0|}{\Gamma(\alpha)} t^{(\alpha-1)} + \frac{L_1 \Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)} t^{(\alpha-\beta_1)}
\end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{|x_0|}{\Gamma(\alpha)}(t+\gamma_1)^{((1/2)(\alpha+\beta_1)-1)} + \frac{L_1\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}(t+\gamma_1)^{(-\gamma_1)}\right]t^{(-\gamma_1)} \\
&\leq \left[\frac{|x_0|}{\Gamma(\alpha)}(\alpha+n_1+\gamma_1)^{((1/2)(\alpha+\beta_1)-1)} + \frac{L_1\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}(\alpha+n_1+\gamma_1)^{(-\gamma_1)}\right]t^{(-\gamma_1)} \\
&\leq t^{(-\gamma_1)}.
\end{aligned}
\tag{3.12}
$$

Thus, $x(t) \in S_1$ for $t \in N_{\alpha+n_1}$. According to Theorem 2.13 and Lemma 3.1, there exists a $x \in S_1$ such that $x = Ax+Bx$, that is, P has a fixed point in S_1 which is a solution of (1.1) for $t \in N_{\alpha+n_1}$. This completes the proof. □

Theorem 3.3. *Assume that condition* (H_1) *holds, then the solutions of* (1.1) *is attractive.*

Proof. By Lemma 3.2, the solutions of (1.1) exist and are in S_1. All functions $x(t)$ in S_1 tend to 0 as $t \to \infty$. Then the solutions of (1.1) tend to zero as $t \to \infty$. This completes the proof. □

Theorem 3.4. *Assume that the following condition is satisfied:*

(H_2) *there exist constants* $\beta_2 \in (\alpha, 1)$ *and* $L_2 \geq 0$ *such that*

$$
\left|f(t,x(t)) - f(t,y(t))\right| \leq L_2 t^{(-\beta_2)}\|x-y\| \quad \text{for } t \in N_\alpha. \tag{3.13}
$$

Then the solutions of (1.1) *are stable provided that*

$$
c := \frac{L_2\Gamma(1+\alpha)\Gamma(1-\beta_2)}{\Gamma(1+\alpha-\beta_2)\Gamma(1+\beta_2)} < 1. \tag{3.14}
$$

Proof. Let $x(t)$ be a solution of (1.1), and let $\tilde{x}(t)$ be a solution of (1.1) satisfying the initial value condition $\tilde{x}(0) = \tilde{x}_0$. For $t \in N_\alpha$, applying condition (H_2), we have

$$
\begin{aligned}
|x(t)-\tilde{x}(t)| &\leq \frac{t^{(\alpha-1)}}{\Gamma(\alpha)}|x_0-\tilde{x}_0| + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{(\alpha-1)} \\
&\qquad \times \left|f(s+\alpha,x(s+\alpha)) - f(s+\alpha,\tilde{x}(s+\alpha))\right| \\
&\leq \frac{t^{(\alpha-1)}}{\Gamma(\alpha)}|x_0-\tilde{x}_0| + \frac{L_2}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{(\alpha-1)}(s+\alpha)^{(-\beta_2)}\|x-\tilde{x}\| \\
&= \frac{t^{(\alpha-1)}}{\Gamma(\alpha)}|x_0-\tilde{x}_0| + L_2\Delta^{-\alpha}(t+\alpha)^{(-\beta_2)}\|x-\tilde{x}\| \\
&= \frac{t^{(\alpha-1)}}{\Gamma(\alpha)}|x_0-\tilde{x}_0| + \frac{L_2\Gamma(1-\beta_2)}{\Gamma(1+\alpha-\beta_2)}(t+\alpha)^{(\alpha-\beta_2)}\|x-\tilde{x}\| \\
&\leq \frac{\alpha^{(\alpha-1)}}{\Gamma(\alpha)}|x_0-\tilde{x}_0| + \frac{L_2\Gamma(1-\beta_2)}{\Gamma(1+\alpha-\beta_2)}\alpha^{(\alpha-\beta_2)}\|x-\tilde{x}\|
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha|x_0 - \tilde{x}_0| + \frac{L_2\Gamma(1+\alpha)\Gamma(1-\beta_2)}{\Gamma(1+\alpha-\beta_2)\Gamma(1+\beta_2)}\|x - \tilde{x}\| \\
&= \alpha|x_0 - \tilde{x}_0| + c\|x - \tilde{x}\|,
\end{aligned}
\tag{3.15}
$$

which implies that

$$
\|x - \tilde{x}\| \le \frac{\alpha}{1-c}|x_0 - \tilde{x}_0|. \tag{3.16}
$$

For any given $\varepsilon > 0$, let $\delta = ((1-c)/\alpha)\varepsilon$, $|x_0 - \tilde{x}_0| < \delta$ follows that $\|x - \tilde{x}\| < \varepsilon$, which yields that the solutions of (1.1) are stable. This completes the proof. □

Theorem 3.5. *Assume that conditions* (H_1) *and* (H_2) *hold, then the solutions of* (1.1) *are asymptotically stable provided that* (3.14) *holds.*

Theorem 3.5 is the simple consequence of Theorems 3.3 and 3.4.

Theorem 3.6. *Assume that the following condition is satisfied:*

(H_3) *there exist constants* $\beta_3 \in (\alpha, (1/2)(1+\alpha))$, $\gamma_2 = (1/2)(1-\alpha)$, *and* $L_3 \ge 0$ *such that*

$$
\left|f(t, x(t))\right| \le L_3\left(t + \gamma_2\right)^{(-\beta_3)}|x(t)| \quad \text{for } t \in N_\alpha. \tag{3.17}
$$

Then the solutions of (1.1) *is attractive.*

Proof. Set

$$
S_2 = \left\{x(t) : |x(t)| \le t^{(-\gamma_2)} \text{ for } t \in N_{\alpha+n_2}\right\}, \tag{3.18}
$$

where $n_2 \in N$ satisfies that

$$
\frac{|x_0|}{\Gamma(\alpha)}\left(\alpha + n_2 + \gamma_2\right)^{(-\gamma_2)} + \frac{L_3\Gamma(1-\beta_3-\gamma_2)}{\Gamma(1+\alpha-\beta_3-\gamma_2)}\left(\alpha + n_2 + \gamma_2\right)^{(\alpha-\beta_3)} \le 1. \tag{3.19}
$$

We first prove condition (c) of Theorem 2.13, that is, for fixed $y \in S_2$ and for all $x \in R$, $x = Ax + By \Rightarrow x \in S_2$ holds.

If $x = Ax + By$, applying condition (H_3) and (3.19), for $t \in N_{\alpha+n_2}$, we have

$$\begin{aligned}
|x(t)| &\le |Ax(t)| + |By(t)| \\
&\le \frac{|x_0|}{\Gamma(\alpha)} t^{(\alpha-1)} + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{(\alpha-1)} |f(s+\alpha, y(s+\alpha))| \\
&\le \frac{|x_0|}{\Gamma(\alpha)} t^{(\alpha-1)} + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{(\alpha-1)} L_3 (s+\alpha+\gamma_2)^{(-\beta_3)} |y(s+\alpha)| \\
&\le \frac{|x_0|}{\Gamma(\alpha)} t^{(\alpha-1)} + \frac{L_3}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{(\alpha-1)} (s+\alpha+\gamma_2)^{(-\beta_3)} (s+\alpha)^{(-\gamma_2)} \\
&\le \frac{|x_0|}{\Gamma(\alpha)} t^{(\alpha-1)} + \frac{L_3}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{(\alpha-1)} (s+\alpha)^{(-\beta_3-\gamma_2)} \\
&\le \frac{|x_0|}{\Gamma(\alpha)} t^{(\alpha-1)} + \frac{L_3 \Gamma(1-\beta_3-\gamma_2)}{\Gamma(1+\alpha-\beta_3-\gamma_2)} (t+\alpha)^{(\alpha-\beta_3-\gamma_2)} \\
&< \frac{|x_0|}{\Gamma(\alpha)} t^{(\alpha-1)} + \frac{L_3 \Gamma(1-\beta_3-\gamma_2)}{\Gamma(1+\alpha-\beta_3-\gamma_2)} t^{(\alpha-\beta_3-\gamma_2)} \\
&= \left[\frac{|x_0|}{\Gamma(\alpha)} (t+\gamma_2)^{(-\gamma_2)} + \frac{L_3 \Gamma(1-\beta_3-\gamma_2)}{\Gamma(1+\alpha-\beta_3-\gamma_2)} (t+\gamma_2)^{(\alpha-\beta_3)} \right] t^{(-\gamma_2)} \\
&\le \left[\frac{|x_0|}{\Gamma(\alpha)} (\alpha+n_2+\gamma_2)^{(-\gamma_2)} + \frac{L_3 \Gamma(1-\beta_3-\gamma_2)}{\Gamma(1+\alpha-\beta_3-\gamma_2)} (\alpha+n_2+\gamma_2)^{(\alpha-\beta_3)} \right] t^{(-\gamma_2)} \\
&\le t^{(-\gamma_2)}.
\end{aligned} \tag{3.20}$$

Thus, condition (c) of Theorem 2.13 holds.

The proof of condition (b) of Theorem 2.13 is similar to that of Lemma 3.1, and we omit it. Therefore, P has a fixed point in S_2 by using Theorem 2.13, that is, the IVP (1.1) has a solution in S_2. Moreover, all functions in S_2 tend to 0 as $t \to \infty$, then the solution of (1.1) tends to zero as $t \to \infty$, which shows that the zero solution of (1.1) is attractive. This completes the proof. □

Theorem 3.7. *Assume that conditions (H_2) and (H_3) hold, then the solutions of* (1.1) *are asymptotically stable provided that* (3.14) *holds.*

Theorem 3.8. *Assume that the following condition is satisfied:*

(H_4) *there exist constants $\eta \in (0,1)$, $\beta_4 \in (\alpha, (2+\alpha\eta)/(2+\eta))$, and $L_4 \ge 0$ such that*

$$|f(t, x(t))| \le L_4 (t+1)^{(-\beta_4)} |x(t)|^{\eta} \quad \text{for } t \in N_{\alpha}. \tag{3.21}$$

Then the solutions of (1.1) *is attractive.*

Proof. Set

$$S_3 = \left\{ x(t) : |x(t)| \le t^{(-\gamma_3)} \text{ for } t \in N_{\alpha+n_3} \right\}, \tag{3.22}$$

where $\gamma_3 = (1/2)(\beta_4 - \alpha)$, and $n_3 \in N$ satisfies that

$$\frac{|x_0|}{\Gamma(\alpha)}(\alpha + n_3 + \gamma_3)^{(\alpha-1+\gamma_3)} + \frac{L_4\Gamma(1-\beta_4-\gamma_3\eta)}{\Gamma(1+\alpha-\beta_4-\gamma_3\eta)}(\alpha + n_3 + \gamma_3)^{-\gamma_3} \le 1. \tag{3.23}$$

Here we only prove that condition (c) of Theorem 2.13 holds, and the remaining part of the proof is similar to that of Theorem 3.6.

Since $\eta \in (0,1)$, $\beta_4 \in (\alpha, (2+\alpha\eta)/(2+\eta))$, and $\gamma_3 = (1/2)(\beta_4 - \alpha)$, then $\gamma_3, \gamma_3\eta, \alpha + \gamma_3 \in (0,1)$, $\beta_4 + \gamma_3\eta \in (\alpha, 1)$.

If $x = Ax + By$, applying condition (H_4), Lemma 2.5 and (3.23), for $t \in N_{\alpha+n_3}$, we have

$$\begin{aligned}
|x(t)| \le |Ax(t)| + |By(t)| &\le \frac{|x_0|}{\Gamma(\alpha)}t^{(\alpha-1)} + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{(\alpha-1)}|f(s+\alpha, y(s+\alpha))| \\
&\le \frac{|x_0|}{\Gamma(\alpha)}t^{(\alpha-1)} + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{(\alpha-1)}L_4(s+\alpha+1)^{(-\beta_4)}|y(s+\alpha)|^{\eta} \\
&\le \frac{|x_0|}{\Gamma(\alpha)}t^{(\alpha-1)} + \frac{L_4}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{(\alpha-1)}(s+\alpha+\gamma_3\eta)^{(-\beta_4)}\left[(s+\alpha)^{(-\gamma_3)}\right]^{\eta} \\
&\le \frac{|x_0|}{\Gamma(\alpha)}t^{(\alpha-1)} + \frac{L_4}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{(\alpha-1)}(s+\alpha+\gamma_3\eta)^{(-\beta_4)}(s+\alpha)^{(-\gamma_3\eta)} \\
&= \frac{|x_0|}{\Gamma(\alpha)}t^{(\alpha-1)} + \frac{L_4}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{(\alpha-1)}(s+\alpha)^{(-\beta_4-\gamma_3\eta)} \\
&\le \frac{|x_0|}{\Gamma(\alpha)}t^{(\alpha-1)} + \frac{L_4\Gamma(1-\beta_4-\gamma_3\eta)}{\Gamma(1+\alpha-\beta_4-\gamma_3\eta)}(t+\alpha)^{(\alpha-\beta_4-\gamma_3\eta)} \\
&< \frac{|x_0|}{\Gamma(\alpha)}t^{(\alpha-1)} + \frac{L_4\Gamma(1-\beta_4-\gamma_3\eta)}{\Gamma(1+\alpha-\beta_4-\gamma_3\eta)}t^{(\alpha-\beta_4-\gamma_3\eta)} \\
&\le \frac{|x_0|}{\Gamma(\alpha)}t^{(\alpha-1)} + \frac{L_4\Gamma(1-\beta_4-\gamma_3\eta)}{\Gamma(1+\alpha-\beta_4-\gamma_3\eta)}t^{(\alpha-\beta_4)} \\
&= \left[\frac{|x_0|}{\Gamma(\alpha)}(t+\gamma_3)^{(\alpha-1+\gamma_3)} + \frac{L_4\Gamma(1-\beta_4-\gamma_3\eta)}{\Gamma(1+\alpha-\beta_4-\gamma_3\eta)}(t+\gamma_3)^{(-\gamma_3)}\right]t^{(-\gamma_3)} \\
&\le \left[\frac{|x_0|}{\Gamma(\alpha)}(\alpha+n_3+\gamma_3)^{(\alpha-1+\gamma_3)}\right. \\
&\qquad \left. + \frac{L_4\Gamma(1-\beta_4-\gamma_3\eta)}{\Gamma(1+\alpha-\beta_4-\gamma_3\eta)}(\alpha+n_3+\gamma_3)^{(-\gamma_3)}\right]t^{(-\gamma_3)} \\
&\le t^{(-\gamma_3)}.
\end{aligned} \tag{3.24}$$

Thus, condition (c) of Theorem 2.13 holds. This completes the proof. □

4. Examples

Example 4.1. Consider

$$\begin{gathered}\Delta^{0.5}x(t) = 0.2(t+0.5)^{(-0.75)}\sin(x(t+0.5)),\quad t \in N_0,\\ \Delta^{-0.5}x(t)|_{t=0} = x_0,\end{gathered} \tag{4.1}$$

where $f(t, x(t)) = 0.2t^{(-0.75)}\sin(x(t))$, $t \in N_{0.5}$.

Since

$$\left|f(t, x(t))\right| = \left|0.2t^{(-0.75)}\sin(x(t))\right| \leq 0.2t^{(-0.75)}, \tag{4.2}$$

this implies that condition (H_1) holds.

In addition,

$$\left|f(t, x(t)) - f(t, y(t))\right| \leq 0.2t^{(-0.75)}\|x - y\|. \tag{4.3}$$

Thus, condition (H_2) is satisfied.

Moreover, from $L_2 = 0.2$, $\alpha = 0.5$, and $\beta_2 = 0.75$, we have

$$c = \frac{L_2\Gamma(1+\alpha)\Gamma(1-\beta_2)}{\Gamma(1+\alpha-\beta_2)\Gamma(1+\beta_2)} = \frac{0.2\Gamma(1.5)\Gamma(0.25)}{\Gamma(1.25)\Gamma(1.75)} \approx 0.7716 < 1, \tag{4.4}$$

which implies that inequality (3.14) holds.

Thus the solutions of (4.1) are asymptotically stable by Theorem 3.5.

Example 4.2. Consider

$$\begin{gathered}\Delta^{0.5}x(t) = 0.2(t+1.5)^{(-0.6)}x(t+0.5),\quad t \in N_0,\\ \Delta^{-0.5}x(t)|_{t=0} = x_0,\end{gathered} \tag{4.5}$$

where $f(t, x(t)) = 0.2(t+1)^{(-0.6)}x(t)$, $t \in N_{0.5}$.

Since $\beta_3 = 0.6$, $\alpha = 0.5$, we have that $\beta_3 \in (\alpha, (1/2)(1+\alpha))$, $\gamma_2 = 0.25$ and

$$\left|f(t, x(t))\right| = \left|0.2(t+1)^{(-0.6)}x(t)\right| \leq 0.2(t+0.25)^{(-0.6)}|x(t)|, \tag{4.6}$$

which implies that condition (H_3) is satisfied.

Meanwhile,

$$\left|f(t, x(t)) - f(t, y(t))\right| \leq 0.2(t+1)^{(-0.6)}\|x - y\| \leq 0.2t^{(-0.6)}\|x - y\|, \tag{4.7}$$

which implies that condition (H_2) is satisfied.

From $L_2 = 0.2$, $\alpha = 0.5$, and $\beta_2 = 0.6$, we have

$$c = \frac{L_2\Gamma(1+\alpha)\Gamma(1-\beta_2)}{\Gamma(1+\alpha-\beta_2)\Gamma(1+\beta_2)} = \frac{0.2\Gamma(1.5)\Gamma(0.4)}{\Gamma(0.9)\Gamma(1.6)} \approx 0.4120 < 1, \tag{4.8}$$

which implies that inequality (3.14) holds.

Thus the solutions of (4.5) are asymptotically stable by Theorem 3.7.

Example 4.3. Consider

$$\begin{gathered} \Delta^{0.5}x(t) = (t+1.5)^{(-0.6)}x^{1/3}(t+0.5), \quad t \in N_0, \\ \Delta^{-0.5}x(t)|_{t=0} = x_0, \end{gathered} \tag{4.9}$$

where $f(t, x(t)) = (t+1)^{(-0.6)}x^{1/3}(t)$, $t \in N_{0.5}$.

Since $\alpha = 0.5$, $\beta_4 = 0.6$, $\eta = 1/3$, we have that $\eta \in (0,1)$, $\beta_4 \in (\alpha, (2+\alpha\eta)/(2+\eta))$ and

$$\left|f(t, x(t))\right| \leq (t+1)^{(-0.6)}|x(t)|^{1/3}, \tag{4.10}$$

then condition (H_4) is satisfied.

The solutions of (4.9) are attractive by Theorem 3.8.

Acknowledgments

This research was supported by the NSF of Hunan Province (10JJ6007, 2011FJ3013), the Scientific Research Foundation of Hunan Provincial Education Department, and the Construct Program of the Key Discipline in Hunan Province.

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