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Research Article

Twin TQFTs and Frobenius Algebras

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We introduce the category of singular 2-dimensional cobordisms and show that it admits a completely algebraic description as the free symmetric monoidal category on a *twin Frobenius algebra*, by providing a description of this category in terms of generators and relations. A twin Frobenius algebra (C, W, z, z^*) consists of a commutative Frobenius algebra C, a symmetric Frobenius algebra W, and an algebra homomorphism $z : C \to W$ with dual $z^* : W \to C$, satisfying some extra conditions. We also introduce a generalized 2-dimensional Topological Quantum Field Theory defined on singular 2-dimensional cobordisms and show that it is equivalent to a twin Frobenius algebra in a symmetric monoidal category.

1. Introduction

A 2-dimensional Topological Quantum Field Theory (TQFT) is a symmetric monoidal functor from the category **2Cob** of 2-dimensional cobordisms to the category $\mathbf{Vect}_k$ of vector spaces over a field k. The objects in **2Cob** are smooth compact 1-manifolds without boundary, and the morphisms are the equivalence classes of smooth compact oriented cobordisms between them, modulo diffeomorphisms that restrict to the identity on the boundary. The category **2Cob** of 2-cobordisms and that of 2D TQFTs are well understood, and it is known that 2D TQFTs are characterized by commutative Frobenius algebras, in the sense that the category of 2D TQFTs is equivalent as a symmetric monoidal category to the category of commutative Frobenius algebras. For the classic results involving these concepts, we refer to [1–3] and the book [4].

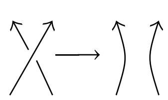
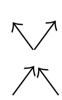
Lauda and Pfeiffer studied in [5] a special type of extended TQFTs defined on *open-closed cobordisms*. These cobordisms are certain smooth oriented 2-manifolds with corners that can be viewed as cobordisms between compact 1-manifolds with boundary, that is, between disjoint unions of circles S^1 and unit intervals $I = [0, 1]$. An *open-closed TQFT* is a symmetric monoidal functor $Z : \mathbf{2Cob}^{\text{ext}} \to \mathbf{Vect}_k$, where $\mathbf{2Cob}^{\text{ext}}$ denotes the category of open-closed cobordisms. Lauda and Pfeiffer showed that open-closed TQFTs are characterized by what they call *knowledgeable Frobenius algebras* (A, C, ι, ι^*), where the vector space $C := Z(S^1)$ associated with the circle has the structure of a commutative Frobenius algebra, the vector space $A := Z(I)$ associated with the interval has the structure of a symmetric Frobenius algebra, and there are linear maps $\iota : C \to A$ and $\iota^* : A \to C$ satisfying certain conditions. This result was obtained by providing a description of the category of open-closed cobordisms in terms of generators and the Moore-Segal relations. They defined a normal form for such cobordisms, characterized by topological invariants, and then proved the sufficiency of the relations by constructing a sequence of *moves* which transforms the given cobordism into the normal form. They also showed that the category $\mathbf{2Cob}^{\text{ext}}$ of open-closed cobordisms is equivalent to the symmetric monoidal category freely generated by a knowledgeable Frobenius algebra. We remark that the entire construction in [5] was given for an arbitrary symmetric monoidal category, and not only for $\mathbf{Vect}_k$.

In [6], the author constructed a bigraded tangle cohomology theory which depends on one parameter, via a setup with webs and dotted foams (singular cobordisms) modulo a finite set of local relations. This work is a blend of Bar-Natan's [7] approach to "local" Khovanov homology and Khovanov's framework [8] using webs and foams. In [9], the author generalized the construction given in [6] to a two-parameter theory for tangles, which we call the *universal* $\mathfrak{sl}(2)$ *foam cohomology* (it is "universal" in the sense of [10]). This two-parameter theory corresponds to a certain Frobenius algebra structure defined on $\mathbb{Z}[i, a, h, X]/(X^2 - hX - a)$, and which, for the case of of links, is a categorification of the quantum $\mathfrak{sl}(2)$-link invariant. Adding the relation $a = h = 0$ yields an isomorphic version of the $\mathfrak{sl}(2)$ Khovanov homology [11, 12], while imposing $a = 1$, $h = 0$ recovers Lee's theory [13]. The advantage of working with foams instead of classical 2-cobordisms is that the construction in [9] (as well as its particular case introduced in [6]) yields a theory that satisfies an honest functoriality property with respect to tangle or link cobordisms, relative boundary, that is, with no sign indeterminacy. In particular, it resolves the sign ambiguity residing in the functoriality property of the Khovanov homology. We note that there is also the work by Clark et al. [14] that fixes the functoriality property of Khovanov's invariant through the use of singular cobordisms (they called these "disoriented cobordisms").

In [15], the author described a method that computes fast and efficient the $\mathfrak{sl}(2)$ foam cohomology groups, and also provided a purely topological version of the $\mathfrak{sl}(2)$ foam theory in which no dots are required on cobordisms.

We briefly review below the gadgets used in [6, 9]. A *web* is a planar graph with bivalent vertices near which the two incident edges are oriented either towards the vertex or away from it. Webs without vertices, thus oriented circles, are also allowed. Examples of webs are depicted in (11). A *foam* (also called *singular cobordism*) is an abstract cobordism between webs—regarded up to boundary-preserving isotopies—and has *singular arcs* (and/or *singular circles*) where orientations disagree, and near which the facets incident with a certain singular arc are compatibly oriented, inducing an orientation on that arc. Examples of such cobordisms are given in (12) (the red curves in a singular cobordism diagram are singular arcs/circles).

The author arrives at webs and foams by considering a link diagram L (we talk here about links instead of tangles just for simplicity) and resolving each crossing in L in one of the following ways:

 and  (1)

Then she associates to L a "formal" chain complex $[L]$, whose objects are formally graded webs, and whose morphisms are formal linear combinations of foams. When considered as an object in the category of complexes of foams modulo a certain set of local relations, $[L]$ is an up-to-homotopy invariant of L. The jump from the geometric setup to an algebraic one, allowing to obtain a cohomology theory, is done via a "tautological functor" similar to the one used in [7].

There are many similarities between the algebraic structure of the category of open-closed cobordisms and certain relations satisfied by the singular cobordisms, although the two types of cobordisms are topologically different. The original Khovanov homology relies on a 2D TQFT, and it would be quite desirable and refreshing to have some kind of TQFT defined on foams/singular cobordisms and use it to obtain another method for defining the universal $\mathfrak{sl}(2)$ foam cohomology theory (specifically, the purely topological one—with no dots on cobordisms—discussed in [15], Section 4), and a generalization of the Khovanov homology. In particular, this would provide us with knowledge of the algebraic structure that governs this cohomology theory that provides a properly functorial Khovanov homology theory.

In this paper we make the first step in achieving this goal. The singular cobordisms considered here are a particular case of those used in [6, 9, 15], in the sense that the 1-manifolds are disjoint unions of oriented circles and biwebs (webs with *exactly* two bivalent vertices). The second step in reaching our goal will be treated in a subsequent paper, where we also show that it suffices to work with biwebs, as opposed to arbitrary webs (webs with an even number of bivalent vertices).

We introduce the category **Sing-2Cob** of singular 2-cobordisms and show that it is equivalent as a symmetric monoidal category to the category freely generated by, what we call, a *twin Frobenius algebra*. A twin Frobenius algebra is almost the same as a knowledgeable Frobenius algebra; specifically, all properties of the latter one are satisfied by the first one, except for the "Cardy relation" which is replaced by what we call the "genus-one relation." The definition of twin Frobenius algebras and their category is given in Section 2.

We present in Section 3 a normal form for an arbitrary singular 2-cobordism and characterize the category **Sing-2Cob** in terms of generators and relations. In Section 4 we define *twin TQFTs* as symmetric monoidal functors **Sing-2Cob** $\to \mathscr{C}$, where $\mathscr{C}$ is an arbitrary symmetric monoidal category, and prove that the category of twin TQFTs in $\mathscr{C}$ is equivalent, as a symmetric monoidal category, to the category of twin Frobenius algebras in $\mathscr{C}$.

In Section 5 we provide examples of twin Frobenius algebras and thus twin TQFTs in $\mathscr{C} = \mathbf{Vect}_k$ and/or $\mathscr{C} = \mathbf{R}\text{-}\mathbf{Mod}$, where k is a field and R a commutative ring.

2. Twin Frobenius Algebras

2.1. Definitions. Throughout the paper, one considers an arbitrary symmetric monoidal (tensor) category $(\mathscr{C}, \otimes, \mathbf{1}, \alpha, \lambda, \rho, \tau)$ with unit object $\mathbf{1} \in \mathscr{C}$, associativity law $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, left-unit and right-unit laws $\lambda_X : \mathbf{1} \otimes X \to X$ and $\rho_X : X \otimes \mathbf{1} \to X$, and with symmetric braiding $\tau_{X,Y} : X \otimes Y \to Y \otimes X$, for X, Y and Z objects in $\mathscr{C}$.

For reader's convenience, we recall a few definitions.

An *algebra object* (C, m, ι) in $\mathscr{C}$ consists of an object C and morphisms $m : C \otimes C \to C$ and $\iota : \mathbf{1} \to C$ in $\mathscr{C}$ such that

$$\begin{gathered} m \circ (\mathrm{id}_C \otimes m) \circ \alpha_{C,C,C} = m \circ (m \otimes \mathrm{id}_C), \\ m \circ (\mathrm{id}_C \otimes \iota) = \rho_C, \qquad m \circ (\iota \otimes \mathrm{id}_C) = \lambda_C. \end{gathered} \tag{2}$$

A *coalgebra object* (C, Δ, ϵ) in $\mathscr{C}$ is an object C and morphisms $\Delta : C \to C \otimes C$ and $\epsilon : C \to \mathbf{1}$ such that

$$\begin{gathered} (\mathrm{id}_C \otimes \Delta) \circ \Delta = \alpha_{C,C,C} \circ (\Delta \otimes \mathrm{id}_C) \circ \Delta, \\ (\mathrm{id}_C \otimes \epsilon) \circ \Delta = \rho_C^{-1}, \qquad (\epsilon \otimes \mathrm{id}_C) \circ \Delta = \lambda_C^{-1}. \end{gathered} \tag{3}$$

A *homomorphism of algebras* $f : C \to C'$ between two algebra objects (C, m, ι) and (C', m', ι') in $\mathscr{C}$ is a morphism f of $\mathscr{C}$ such that

$$f \circ m = m' \circ (f \otimes f), \qquad f \circ \iota = \iota'. \tag{4}$$

A *homomorphism of coalgebras* $f : C \to C'$ between two coalgebra objects (C, Δ, ϵ) and (C', Δ', ϵ') in $\mathscr{C}$ is a morphism f of $\mathscr{C}$ such that

$$(f \otimes f) \circ \Delta = \Delta' \circ f, \qquad \epsilon' \circ f = \epsilon. \tag{5}$$

A *Frobenius algebra object* $(C, m, \iota, \Delta, \epsilon)$ in $\mathscr{C}$ consists of an object C together with morphisms $m, \iota, \Delta, \epsilon$ such that:

(i) (C, m, ι) is an algebra object and (C, Δ, ϵ) is a coalgebra object in $\mathscr{C}$,

(ii) $(m \otimes \mathrm{id}_C) \circ \alpha^{-1}_{C,C,C} \circ (\mathrm{id}_C \otimes \Delta) = \Delta \circ m = (\mathrm{id}_C \otimes m) \circ \alpha_{C,C,C} \circ (\Delta \otimes \mathrm{id}_C)$.

A Frobenius object $(C, m, \iota, \Delta, \epsilon)$ in $\mathscr{C}$ is called *commutative* if $m \circ \tau = m$, and it is called *symmetric* if $\epsilon \circ m = \epsilon \circ m \circ \tau$.

Given two Frobenius algebra objects $(C, m, \iota, \Delta, \epsilon)$ and $(C', m', \iota', \Delta', \epsilon')$, a *homomorphism of Frobenius algebras* $f : C \to C'$ is a morphism f in $\mathscr{C}$ which is both a homomorphism of algebra and coalgebra objects.

Definition 1. A twin Frobenius algebra $\mathbf{T} := (C, W, z, z^*)$ in $\mathscr{C}$ consists of

(i) a commutative Frobenius algebra $C = (C, m_C, \iota_C, \Delta_C, \epsilon_C)$,

(ii) a symmetric Frobenius algebra $W = (W, m_W, \iota_W, \Delta_W, \epsilon_W)$,

(iii) two morphisms $z : C \to W$ and $z^* : W \to C$ of $\mathscr{C}$

such that z is a homomorphism of algebra objects in $\mathscr{C}$ and

$$\begin{gathered} \epsilon_C \circ m_C \circ (\mathrm{id}_C \otimes z^*) = \epsilon_W \circ m_W \circ (z \otimes \mathrm{id}_W), \\ \text{(duality)}, \\ m_W \circ (\mathrm{id}_W \otimes z) = m_W \circ \tau_{W,W} \circ (\mathrm{id}_W \otimes z), \\ \text{(centrality condition)}, \\ z \circ m_C \circ \Delta_C \circ z^* = m_W \circ \tau_{W,W} \circ \Delta_W, \\ \text{(genus-one condition)}. \end{gathered} \tag{6}$$

The first equality says that z^* is the morphism dual to z (which implies that z^* is a homomorphism of coalgebras in $\mathscr{C}$). If $\mathscr{C} = \mathbf{Vect}_k$, the second equality says that $z(C)$ is contained in the center of the algebra W.

The reader will notice the similarities between twin and knowledgeable Frobenius algebras: their properties are almost the same, except that the *Cardy condition* for a knowledgeable Frobenius algebra is replaced by the *genus-one condition* in the definition of a twin Frobenius algebra.

Definition 2. A *homomorphism of twin Frobenius algebras*

$$f : (C_1, W_1, z_1, z_1^*) \longrightarrow (C_2, W_2, z_2, z_2^*) \tag{7}$$

in a symmetric monoidal category $\mathscr{C}$ consists of a pair $f = (f_1, f_2)$ of Frobenius algebra homomorphisms $f_1 : C_1 \to C_2$ and $f_2 : W_1 \to W_2$ such that $z_2 \circ f_1 = f_2 \circ z_1$ and $z_2^* \circ f_2 = f_1 \circ z_1^*$.

Definition 3. We denote by $\mathbf{T}$-$\mathbf{Frob}(\mathscr{C})$ the category whose objects are twin Frobenius algebras in $\mathscr{C}$ and whose morphisms are twin Frobenius algebra homomorphisms.

Proposition 4. *The category* $\mathbf{T}$-$\mathbf{Frob}(\mathscr{C})$ *forms a symmetric monoidal category in the following sense:*

(i) *the tensor product of two twin Frobenius algebra objects* $\mathbf{T_1} = (C_1, W_1, z_1, z_1^*)$ *and* $\mathbf{T_2} = (C_2, W_2, z_2, z_2^*)$ *is defined as*

$$\mathbf{T_1} \otimes \mathbf{T_2} := (C_1 \otimes C_2, W_1 \otimes W_2, z_1 \otimes z_2, z_1^* \otimes z_2^*); \tag{8}$$

(ii) *the unit object is given by* $\bar{\mathbf{1}} := (\mathbf{1}, \mathbf{1}, \mathrm{id}_{\mathbf{1}}, \mathrm{id}_{\mathbf{1}})$;

(iii) *the associativity and unit laws and the symmetric braiding are induced by those of* $\mathscr{C}$;

(iv) *the tensor product of two homomorphisms* $f = (f_1, f_2)$ *and* $g = (g_1, g_2)$ *of twin Frobenius algebras is defined as* $f \otimes g := (f_1 \otimes g_1, f_2 \otimes g_2)$.

2.2. The Category **Th(T-Frob)**. The definition of the category called the *theory of twin Frobenius algebras*, denoted by **Th**(**T-Frob**), follows Laplaza's [16] construction of the "free category with group structure." The objects of this category are elements of the free $\{\mathbf{1}, \otimes\}$-algebra over the two element set $\{C, W\}$ and is the analogue of the category **Th**(**K-Frob**) introduced in [5, Section 2.2].

The objects of **Th**(**T-Frob**) are words generated by the symbols $\mathbf{1}, C$, and W, which are objects by themselves. If X and Y are objects of **Th**(**T-Frob**) then $(X \otimes Y)$ is also an object.

Consider a graph $\mathscr{G}$ whose vertices are the objects of **Th**(**T-Frob**). Then, there are the following edges:

$$\begin{gathered} m_C : C \otimes C \longrightarrow C, \qquad \iota_C : \mathbf{1} \longrightarrow C, \\ \Delta_C : C \longrightarrow C \otimes C, \qquad \epsilon_C : C \longrightarrow \mathbf{1}, \\ m_W : W \otimes W \longrightarrow W, \qquad \iota_W : \mathbf{1} \longrightarrow W, \\ \Delta_W : W \longrightarrow W \otimes W, \qquad \epsilon_W : W \longrightarrow \mathbf{1}, \\ z : C \longrightarrow W, \qquad z^* : W \longrightarrow C. \end{gathered} \tag{9}$$

For all objects X, Y, Z there are the following edges:

$$
\begin{aligned}
\alpha_{X,Y,Z} &: (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z), \\
\overline{\alpha}_{X,Y,Z} &: X \otimes (Y \otimes Z) \longrightarrow (X \otimes Y) \otimes Z, \\
\tau_{X,Y} &: X \otimes Y \longrightarrow Y \otimes X, \\
\overline{\tau}_{X,Y} &: Y \otimes X \longrightarrow X \otimes Y, \\
\lambda_X : \mathbf{1} \otimes X \longrightarrow X, &\qquad \overline{\lambda}_X : X \otimes \mathbf{1} \longrightarrow X, \\
\rho_X : X \otimes \mathbf{1} \longrightarrow X, &\qquad \overline{\rho}_X : X \longrightarrow X \otimes \mathbf{1}.
\end{aligned}
\tag{10}
$$

For every edge $f : X \to Y$ and for every object Z, there are edges $f \otimes Z : X \otimes Z \to Y \otimes Z$ and $Z \otimes f : Z \otimes X \to Z \otimes Y$.

We denote by $\mathscr{H}$ the category freely generated by the graph $\mathscr{G}$, and define a relation $\sim$ on $\mathscr{H}$ by requiring the following:

(i) that each pair of edges e and $\overline{e}$ are inverses of each other;

(ii) the relations that make $(C, m_C, \iota_C, \Delta_C, \epsilon_C)$ a commutative Frobenius algebra and $(W, m_W, \iota_W, \Delta_W, \epsilon_W)$ a symmetric Frobenius algebra;

(iii) the relations that make $z : C \to W$ an algebra homomorphism and those given in (6);

(iv) the relations that make $\alpha_{X,Y,Z}$, λ_X, and ρ_X satisfy the pentagon and triangle axioms of a monoidal category and those that make $\tau_{X,Y}$ a symmetric braiding, for all objects X, Y, Z;

(v) the relations that state the naturality of α, λ, ρ, τ, $\overline{\alpha}$, $\overline{\lambda}$, $\overline{\rho}$, $\overline{\tau}$ and those that make $\otimes$ a functor.

We also require that for each relation $a \sim b$, we have as well the relations $a \otimes X \sim b \otimes X$ and $X \otimes a \sim X \otimes b$ for all objects X in $\mathscr{H}$, as well as the relations obtained from these by applying this process a finite number of times.

We define the category $\mathbf{Th}(\mathbf{T}\text{-}\mathbf{Frob}) := \mathscr{H}_{/\sim}$, that is, the category $\mathscr{H}$ modulo the category congruence generated by $\sim$ defined above. The category $\mathbf{Th}(\mathbf{T}\text{-}\mathbf{Frob})$ contains a twin Frobenius algebra object $\mathbf{T} = (C, W, z, z^*)$ and is the symmetric monoidal category freely generated by $\mathbf{T}$. For each twin Frobenius algebra $\mathbf{T}' = (C', W', z', z'^*)$ in $\mathscr{C}$, there is exactly one strict symmetric monoidal functor $F_{\mathbf{T}'} : \mathbf{Th}(\mathbf{T}\text{-}\mathbf{Frob}) \to \mathscr{C}$ that maps $\mathbf{T}$ to $\mathbf{T}'$ and $\mathbf{1} \in \mathbf{Th}(\mathbf{T}\text{-}\mathbf{Frob})$ to $\mathbf{1} \in \mathscr{C}$.

Proposition 5. *The category* $\mathbf{T}\text{-}\mathbf{Frob}(\mathscr{C})$ *is equivalent as a symmetric monoidal category to the category of symmetric monoidal functors* $\mathbf{Th}(\mathbf{T}\text{-}\mathbf{Frob}) \to \mathscr{C}$ *and their monoidal natural transformations.*

Proof. The monoidal equivalence of the two categories is constructed identical to that in the proof of [5, Proposition 2.8]. One has only to replace the category $\mathbf{Th}(\mathbf{K}\text{-}\mathbf{Frob})$ used in [5] with our category $\mathbf{Th}(\mathbf{T}\text{-}\mathbf{Frob})$. □

3. Singular Cobordisms and the Category Sing-2Cob

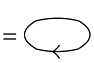
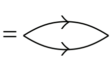
In this section we define the category of singular 2-cobordisms and give a presentation of it in terms of generators and relations. Singular 2-cobordisms form a special type of compact, globally oriented, smooth 2-manifolds. What we call **Sing-2Cob** in the following is in fact a skeleton of the category of singular 2-cobordisms; we choose particular embedded 1-manifolds as the objects of this category.

3.1. Description and Topological Invariants

Definition 6. A *singular 2-cobordism* (or shortly, singular cobordisms) is an abstract, piecewise oriented (but globally orientable) smooth 2-manifold Σ with boundary $\partial\Sigma = \overline{\partial^-\Sigma} \cup \partial^+\Sigma$, where $\overline{\partial^-\Sigma}$ is $\partial^-\Sigma$ with opposite orientation. Both $\partial^-\Sigma$ and $\partial^+\Sigma$ are embedded, closed 1-manifolds, called the source and target boundary, respectively. A singular cobordism has *singular arcs* and/or *singular circles* where orientations disagree. There are exactly two compatibly oriented 2-cells of the underlying 2-dimensional CW-complex Σ that meet at a singular arc/circle, and orientations of two neighboring 2-cells induce an orientation on the singular arc/circle that they share. In our diagrams we draw the singular arcs/circles using red oriented curves.

Two singular cobordisms Σ_1 and Σ_2 are considered *equivalent*, and we write $\Sigma_1 \cong \Sigma_2$, if there exists an orientation-preserving diffeomorphism $\Sigma_1 \to \Sigma_2$ which restricts to the identity on the boundary and which preserves the singular arcs and circles.

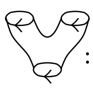
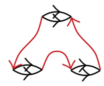
The boundary $\partial\Sigma$ of a singular cobordism Σ is a disjoint union of (clockwise) oriented circles and *biwebs*. In this paper, a *biweb* is a closed oriented graph with two bivalent vertices, such that each vertex is either a source or a sink. Since we want to work with the skeleton of the category of singular cobordisms, we fix one specific oriented circle, and a specific biweb. We denote the circle by 0 and the biweb by 1:

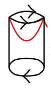
0 =  1 =  (11)

Definition 7. An object in the category **Sing-2Cob** consists of a finite sequence $\mathbf{n} = (n_1, n_2, \dots, n_k)$, where $n_j \in \{0, 1\}$. The length of the sequence, denoted by $|\mathbf{n}| = k$, can be any nonnegative integer and equals the number of disjoint connected components of the corresponding object. Hence $\mathbf{n}$ is the disjoint union of s copies of the fixed circle and t copies of the fixed biweb, for some nonnegative integers s and t with $s + t = |\mathbf{n}|$. If $|\mathbf{n}| = 0$, then $\mathbf{n}$ is the empty 1-manifold.

A morphism $\Sigma : \mathbf{n} \to \mathbf{m}$ in **Sing-2Cob** is an equivalence class $[\Sigma]$ (induced by $\cong$) of singular 2-cobordisms with source boundary $\mathbf{n}$ and target boundary $\mathbf{m}$. The composition of morphisms is obtained in the standard way, namely, by gluing along the common boundary.

Although the objects of **Sing-2Cob** are embedded 1-manifolds, the morphisms are not embedded. Examples of morphisms of **Sing-2Cob** are given below. The source of our cobordisms is at the top and the target at the bottom of drawings; in other words, we read morphisms as cobordisms from top to bottom, by convention.

 $: (0,0) \longrightarrow (0)$  $: (1) \longrightarrow (1,1)$

 $: (1) \longrightarrow (0)$ (12)

The concatenation $\mathbf{n} \coprod \mathbf{m} := (n_1, n_2, \ldots, n_{|\mathbf{n}|}, m_1, m_2, \ldots, m_{|\mathbf{m}|})$ of sequences together with the free union of singular cobordisms, which we also denote by $\coprod$, endows the category **Sing-2Cob** with the structure of a symmetric monoidal category.

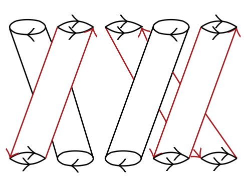
For each $k \in \mathbb{N}$, there is an action of the symmetric group S_k on the subset of objects $\mathbf{n}$ in **Sing-2Cob** for which $|n| = k$, defined by

$$\sigma * \mathbf{n} := \left(n_{\sigma^{-1}(1)}, n_{\sigma^{-1}(2)}, \ldots, n_{\sigma^{-1}(k)}\right). \quad (13)$$

Given any object $\mathbf{n}$ in **Sing-2Cob** and any permutation $\sigma \in S_{|n|}$, there is an obvious induced *permutation cobordism*

$$\sigma^{\mathbf{n}} : \mathbf{n} \longrightarrow \sigma * \mathbf{n}. \quad (14)$$

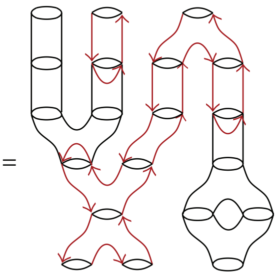
For example, if $\mathbf{n} = (0, 1, 1, 0, 1)$ and $\sigma = (12)(354) \in S_5$, the corresponding morphism $\sigma^{\mathbf{n}}$ is the singular cobordism given in

$\sigma^{\mathbf{n}} =$  (15)

We remark that as morphisms of **Sing-2Cob**, these cobordisms satisfy $\tau^{\sigma * \mathbf{n}} \circ \sigma^{\mathbf{n}} = (\tau \circ \sigma)^{\mathbf{n}}$, for any object $\mathbf{n}$ and σ, $\tau \in S_{|n|}$.

Definition 8. Let $\Sigma : \mathbf{n} \rightarrow \mathbf{m}$ be a morphism in **Sing-2Cob** and let l be the number of its boundary components representing the biweb. In other words, l is the number of 1 entries of $\mathbf{n} \coprod \mathbf{m}$. Number these components by $1, 2, \ldots, l$. The orientation of Σ induces an orientation on all singular arcs of Σ and defines a permutation $\sigma(\Sigma) \in S_l$, called the *singular boundary permutation* of Σ.

For exemplification, we consider the morphism Σ depicted in (16) and we number the biwebs in its boundary by 1, 2, 3, and 4 from left to right, starting with those in the source and followed by those in the target. The singular boundary permutation of this morphism is $\sigma(\Sigma) = (1)(234) = (234) \in S_4$

$\Sigma =$  (16)

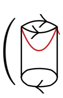
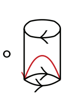
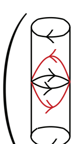
Since a singular cobordism determines a boundary permutation, we need to refine the definition of the morphisms in **Sing-2Cob**.

Definition 9. A morphism $\Sigma : \mathbf{n} \rightarrow \mathbf{m}$ is a pair $\Sigma = ([\Sigma], \sigma)$ consisting of an equivalence class $[\Sigma]$ of singular cobordisms with source boundary $\mathbf{n}$ and target boundary $\mathbf{m}$, and with singular boundary permutation $\sigma(\Sigma) = \sigma$.

Definition 10. Let $\Sigma : \mathbf{n} \rightarrow \mathbf{m}$ be a morphism of **Sing-2Cob**.

(1) The *genus* $g(\Sigma)$ is the genus of the topological 2-manifold underlying Σ.

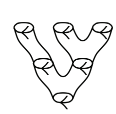
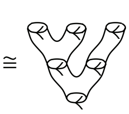
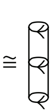
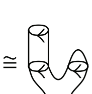
(2) The *singular number* $s(\Sigma)$ is the number of singular circles that Σ contains. (Note that each such circle is homotopic to a point within Σ.)

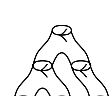
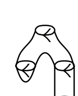
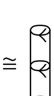
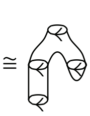
For example, $s\Big($  $\circ$  $\Big) = s\Big($  $\Big) = 1.$

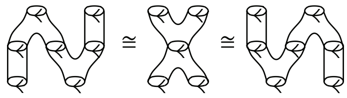
3.2. Structural Relations. In this subsection we provide a list of diffeomorphisms which describe the algebraic structure of the category **Sing-2Cob**.

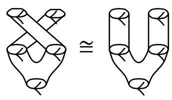
Proposition 11. *The following diffeomorphisms hold in the symmetric monoidal category* **Sing-2Cob**.

(1) *The object* $\mathbf{n} = (0)$ *forms a commutative Frobenius algebra object*

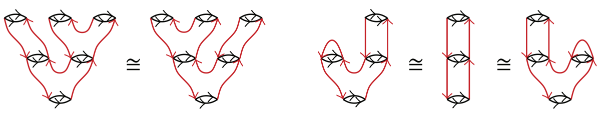
 $\cong$   $\cong$  $\cong$  (17)

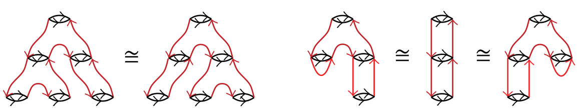
 $\cong$   $\cong$  $\cong$  (18)

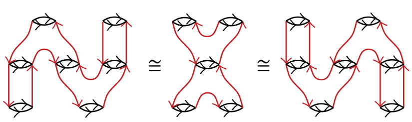
 (19)

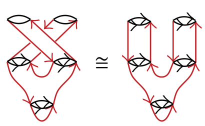
 (20)

(2) *The object* $\mathbf{n}$ = (1) *forms a symmetric Frobenius algebra object*

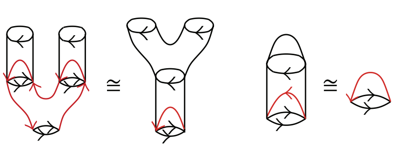
 (21)

 (22)

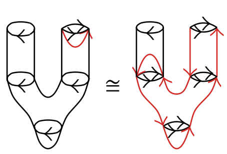
 (23)

 (24)

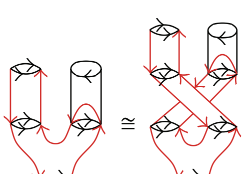
(3) *The zipper*  *forms an algebra homomorphism*

 (25)

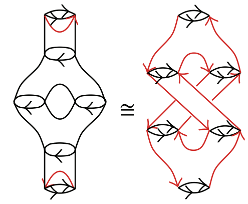
(4) *The cozipper*  *is dual to the zipper*

 (26)

(5) *Centrality relation*

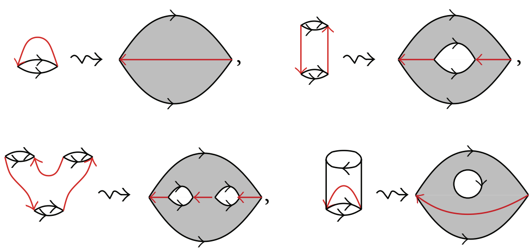
 (27)

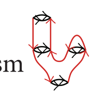
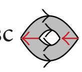
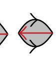
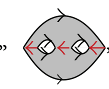
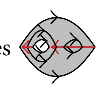
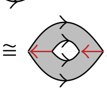
(6) *The genus-one relation*

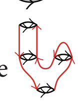
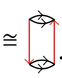
 (28)

Proof. It is well known that the first set of equivalences of cobordisms depicted in (17)–(20) hold, and it is not hard to see that also the remaining diffeomorphisms hold. We prove these using nested discs.

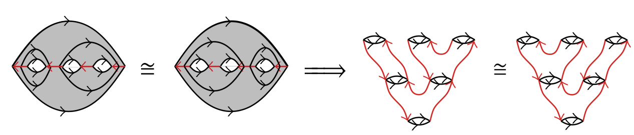
We can use punctured discs to represent singular cobordisms. For example, we have the following graphical representations:

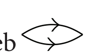
 (29)

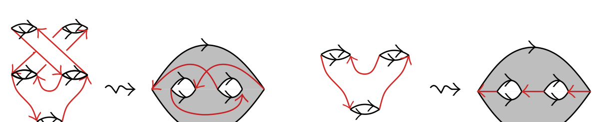
Then the singular cobordism  is interpreted as sewing in an annulus and a disc   in a "pair-of-pants" , which produces  $\cong$ .

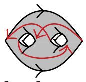
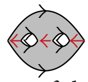
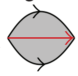
Thus we have  $\cong$ .

The associativity property depicted in (21) is obtained by making the following different decompositions of a disc with three punctures:

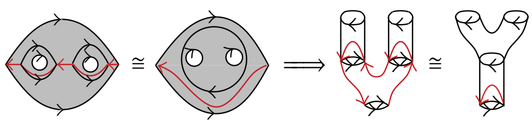
 (30)

We explain now why the biweb  = (1) forms a *symmetric* Frobenius algebra object, instead of a commutative Frobenius algebra object. For this, let us consider the following singular cobordisms and their graphical representation:

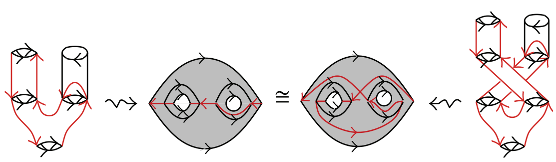
 (31)

We observe that  $\ncong$ , since the vertices of the biwebs forming the boundary of the two discs have different connection types. However, connecting the vertices of the outside biweb, thus gluing in the disc  (corresponding to the singular "cup" cobordism ) along the outside biweb, we obtain the diffeomorphism in (24).

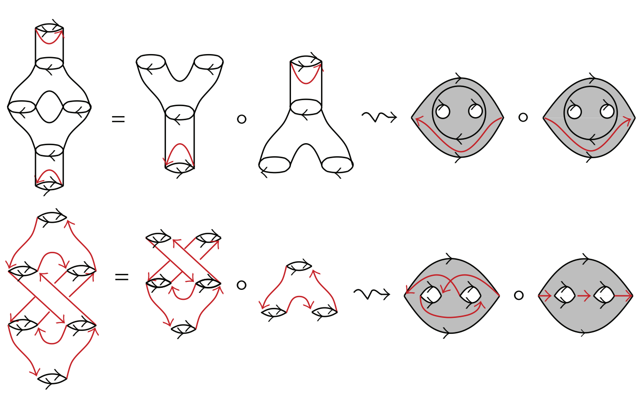
The first relation in (25) is obtained from the following two different decompositions of a disc with two punctures:

 (32)

We verify below the centrality relation given in (27) as follows:

 (33)

Finally, we prove the genus-one relation depicted in (28). The graphical representations of the two involved cobordisms are hard to draw. However, we have the following:



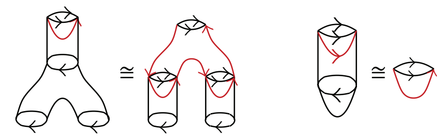
(34)

Since the two compositions of nested disks above are diffeomorphic, the genus-one relation follows.

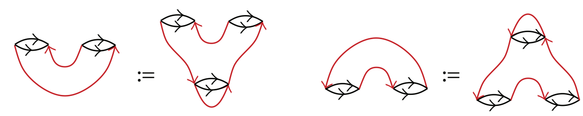
We leave to the interested reader the proof of the remaining diffeomorphisms. □

3.3. Consequences of Relations. We provide now additional diffeomorphisms implied by those described in Proposition 11, and which will be useful for the remaining of the paper.

Proposition 12. *The cozipper*  *is a coalgebra homomorphism, that is, the following singular cobordisms are equivalent:*

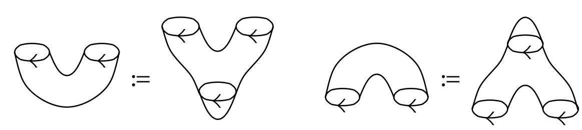
 (35)

It will be useful to define the following singular cobordisms called singular pairing and singular copairing:



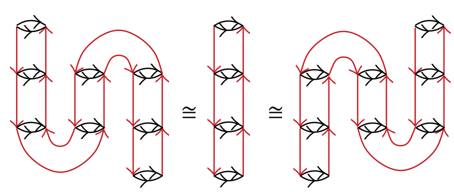
(36)

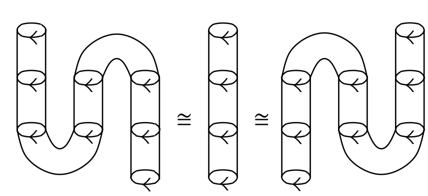
Similarly, one defines the cobordisms which one calls the ordinary pairing and ordinary copairing:



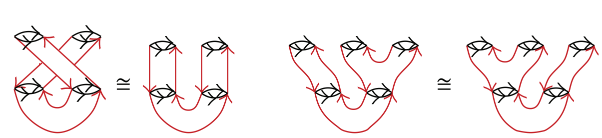
(37)

These cobordisms satisfy the zig-zag identities:

 (38)

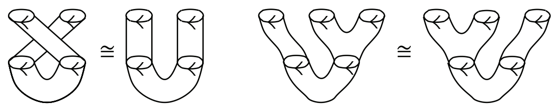
 (39)

It follows from (24) *and* (21) *that the singular pairing is invariant and symmetric:*



(40)

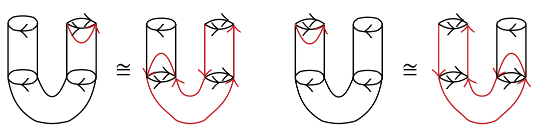
Similarly, it follows from (20) *and* (17) *that the ordinary pairing is invariant and symmetric:*



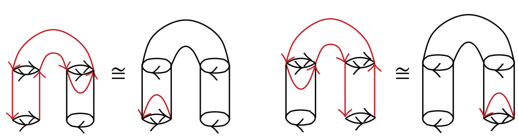
(41)

It is easy to see that similar results hold for both singular and ordinary copairings.

Proposition 13. *The following singular cobordisms are equivalent:*

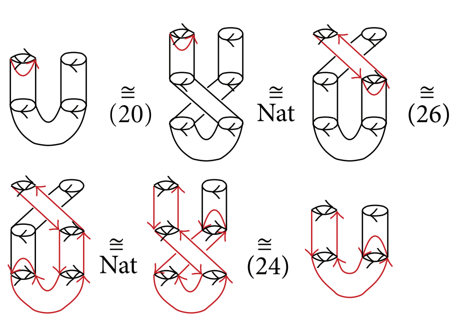


(42)



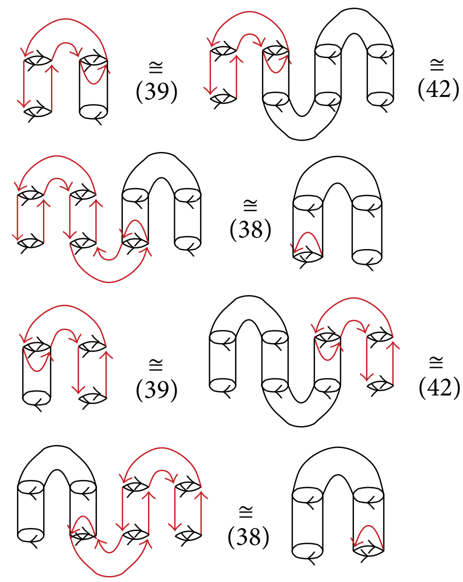
(43)

Proof. The first equivalence of cobordisms in (42) is the same as the equivalence in (26). The proof of the second diffeomorphism in (42) is given below, where by "Nat" we denote the diffeomorphisms which express the natural behavior of the symmetric twist:



(44)

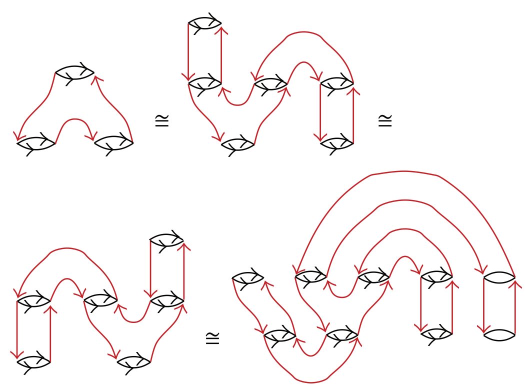
The proof of (43) is given below:



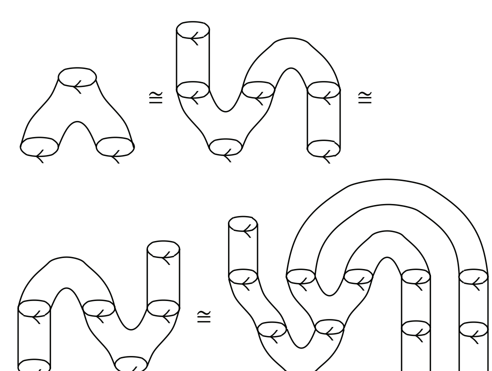
(45)

□

Proposition 14. *The following singular cobordisms are equivalent:*

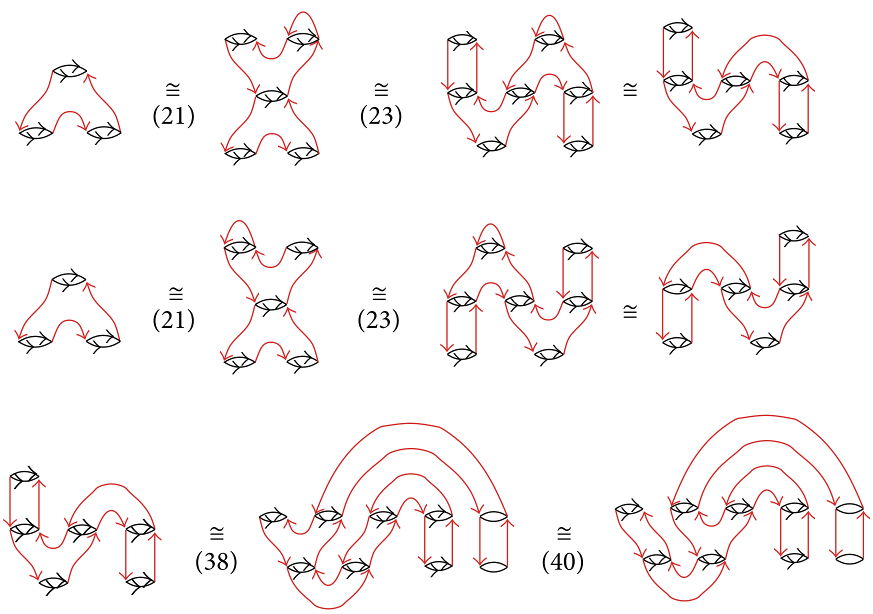


(46)



(47)

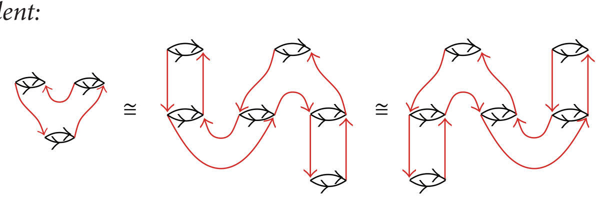
Proof. The diffeomorphisms given in (46) follow from the following sequences of diffeomorphisms:



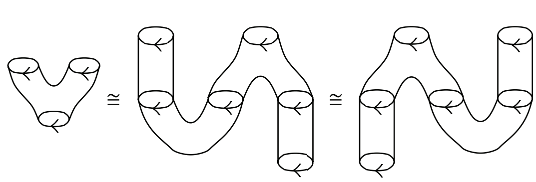
(48)

The proof of (47) is done similarly by replacing the biweb with an oriented circle, thus replacing the singular cobordisms above with their ordinary cobordisms counterparts. □

Proposition 15. *The following singular cobordisms are equivalent:*



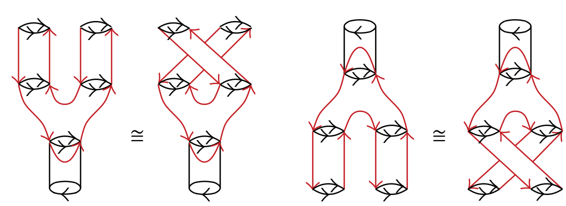
(49)



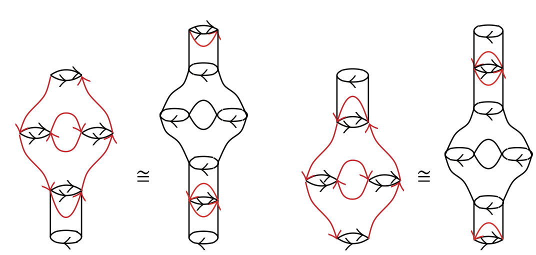
(50)

Proof. The proof of these equivalences is done in a similar manner as in the previous proposition. □

Proposition 16. *The following singular cobordisms are equivalent:*

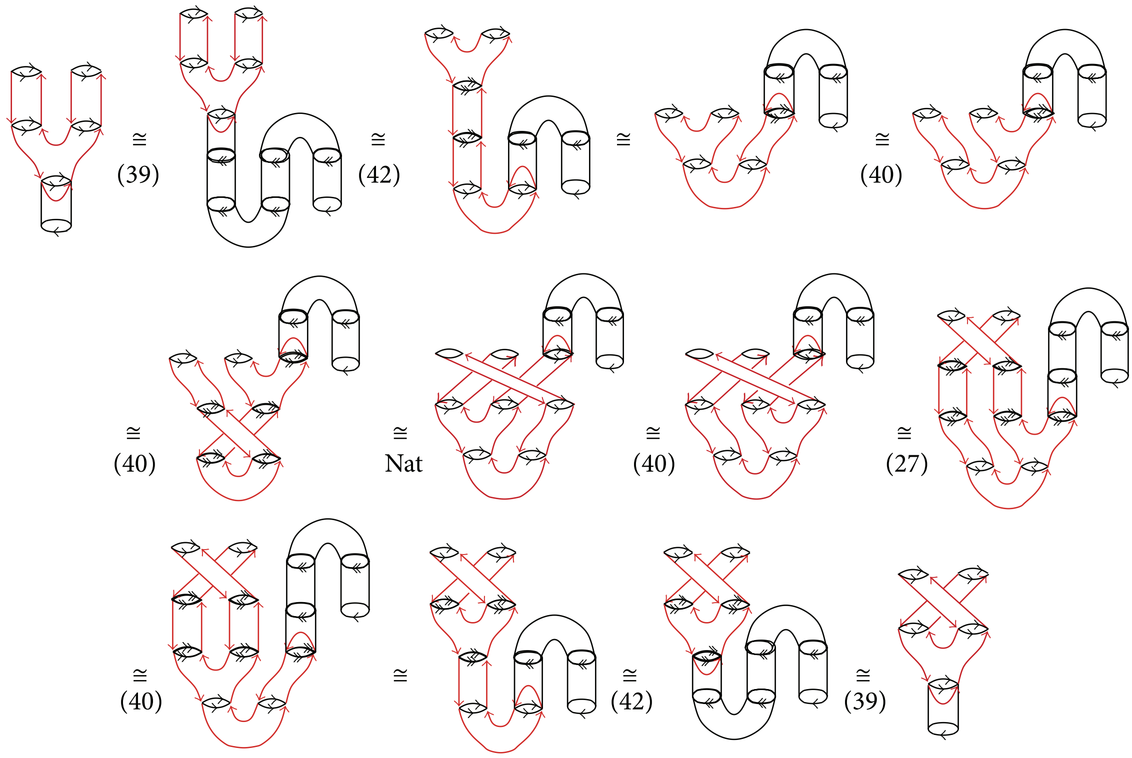


(51)



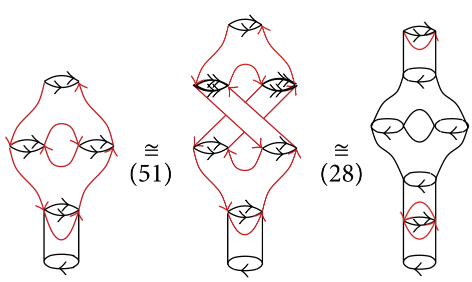
(52)

Proof. The first diffeomorphism in (51) is obtained from the following sequence of diffeomorphisms:



(53)

The first diffeomorphism depicted in (52) follows from the following sequence of diffeomorphisms:

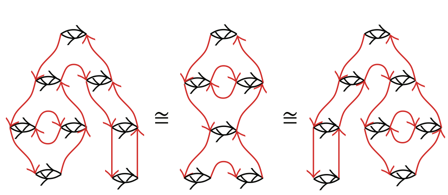


(54)

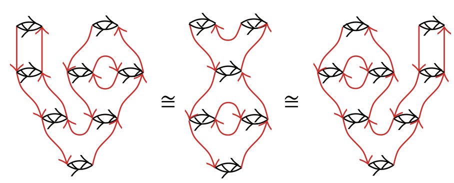
The second equivalences of singular cobordisms in (51) and (52) are obtained in a similar manner. □

It is easy to see that the next three propositions hold.

Proposition 17. *The singular genus-one operator*  *can be moved around freely in any diagram. Specifically, the following cobordisms are equivalent:*

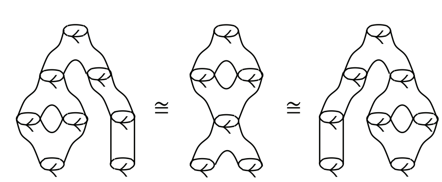


(55)

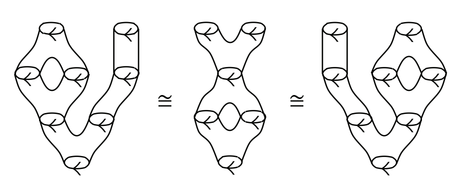


(56)

Proposition 18. *The genus-one operator*  *can be moved around freely in any diagram. Specifically, the following cobordisms are equivalent:*

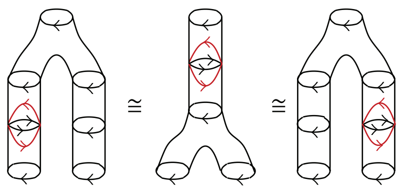


(57)

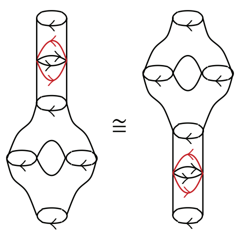


(58)

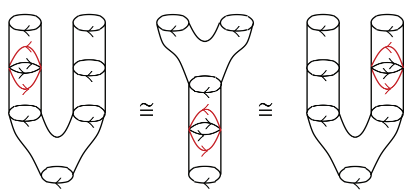
Proposition 19. *Singular cobordisms of the form*  *can be moved freely in any diagram. That is, the following singular cobordisms are equivalent:*



(59)

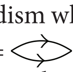
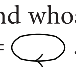


(60)



(61)

3.4. The Normal Form of a Connected Singular 2-Cobordisms. In this subsection we define the *normal form* of a connected singular cobordism Σ with a given topological structure, namely, genus, singular number, and singular boundary permutation. The reader will find many similarities with the description of the normal form of an open-closed cobordism given in [5].

3.4.1. Particular Case. We define first the normal form of a connected singular cobordism whose source consists entirely of copies of the biweb 1 = , and whose target consists entirely of copies of the circle 0 = . Specifically, we consider singular cobordisms $\Sigma : \mathbf{n} \to \mathbf{m}$ for which $\mathbf{n} = (1, 1, \ldots, 1)$ and $\mathbf{m} = (0, 0, \ldots, 0)$, and denote the set of all such cobordisms by $\mathbf{Sing\text{-}2Cob}_{W\to C}(\mathbf{n}, \mathbf{m})$. Then we give the normal form for an arbitrary connected singular cobordism by using the zig-zag identities (38) and (39).

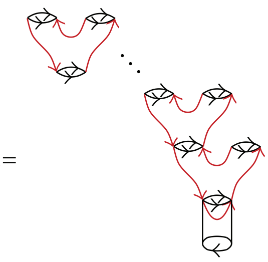
Notice that relations of the form $(\Sigma' \coprod \mathrm{id}_{\mathbf{m}}) \circ (\mathrm{id}_{\mathbf{n}'} \coprod \Sigma) = \Sigma' \coprod \Sigma$ hold in **Sing-2Cob** for any $\Sigma : \mathbf{n} \to \mathbf{m}$ and $\Sigma' : \mathbf{n}' \to \mathbf{m}'$, and we will make use of them in order to have small heights for diagrams.

Definition 20. Let $\Sigma \in \mathbf{Sing\text{-}2Cob}_{W\to C}(\mathbf{n}, \mathbf{m})$ be a connected cobordism with singular boundary permutation $\sigma(\Sigma)$, genus $g(\Sigma)$, and singular number $s(\Sigma)$ and write the singular boundary permutation as a product of disjoint cycles $\sigma(\Sigma) = \sigma_1\sigma_2\cdots\sigma_r$, $r \in \mathbb{N} \cup \{0\}$, where σ_k has length $q_k \in \mathbb{N}$, $1 \le k \le r$. The normal form of Σ is the composition

$$\mathrm{NF}_{W\to C}(\Sigma) = E_{|\mathbf{m}|} \circ D_{g(\Sigma)} \circ C_{s(\Sigma)} \circ B_r \circ \left(\coprod_{k=1}^{r} A(q_k)\right) \circ \Sigma_{\overline{\sigma(\Sigma)}} \qquad (62)$$

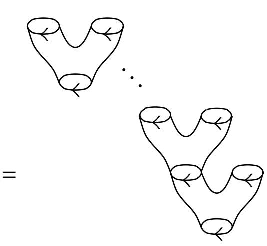
of the following singular cobordisms.

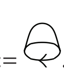
(1) For each cycle σ_k, the singular cobordism $A(q_k)$ consists of $q_k - 1$ singular multiplications followed by a cozipper, as depicted below:

$A(q_k) :=$  (63)

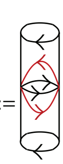
The normal form contains the free union of such cobordisms for each cycle σ_k, $1 \le k \le r$. If $q_k = 1$ then $A(q_k)$ is a cozipper, and if $|\mathbf{n}| = 0$ then $r = 0$, and the free union $\coprod_{k=1}^{r} A(q_k)$ is replaced by the empty set.

(2) If $r \ge 1$, then the singular cobordism B_r consists of $r - 1$ multiplications

$B_r :=$  (64)

If $r = 0$ then $B_0 :=$ .

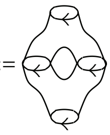
(3) If the singular number $s(\sigma) \ge 1$, the singular cobordism $C_{s(\sigma)}$ is the composite

$$C_{s(\Sigma)} := \underbrace{S \circ S \circ \cdots \circ S}_{s(\Sigma)}, \quad \text{where } S :=$$ . (65)

If $s(\Sigma) = 0$ then $C_{s(\Sigma)} = \emptyset$.

(4) If $g(\Sigma) \geq 1$, the singular cobordism $D_{g(\Sigma)}$ is the composite

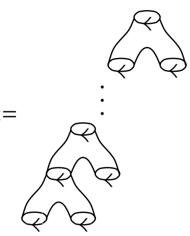
$$D_{g(\Sigma)} := \underbrace{G \circ G \circ \cdots \circ G}_{g(\Sigma)}, \quad \text{where } G := \tag{66}$$



If $g(\Sigma) = 0$ then $D_{s(\Sigma)} = \emptyset$.

(5) If $|\mathbf{m}| \geq 1$, then the singular cobordism $E_{|\mathbf{m}|}$ consists of $|\mathbf{m}| - 1$ comultiplications, as depicted below:

$$E_{|\mathbf{m}|} := \tag{67}$$



If $|\mathbf{m}| = 0$ then $E_0 :=$ .

(6) $\Sigma_{\overline{\sigma(\Sigma)}}$ represents the permutation singular cobordism induced by the permutation $\overline{\sigma(\Sigma)}$ given below (permutation cobordisms were defined in (14)). Let $\tau(\Sigma)$ be the singular boundary permutation of the cobordism

$$E_{|\mathbf{m}|} \circ D_{g(\Sigma)} \circ C_{s(\Sigma)} \circ B_r \circ \left(\coprod_{k=1}^{r} A(q_k) \right). \tag{68}$$

Then $\overline{\sigma(\Sigma)}$ is the permutation that satisfies

$$\sigma(\Sigma) = \overline{\sigma(\Sigma)}^{-1} \cdot \tau(\Sigma) \cdot \overline{\sigma(\Sigma)}. \tag{69}$$

Note that precomposing $E_{|\mathbf{m}|} \circ D_{g(\Sigma)} \circ C_{s(\Sigma)} \circ B_r \circ (\coprod_{k=1}^{r} A(q_k))$ with $\Sigma_{\overline{\sigma(\Sigma)}}$ yields a singular cobordism whose singular boundary permutation is $\sigma(\Sigma)$.

In Figure 1 we show a cobordism of the form (68), that is, the normal form of a cobordism in $\mathbf{Sing\text{-}2Cob}_{W \to C}(\mathbf{n}, \mathbf{m})$, without precomposition with $\Sigma_{\overline{\sigma(\Sigma)}}$.

The following two results say that a cobordism given in its normal form is invariant, up to equivalence, under composition with certain permutation morphisms.

Proposition 21. *Let* $[\Sigma] \in \mathbf{Sing\text{-}2Cob}_{W \to C}(\mathbf{n}, \mathbf{m})$. *Then*

$$\begin{aligned} [\sigma^{\mathbf{m}} \circ NF_{W \to C}(\Sigma)] &= [NF_{W \to C}(\Sigma)] \\ &= [NF_{W \to C}(\Sigma) \circ \sigma_k^{\mathbf{n}}] \end{aligned} \tag{70}$$

for any $\sigma \in S_{|\mathbf{m}|}$ *and for all cycles* $\sigma_k \in S_{|\mathbf{n}|}$, $1 \leq k \leq r$, *that appear in the decomposition of* $\sigma(\Sigma) = \sigma_1 \sigma_2 \cdots \sigma_r$ *into disjoint cycles.*

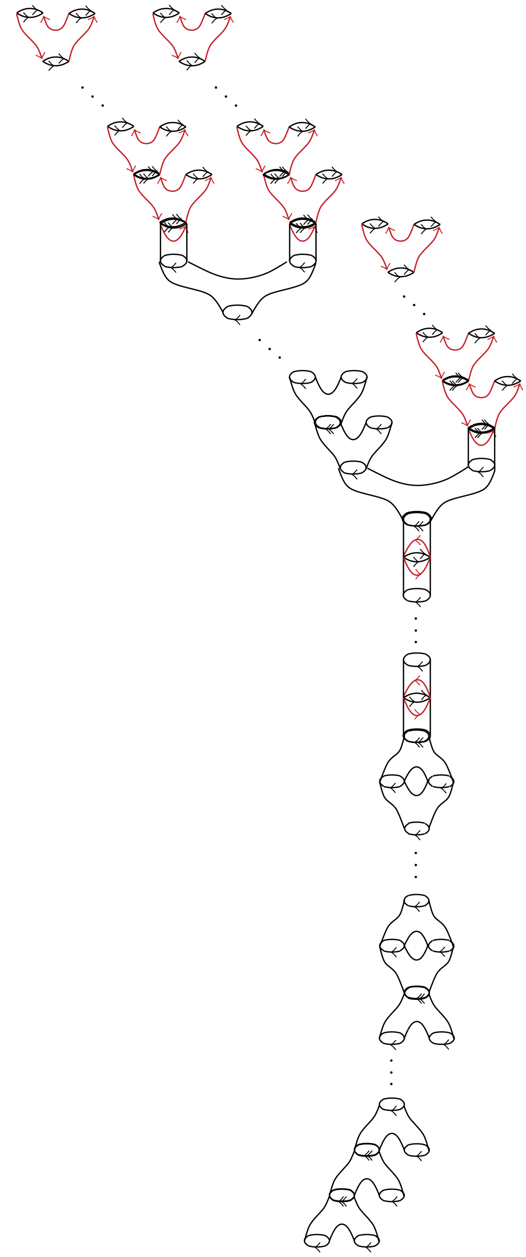


Figure 1: Normal form of a cobordism in $\mathbf{Sing\text{-}2Cob}_{W \to C}(\mathbf{n}, \mathbf{m})$, without precomposition with a permutation.

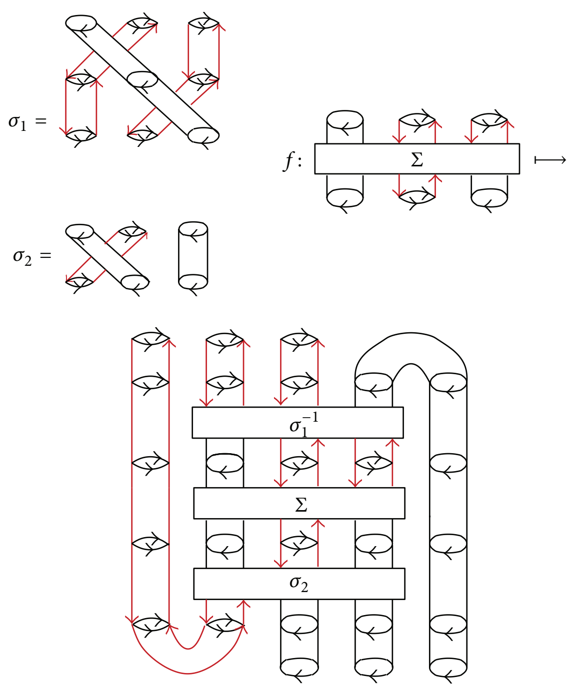
3.4.2. General Case. We use the normal form for a connected singular cobordism in $\mathbf{Sing\text{-}2Cob}_{W \to C}(\mathbf{n}, \mathbf{m})$ and the duality property for the biweb and the circle to obtain the normal form of a generic connected morphism $[\Sigma] \in \mathbf{Sing\text{-}2Cob}(\mathbf{n}, \mathbf{m})$.

Let Σ be a representative of the equivalence class $[\Sigma]$ and let $\mathbf{n}_0 \coprod \mathbf{n}_1$ be the permutation of $\mathbf{n}$ such that $\mathbf{n}_0 = (0, 0, \ldots, 0)$ and $\mathbf{n}_1 = (1, 1, \ldots, 1)$. Similarly, let $\mathbf{m}_0 \coprod \mathbf{m}_1$ be the permutation of $\mathbf{m}$ such that $\mathbf{m}_0 = (0, 0, \ldots, 0)$ and $\mathbf{m}_1 = (1, 1, \ldots, 1)$. In order to use the normal form described above, we need to associate to $[\Sigma]$ a singular cobordism whose source contains only copies of the biweb and whose target contains only copies of the circle. We define the map

$$\begin{aligned} f &: \mathbf{Sing\text{-}2Cob}\,(\mathbf{n}, \mathbf{m}) \\ &\longrightarrow \mathbf{Sing\text{-}2Cob}_{W\to C}\left(\mathbf{m}_1 \coprod \mathbf{n}_1, \mathbf{m}_0 \coprod \mathbf{n}_0\right), \\ &\quad [\Sigma] \longmapsto f\,([\Sigma]), \end{aligned} \tag{71}$$

where the singular cobordism $f([\Sigma])$ is defined as follows. Let σ_1 be the permutation cobordism corresponding to $\sigma_1 \in S_{|\mathbf{n}|}$ that sends $\mathbf{n}$ to $\mathbf{n}_1 \coprod \mathbf{n}_0$. Similarly, denote by σ_2 the cobordism corresponding to the permutation $\sigma_2 \in S_{|\mathbf{m}|}$ that sends $\mathbf{m}$ to $\mathbf{m}_1 \coprod \mathbf{m}_0$. We define $f([\Sigma])$ as the singular cobordism obtained from $[\Sigma]$ by precomposing with σ_1^{-1}, postcomposing with σ_2 and gluing copairings on every circle that $\mathbf{n}_0$ contains, and singular pairings on every biweb that $\mathbf{m}_1$ contains.

For exemplification, consider $\mathbf{n} = (0, 1, 1)$, $\mathbf{m} = (0, 1, 0)$ and $[\Sigma]$ an arbitrary singular cobordism from $\mathbf{n}$ to $\mathbf{m}$ (note that $|\mathbf{n}|$ and $|\mathbf{m}|$ do not have to be equal). The corresponding permutation cobordisms σ_1, σ_2 and the image of $[\Sigma]$ under f are given in (72).



(72)

Notice that the mapping f is well defined; namely, $f([\Sigma])$ is a morphism in the category $\mathbf{Sing\text{-}2Cob}_{W\to C}(\mathbf{m}_1 \coprod \mathbf{n}_1, \mathbf{m}_0 \coprod \mathbf{n}_0)$, and $[f([\Sigma])] = [f([\Sigma'])]$ whenever $[\Sigma] = [\Sigma']$. Therefore it makes sense to consider the normal form $\mathrm{NF}_{W\to C}(f([\Sigma]))$.

We also remark that $f([\Sigma])$ has a certain structure, in the sense that its source $\mathbf{n}'$ and target $\mathbf{m}'$ can be decomposed into free unions $\mathbf{n}' = \mathbf{n}'_t \coprod \mathbf{n}'_s$ and $\mathbf{m}' = \mathbf{m}'_t \coprod \mathbf{m}'_s$, such that the copies of the biweb in $\mathbf{n}'_t$ (or $\mathbf{n}'_s$) and the copies of the circle in $\mathbf{m}'_t$ (or $\mathbf{m}'_s$) correspond to the copies of the biweb and of the circle coming from the target (or source) of $\sigma_2 \circ \Sigma \circ \sigma_1^{-1}$. The permutation σ_1 is an element of $S_{|\mathbf{n}'_s|+|\mathbf{m}'_s|}$, while σ_2 is an element of $S_{|\mathbf{n}'_t|+|\mathbf{m}'_t|}$.

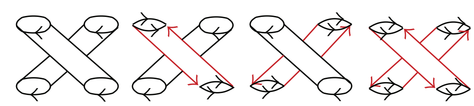
We define an inverse mapping f^{-1} that associates to $[\Phi] \in \mathbf{Sing\text{-}2Cob}_{W\to C}(\mathbf{n}', \mathbf{m}')$ the singular cobordism $f^{-1}([\Phi]) \in \mathbf{Sing\text{-}2Cob}_{W\to C}(\sigma_2(\mathbf{n}'_t \coprod \mathbf{m}'_t), \sigma_1(\mathbf{n}'_s \coprod \mathbf{m}'_s))$. The cobordism $f^{-1}([\Phi])$ is obtained by gluing singular copairings to the biwebs in $\mathbf{n}'_t$ and ordinary pairings to the circles in $\mathbf{m}'_s$, and then by precomposing the resulting cobordism with the cobordism corresponding to σ_1 and by postcomposing it with the cobordism corresponding to σ_2^{-1}. This map is well defined and provides a bijection between the morphisms in $\mathbf{Sing\text{-}2Cob}(\mathbf{n}, \mathbf{m})$ and those in $\mathbf{Sing\text{-}2Cob}_{W\to C}(\mathbf{m}_1 \coprod \mathbf{n}_1, \mathbf{m}_0 \coprod \mathbf{n}_0)$.

Going back to the example in (72), we give in **Figure 2** the singular cobordism $[f^{-1}([f([\Sigma])])] \cong [\Sigma]$.

Definition 22. Let $[\Sigma] \in \mathbf{Sing\text{-}2Cob}(\mathbf{n}, \mathbf{m})$ where Σ is a connected cobordism. One defines the normal form of $[\Sigma]$ by

$$[\mathrm{NF}\,(\Sigma)] := f^{-1}\left([\mathrm{NF}_{W\to C}\left(f\,([\Sigma])\right)]\right). \tag{73}$$

3.5. Nonconnected Singular 2-Cobordisms. We treat the case of nonconnected cobordisms via disjoint unions and permutations of the factors of disjoint unions, following Kock's work [4] for the case of ordinary 2-cobordisms. Since every permutation can be written as a product of transpositions, the following singular cobordisms are sufficient to do this:



(74)

There is no need to talk about crossing over or under, since our cobordisms are abstract manifolds, thus not embedded anywhere.

Without loss of generality, we assume that $\Sigma : \mathbf{n} \to \mathbf{m}$ has two connected components, Σ_1 and Σ_2, and that $\mathbf{n} = (n_1, n_2, \ldots, n_{|n|})$. The source boundary of Σ_1 is a tuple $\mathbf{p}$ whose components form a subset of $\{n_1, n_2, \ldots, n_{|n|}\}$, and the source boundary of Σ_2 is the tuple $\mathbf{q}$, which is the complement of $\mathbf{p}$ in $\{n_1, n_2, \ldots, n_{|n|}\}$.

We can permute the components of $\mathbf{n}$ by applying a diffeomorphism $\mathbf{n} \to \mathbf{n}$, so that the components of $\mathbf{p}$ come before those of $\mathbf{q}$. This diffeomorphism induces a cobordism S, and we can consider the singular cobordism $S\Sigma$. Applying the same method to the target boundary of Σ, which is also the target boundary of $S\Sigma$, there is a permutation singular cobordism $T : \mathbf{m} \to \mathbf{m}$ so that $\Sigma' = S\Sigma T : \mathbf{n} \to \mathbf{m}$ is a singular cobordism which is the disjoint union (as a cobordism) of Σ_1 and Σ_2. Then $\Sigma \cong S^{-1}\Sigma' T^{-1}$, where S^{-1} and T^{-1} are the permutation cobordisms which are the inverses of S and T, respectively. For example, S^{-1} is the diffeomorphism that permutes the components of $\mathbf{n}$ such that the components of $\mathbf{p}$ come after those of $\mathbf{q}$.

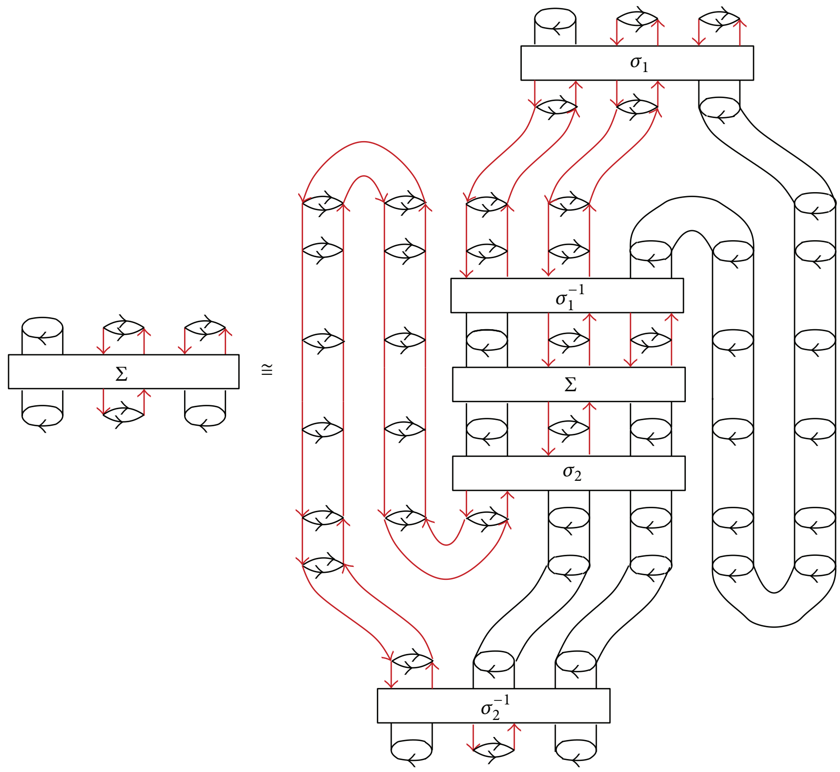
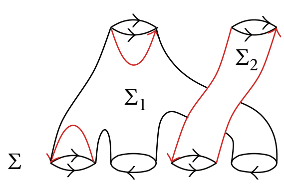


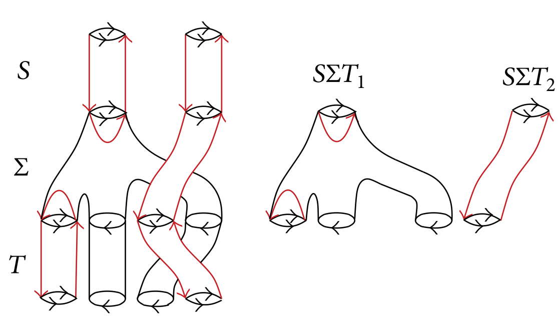
Figure 2

As an example, we consider the following singular cobordism:



(75)

For the given cobordism we do not need to permute the source boundary of Σ, thus S is the disjoint union (as a cobordism) of two cylinders, but we do permute the target boundary of Σ by composing with a cobordism T. The composed cobordism $S\Sigma T$ is the disjoint union of its connected components $(S\Sigma T)_1$ and $(S\Sigma T)_2$:



(76)

We have proved the following.

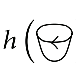
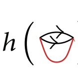
Lemma 23. *Every singular cobordism is equivalent to a composition of a permutation cobordism with a disjoint union of connected cobordisms, followed by a permutation cobordism.*

3.6. Sufficiency of the Relations. In this subsection, we show that the relations described in Proposition 11 are sufficient in order to relate any connected singular cobordism $[\Sigma] \in \mathbf{Sing}\text{-}\mathbf{2Cob}_{W\to C}(\mathbf{n},\mathbf{m})$ to its normal form $\mathrm{NF}_{W\to C}(\Sigma)$.

We use the notation $\underline{\boxed{\mathbf{X}}}$ (or $\overline{\boxed{\mathbf{X}}}$) for an arbitrary singular cobordism X whose target (or source) is not glued to any other cobordism in the decomposition of Σ.

The following terminology is borrowed from [5, Definition 3.21].

Definition 24. Let $[\Sigma] \in \mathbf{Sing}\text{-}\mathbf{2Cob}(\mathbf{n},\mathbf{m})$ be connected. The *height* of a generator G in the decomposition of Σ is the following number defined inductively:

$h\Big($  $\Big) = h\Big($  $\Big) = h\left(\underline{\boxed{\mathbf{X}}}\right) := 0,$

$$h\left(\begin{array}{c}\boxed{\mathbf{G}}\\ \boxed{\mathbf{X}}\end{array}\right) = 1 + h(X),$$

$$h\left(\begin{array}{c}\boxed{\mathbf{G}}\\ \boxed{\mathbf{X}}\;\boxed{\mathbf{Y}}\end{array}\right) = 1 + h(X) + h(Y), \tag{77}$$

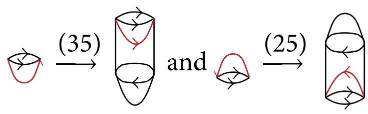
where X and Y are arbitrary cobordisms in the decomposition of Σ.

Theorem 25. *Let* $[\Sigma] \in \mathbf{Sing\text{-}2Cob}_{W\to C}(\mathbf{n}, \mathbf{m})$ *be a connected singular 2-cobordism. Then* Σ *is equivalent to its normal form; namely, one has*

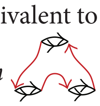
$$[\Sigma] = [NF_{W\to C}(\Sigma)]. \tag{78}$$

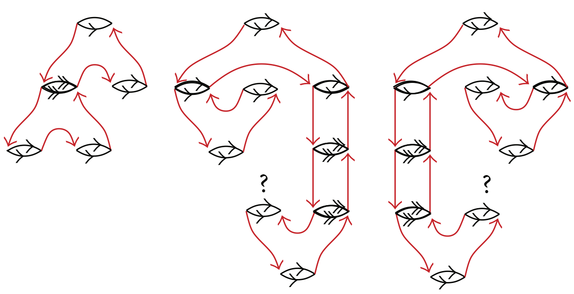
Proof. The proof is similar in spirit to that of [5, Theorem 3.22], with the difference that it uses our cobordisms and their topological structure. We consider Σ be given in an arbitrary decomposition and construct a step by step diffeomorphism (relative to boundary) from this decomposition to the normal form $\mathrm{NF}_{W\to C}(\Sigma)$.

(I) The decomposition of Σ is equivalent to one without *singular cups*  and *singular caps* , by applying the following diffeomorphism:



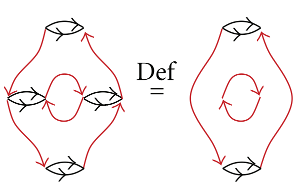
(79)

(II) The decomposition of Σ is equivalent to one in which every *singular comultiplication*  has its target in one of the following situations:



(80)

or

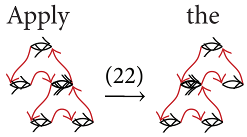


(81)

where the symbol "?" is any singular cobordism which may or may not be connected to the singular multiplication at the bottom of the diagram. This is proved by considering every possible situation in which the singular comultiplication may appear.

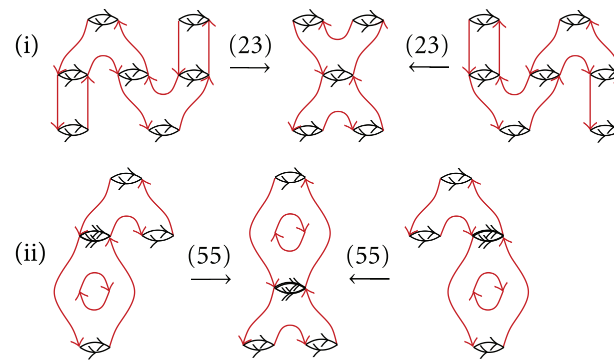
(a) We can exclude the cases  and  by step (I).

(b) Apply the diffeomorphism



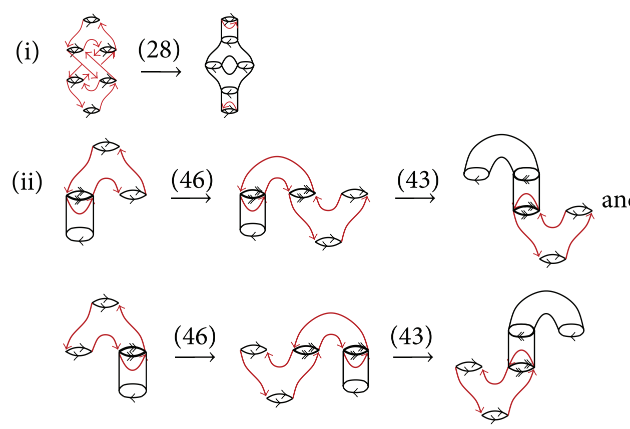
whenever possible.

(c) The following diffeomorphisms reduce the height of the singular comultiplication:



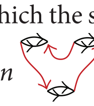
(82)

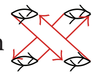
(d) The following diffeomorphisms eliminate the singular comultiplication:

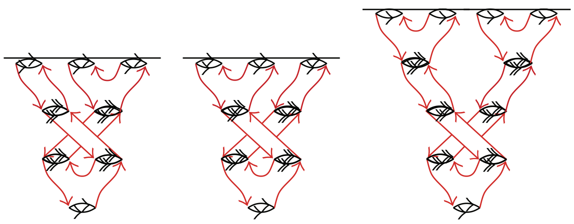


(83)

(e) Iterate steps (II)(a)–(II)(d). Since each step either removes the singular comultiplication or reduces its height, and since the target of Σ does not contain biwebs, this process terminates with every singular comultiplication in one of the situations described above.

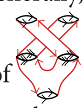
(III) We look now at the possible cases in which the source and target of the *singular multiplication*  may appear.

We remark first that we do not need to consider the case , since the part $\Sigma_{\overline{\sigma(\Sigma)}}$ in the definition of the normal form of Σ (see (62)) is taking care of the twist cobordism . Similarly, the cases

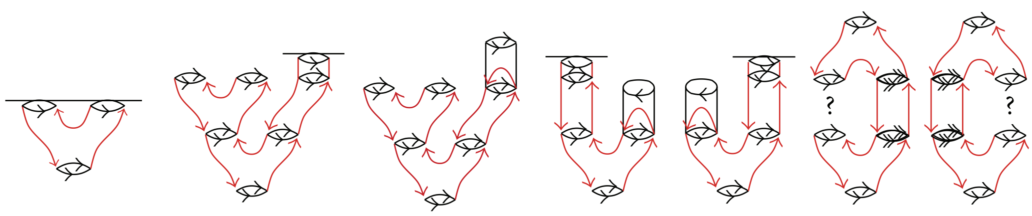


(84)

can be excluded by moving upwards the twist past the singular multiplication(s) (i.e., by the naturality of the twist). In each case, the corresponding diffeomorphism yields a cobordism equivalent to Σ, whose decomposition starts with a permutation cobordism, which will be reflected in the $\Sigma_{\overline{\sigma(\Sigma)}}$ part of the normal form.

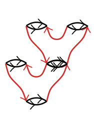
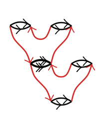
More generally, we use this method whenever the source of  has only singular multiplications and identities above it. If there are also singular comultiplications, we apply step (IV) below. Therefore, we have the following.

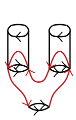
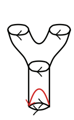
(a) The decomposition of Σ is equivalent to one in which the source of the singular multiplication appears in one of the following situations:

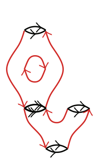
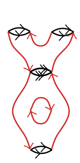
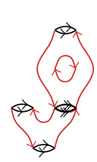


(85)

or that depicted in (81). One can see that this claim holds by considering all possible situations of the singular comultiplication and then applying the following diffeomorphisms. Each iteration of the following steps either removes the singular multiplication or increases its height. As a result, each singular multiplication ends up into one of the situations above.

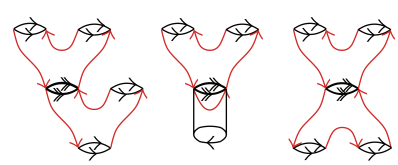
(i)  $\xrightarrow{(21)}$ 

(ii)  $\xrightarrow{(25)}$ 

(iii)  $\xrightarrow{(56)}$  $\xleftarrow{(56)}$ 

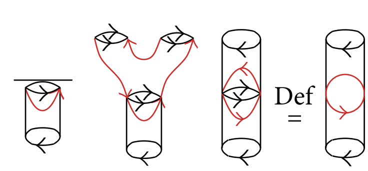
(iv) The diffeomorphisms of (IIc)(i) and (IId)(i).

(b) The decomposition of Σ is equivalent to one in which the target of the singular multiplication appears in one of the following situations:



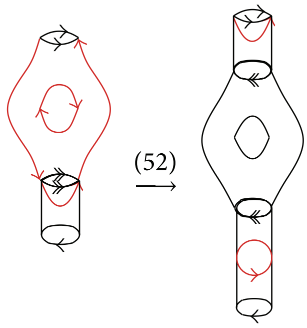
(86)

(i) We first show that the source of every cozipper  can be put so that it appears in one of the following situations:



(87)

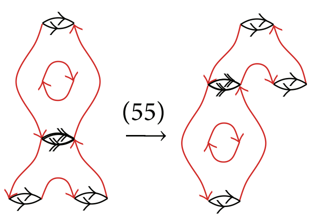
This claim is proved by applying the steps (I), (IId)(ii), and the diffeomorphism:



(88)

whenever it is possible.

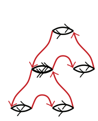
(ii) The singular genus-one operator  can be removed by iterating the step (III)(a)(iii), the diffeomorphism given in (88) and the following diffeomorphism:

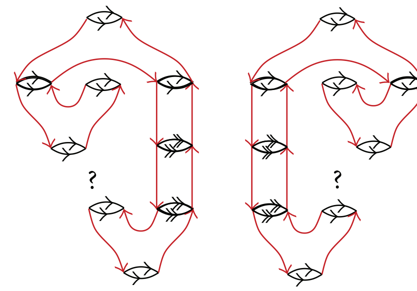


(89)

This process either reduces the height of the singular genus-one operator or removes it. Since the height of the operator cannot be zero, the process guarantees to remove the singular genus-one operator.

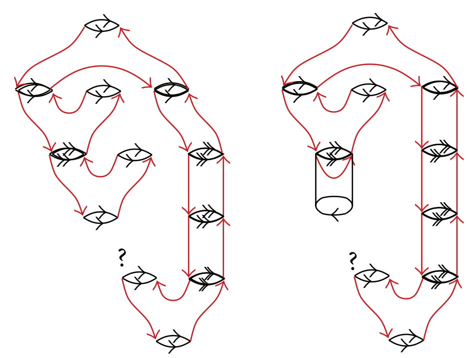
(IV) In this step we show that there exists a sequence of diffeomorphisms that removes all singular comultiplications. Consider the set of all such comultiplications that appear in the decomposition of Σ and choose one of minimal height.

We can exclude the case  since the singular comultiplication is of minimal height. From steps (II) and (III)(b)(ii) we know that the remaining situations to consider are

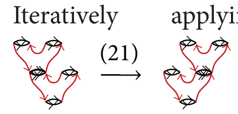


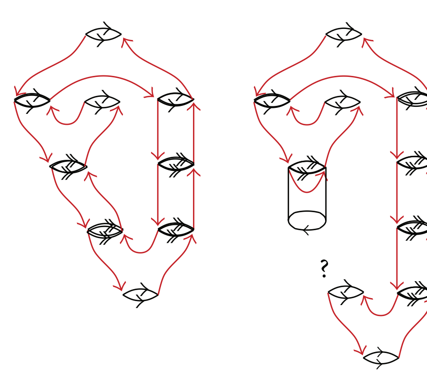
(90)

where "?" may be any singular cobordism that contains no singular comultiplication. Since the above cobordisms are symmetric, it is enough to consider only one case, say the first one. Using step (III)(b) and the assumption that the singular comultiplication is of minimal height, there are exactly two possible situations for the first generator in the decomposition of "?," namely,



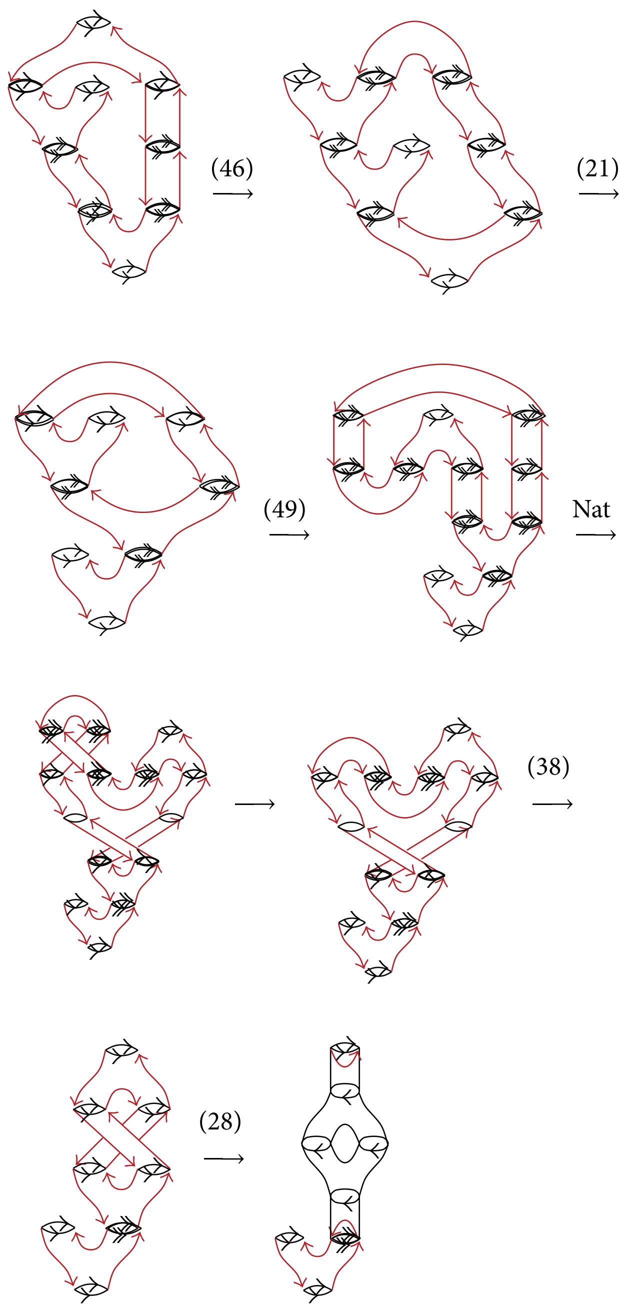
(91)

Iteratively applying the diffeomorphism  and considering again the next two possible situations in the decomposition of "?," we see that after all, there are the following two possible cases:



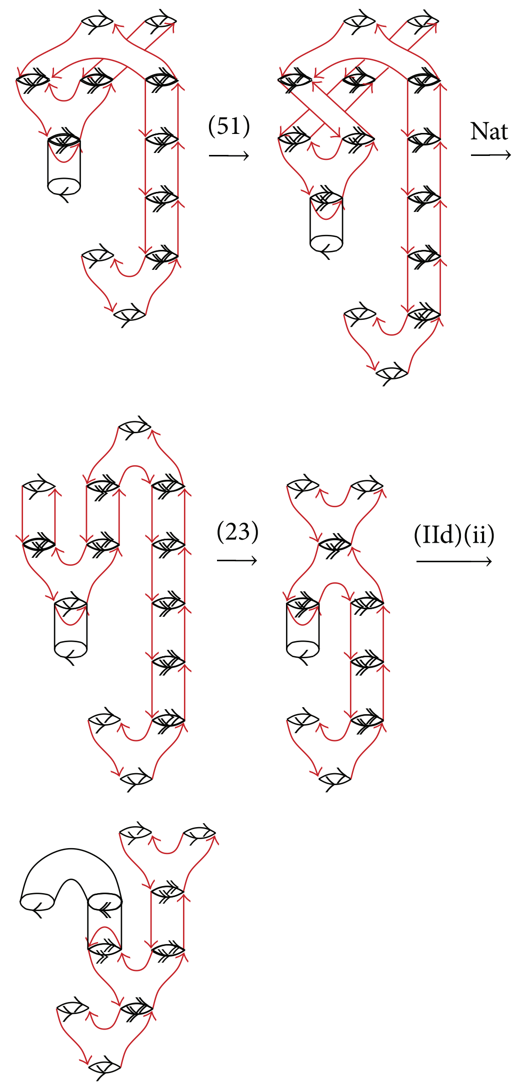
(92)

(a) In the first case, the singular comultiplication is eliminated by applying the following sequence of diffeomorphisms:



(93)

(b) To remove the singular comultiplication in the second case above, we apply the following sequence of diffeomorphisms:



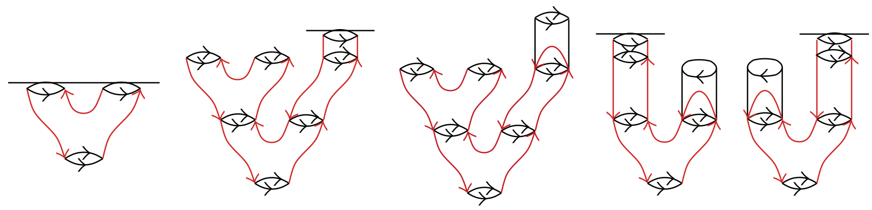
(94)

(c) We reapply steps (II) and (III) if needed.

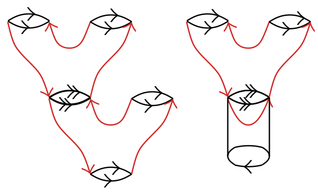
(d) We iterate steps (IV)(a)-(b) and (IV)(c) until the last singular comultiplication has been eliminated. We remark that this process will terminate after a finite number of iterations, since steps (II) and (III) do not increase the number of singular comultiplications.

(V) After the first four steps of the proof, all singular cups, caps, and comultiplications have been eliminated from the decomposition of Σ, and the resulting decomposition has the following properties.

(a) Every singular multiplication has its source in one of the following situations:


(95)

and its target in one of the following situations:

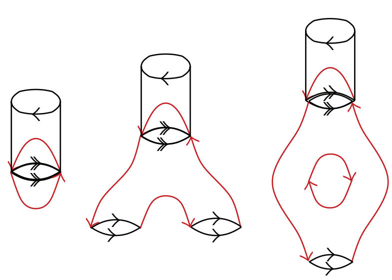

(96)

(b) Every cozipper  appears in one of the situations explained in (IIIb)(i).

(c) Every singular genus-one operator  has been eliminated from the decomposition.

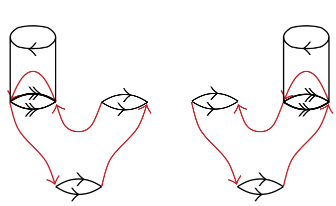
(VI) We show now that the zipper  can be eliminated from the decomposition of Σ or that its target can be put in the situation , thus eliminated completely by defining  := .

The following situations:

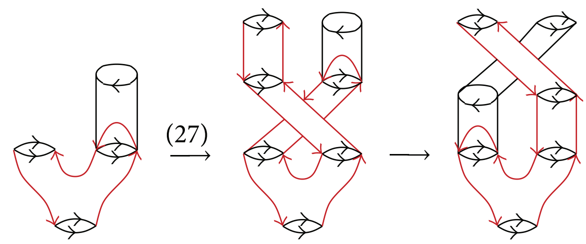

(97)

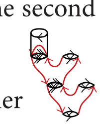
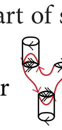
□

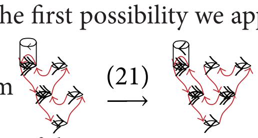
are excluded by steps (I), (IV), and (IIIb)(ii), respectively. It remains to consider the cases:


(98)

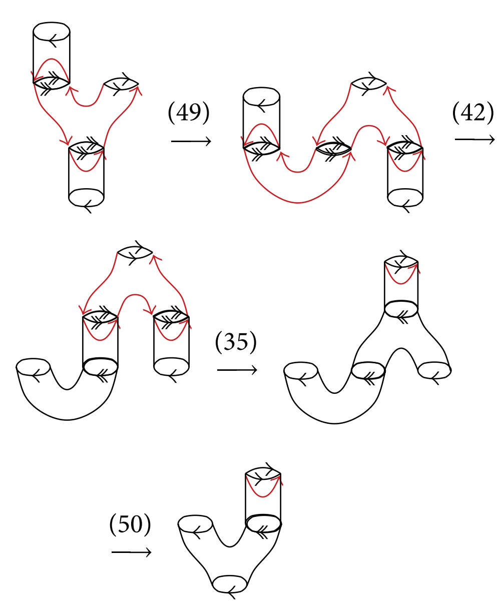
The second case can be reduced to the first one by applying the following sequence of diffeomorphisms:

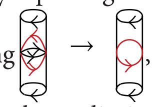

(99)

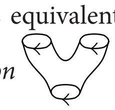
Then, by taking into account the second part of step (V)(a), we need to consider either  or .

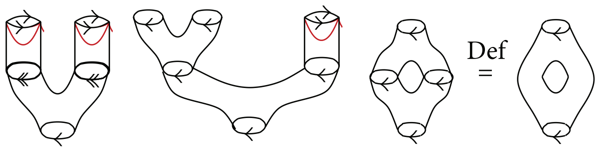
(a) For the first possibility we apply the diffeomorphism  which reduces the height of the zipper.

(b) For the second possibility we employ a sequence of diffeomorphisms that removes the zipper:


(100)

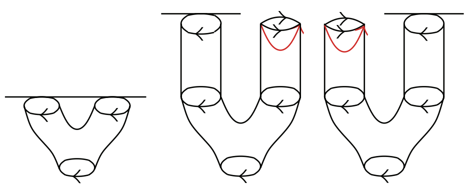
By repeating these steps if necessary and applying , we guarantee that the zipper  has been eliminated from the decomposition of Σ.

(VII) The (resulting) decomposition of Σ is equivalent to one in which the *ordinary multiplication*  has its source in one of the following situations:



(101)

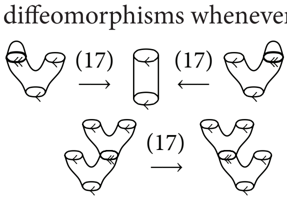
(a) We can exclude the cases



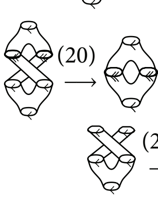
(102)

since we assume that the source of Σ is a free union of biwebs.

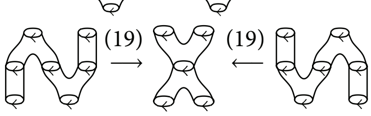
To prove the claim we iterate the following diffeomorphisms whenever possible:

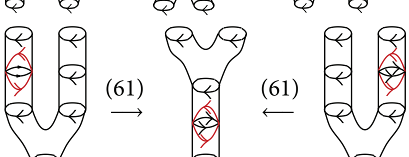
(b) 

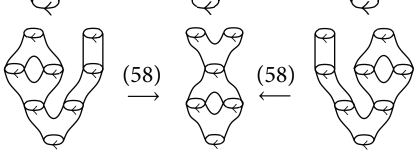
and

(c) 

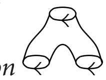
and, more generally,

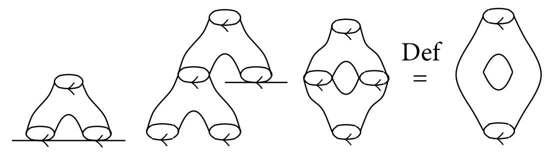
(d) 

(e) 

(f) 

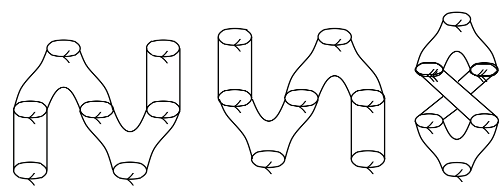
Each of the above diffeomorphisms either removes the ordinary multiplication or increases its height, therefore applying these moves whenever possible assures that the process ends with each ordinary multiplication in the decomposition of Σ in one of the claimed situations.

(VIII) The resulting decomposition of Σ is equivalent to one in which each *ordinary comultiplication*  is in one of the following situations:



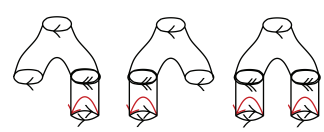
(103)

(a) Employing step (VII), we can exclude the cases:



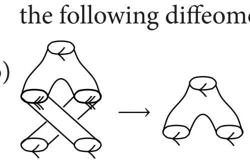
(104)

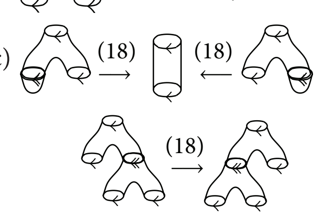
Moreover, every zipper has been eliminated at step (VI), thus we can also exclude

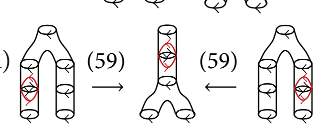


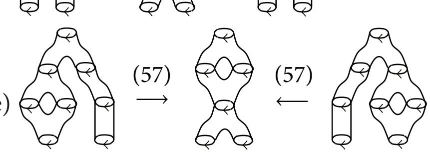
(105)

The claim follows by iterating whenever possible the following diffeomorphisms:

(b) 

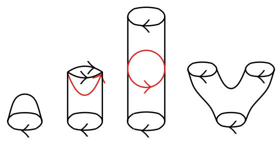
(c) 

(d) 

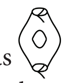
(e) 

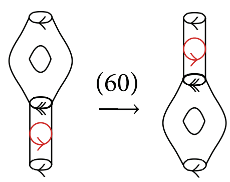
Notice that each of the above diffeomorphisms either decreases the height of the ordinary comultiplication or removes it, thus the process must end after a finite number of iterations.

(IX) The cobordism with a singular circle  can be put in an equivalent decomposition of Σ, so that it has above it one of the following cobordisms:



(106)

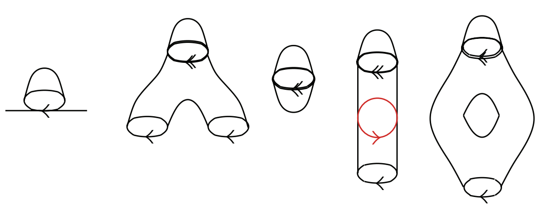
The case in which  has above it the ordinary comultiplication is excluded by step (VIII)(d), while the case in which it has  above it is eliminated by iterating the diffeomorphism



(107)

(X) We claim that the resulting decomposition of Σ is now in the normal form. This follows from steps (V)(a), (VI), (VII), (VIII), and (IX), and the following two remarks.

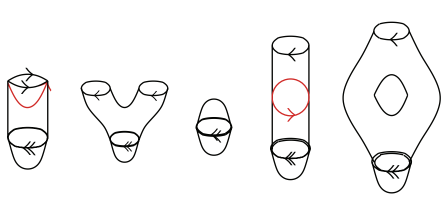
(a) Whenever an *ordinary cap*  appears in the resulting decomposition of Σ, then  has its target in one of the following situations:



(108)

The other situations are excluded by steps (VI) and (VII)(b).

(b) Whenever an *ordinary cup*  appears in the resulting decomposition of Σ, then the source of  is in one of the following situations:



(109)

The other situations are excluded by step (VIII)(c). This completes the proof.

Corollary 26. *Let* $[\Sigma] \in$ **Sing-2Cob(n, m)** *be connected. Then* $[\Sigma] = [NF(\Sigma)]$.

Proof.

$$\begin{aligned} [\mathrm{NF}(\Sigma)] &= \left[f^{-1}\left(\left[\mathrm{NF}_{W\to C}\left(f\left([\Sigma]\right)\right)\right]\right)\right] \\ &= \left[f^{-1}\left(\left[f\left([\Sigma]\right)\right]\right)\right] = [\Sigma]. \end{aligned} \tag{110}$$

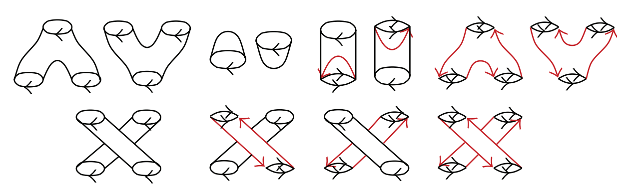
□

Corollary 27. *If* $[\Sigma], [\Sigma'] \in$ **Sing-2Cob(n, m)** *are connected cobordisms with the same singular boundary permutation, genus, and singular number, then* $[\Sigma] = [\Sigma']$.

Proof. This follows at once from the fact that the normal form of a singular cobordism is characterized by the singular boundary permutation, genus, and singular number of the cobordism. □

Putting together the results of this section, we obtain the following.

Theorem 28. *The symmetric monoidal category* **Sing-2Cob** *is generated (under composition and disjoint union) by the following singular 2-cobordisms:*



(111)

with relations given in Proposition 11.

4. Twin TQFTs

In this section we define the notion of twin TQFTs and show that the category of twin TQFTs is equivalent to the category of twin Frobenius algebras.

Definition 29. Let $\mathscr{C}$ be a symmetric monoidal category. A *twin Topological Quantum Field* Theory (TQFT) in $\mathscr{C}$ is a symmetric monoidal functor **Sing-2Cob** $\to \mathscr{C}$. A *homomorphism of twin TQFTs* is a monoidal natural transformation of such functors. One denotes by **T-TQFT**(C) the category of twin TQFTs in $\mathscr{C}$.

Theorem 30. *The category* **Sing-2Cob** *is equivalent as a symmetric monoidal category to the category* **Th(T-Frob)**.

Proof. We need to construct a functor Λ : **Sing-2Cob** $\to$ **Th(T-Frob)**. In general, a monoidal functor is completely determined, up to equivalence, by its values on the generators

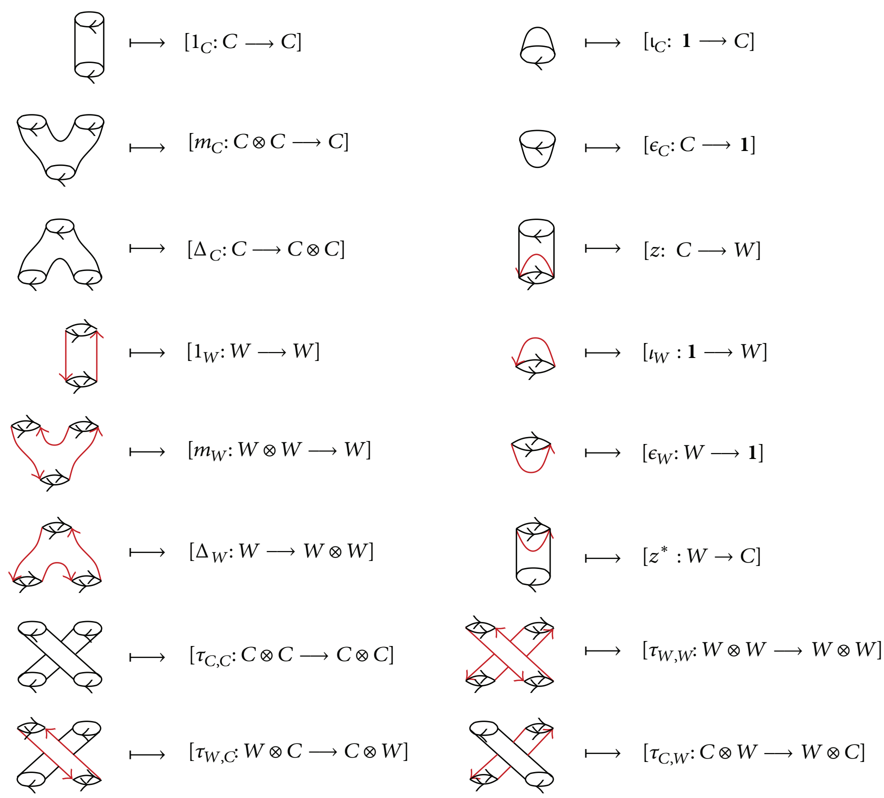


Figure 3: Assignments of Λ on the generating morphisms.

of the source category. On the generating objects of **Sing-2Cob**, Λ is defined as follows:

$$\Lambda : \begin{cases} \varnothing \longmapsto \mathbf{1} \\ \bigcirc \longmapsto C \\ \diamond \longmapsto W \end{cases} \tag{112}$$

Given a general object $\mathbf{n} = (n_1, n_2, n_3, \dots, n_k)$ in **Sing-2Cob**, the functor Λ associates the tensor product in **Th**(**T-Frob**) of copies of C and W, with all parenthesis to the left. That is, $\Lambda(\mathbf{n}) = (((\Lambda(n_1) \otimes \Lambda(n_2)) \otimes \Lambda(n_3)) \cdots \Lambda(n_k))$ with $n_i \in \{0, 1\}$ and $\Lambda(0) := C$ and $\Lambda(1) := W$.

On the generating morphisms in **Sing-2Cob**, Λ is defined as explained in Figure 3.

There is an obvious way to extend inductively Λ to a map defined on all morphisms of **Sing-2Cob**. From the coherence theorems for symmetric monoidal categories, it follows that this assignment is well defined and extends to all general morphisms in **Sing-2Cob**. Moreover, the relations given in Proposition 11 and the proof that these are all the required relations in **Sing-2Cob** imply that the image of Λ is a twin Frobenius algebra in **Th**(**T-Frob**) and, in particular, that Λ defines a functor **Sing-2Cob** $\to$ **Th**(**T-Frob**).

Given $\mathbf{n} = (n_1, n_2, n_3, \dots, n_k)$, $\mathbf{m} = (m_1, m_2, m_3, \dots, m_l) \in$ **Sing-2Cob** we construct a natural isomorphism $\Lambda_2 : \Lambda(\mathbf{n}) \otimes \Lambda(\mathbf{m}) \to \Lambda(\mathbf{n} \coprod \mathbf{m})$ as follows:

$$\begin{aligned} \Lambda(\mathbf{n}) &= (((\Lambda(n_1) \otimes \Lambda(n_2)) \otimes \Lambda(n_3)) \cdots \Lambda(n_k)), \\ \Lambda(\mathbf{m}) &= (((\Lambda(m_1) \otimes \Lambda(m_2)) \otimes \Lambda(m_3)) \cdots \Lambda(m_l)), \\ \Lambda\left(\mathbf{n} \coprod \mathbf{m}\right) &= ((((((\Lambda(n_1) \otimes \Lambda(n_2)) \otimes \Lambda(n_3)) \cdots \Lambda(n_k)) \\ &\quad \otimes \Lambda(m_1)) \otimes \Lambda(m_2)) \cdots \Lambda(m_l)). \end{aligned} \tag{113}$$

Define $\Lambda_0 := 1_{\mathbf{1}}$. It can be easily verified that the triple $(\Lambda, \Lambda_2, \Lambda_0)$ defines a symmetric monoidal functor.

On the other hand, by reversing the arrows in the assignments of Λ on the generating objects and morphisms in **Sing-2Cob**, we see that the singular cobordisms define a twin Frobenius algebra structure on the oriented circle and bi-web (thus we obtain the relations provided in Proposition 11). Therefore, from the results given in Section 2.2, we obtain a strict symmetric monoidal functor $\overline{\Lambda}$: **Th**(**T-Frob**) $\to$ **Sing-2Cob**. If two objects in **Th**(**T-Frob**) are related by a sequence of associators and unit constraints, then they are mapped to the same object in **Sing-2Cob**.

Given a general object $\mathbf{n} \in$ **Sing-2Cob** we have that $\overline{\Lambda}\Lambda(\mathbf{n}) = \mathbf{n}$. Thus $\overline{\Lambda}\Lambda = 1_{\textbf{Sing-2Cob}}$. If X is an object of **Th**(**T-Frob**), then X is a parenthesized word made up of symbols $\mathbf{1}, C, W$, and $\otimes$. It is not hard to see that $\Lambda\overline{\Lambda}(X)$ is isomorphic to X by a sequence of associators and unit constraints. In conclusion, Λ and $\overline{\Lambda}$ define an equivalence of categories. □

Corollary 31. *The category* $\mathbf{T}$-$\mathbf{Frob}(\mathscr{C})$ *of twin Frobenius algebras in* $\mathscr{C}$ *is equivalent, as a symmetric monoidal category, to the category* $\mathbf{T}$-$\mathbf{TQFT}(\mathscr{C})$ *of twin TQFTs in* $\mathscr{C}$.

5. Examples of Twin Frobenius Algebras

Example 32. Let i be the primitive fourth root of unity and let $R = \mathbb{Z}[i][a,h]$ be the ring of polynomials in indeterminates a and h and with Gaussian integer coefficients. Consider also the ring $\mathscr{A} = R[X]/(X^2 - hX - a) = \langle 1, X\rangle_R$ with inclusion map $\iota : R \rightarrow \mathscr{A}$, $\iota(1) = 1$. We remark that we consider these two rings for our first example, because they play an important role in [6, 9, 15].

It is well known (see [17, Section 4]) that ι is a Frobenius extension if and only if there exists an $\mathscr{A}$-bimodule map $\Delta : \mathscr{A} \rightarrow \mathscr{A}\otimes_R\mathscr{A}$ and an R-module map $\epsilon : \mathscr{A} \rightarrow R$ such that Δ is coassociative and cocommutative, and $(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}$.

A *Frobenius system* is a Frobenius extension together with a choice of the maps ϵ and Δ. We denote a Frobenius system by $\mathscr{F} = (R, \mathscr{A}, \epsilon, \Delta)$ (following [10, 17]).

We consider two Frobenius systems $\mathscr{F}_C = (R, \mathscr{A}, \epsilon_C, \Delta_C)$ and $\mathscr{F}_W = (R, \mathscr{A}, \epsilon_W, \Delta_W)$, with

$$\begin{aligned} \epsilon_C(1) &= 0, \\ \epsilon_C(X) &= 1, \\ \epsilon_W(1) &= 0, \\ \epsilon_W(X) &= -i, \\ \Delta_C(1) &= 1\otimes X + X\otimes 1 - h1\otimes 1, \\ \Delta_C(X) &= X\otimes X + a1\otimes 1, \\ \Delta_W(1) &= i(1\otimes X + X\otimes 1 - h1\otimes 1), \\ \Delta_W(X) &= i(X\otimes X + a1\otimes 1). \end{aligned} \tag{114}$$

$\mathscr{F}_W$ is a *twisting* of $\mathscr{F}_C$; that is, the comultiplication Δ_W and counit ϵ_W are obtained from Δ_C and ϵ_C by "twisting" them with invertible element $-i \in \mathscr{A}$:

$$\begin{aligned} \epsilon_W(x) &= \epsilon_C(-ix), \\ \Delta_W(x) &= \Delta_C\left((-i)^{-1}x\right) = \Delta_C(ix), \quad \forall x \in \mathscr{A}. \end{aligned} \tag{115}$$

The fact that the above Frobenius systems differ by a twist is not surprising. Kadison showed that twisting by invertible elements of $\mathscr{A}$ is the only way to modify the counit and comultiplication in Frobenius systems (see [17, Theorem 1.6]).

We obtain two commutative Frobenius structures on $\mathscr{A}$:

$$\begin{aligned} \mathscr{A}_C &= (\mathscr{A}, m_C, \iota_C, \Delta_C, \epsilon_C), \\ \mathscr{A}_W &= (\mathscr{A}, m_W, \iota_W, \Delta_W, \epsilon_W), \end{aligned} \tag{116}$$

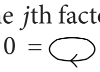
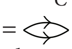
where $\iota_C = \iota_W = \iota$. Multiplication maps $m_C : \mathscr{A}\otimes\mathscr{A} \rightarrow \mathscr{A}$ and $m_W : \mathscr{A}\otimes\mathscr{A} \rightarrow \mathscr{A}$ are defined by the same rules:

$$\begin{aligned} m_C(1\otimes X) = m_C(X\otimes 1) &= m_W(1\otimes X) \\ &= m_W(X\otimes 1) = X, \\ m_C(1\otimes 1) &= m_W(1\otimes 1) = 1, \\ m_C(X\otimes X) &= m_W(X\otimes X) = hX + a. \end{aligned} \tag{117}$$

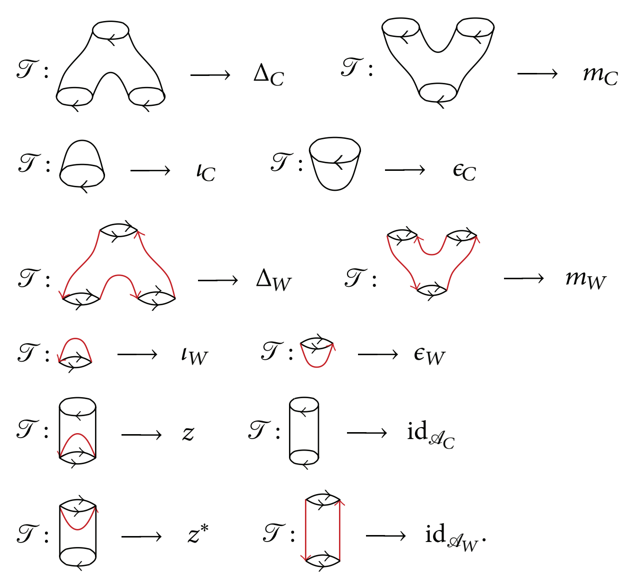
We define the following homomorphisms:

$$\begin{aligned} z : \mathscr{A}_C \longrightarrow \mathscr{A}_W, \quad & \begin{aligned} z(1) &= 1, \\ z(X) &= X, \end{aligned} \\ z^* : \mathscr{A}_W \longrightarrow \mathscr{A}_C, \quad & \begin{aligned} z^*(1) &= -i, \\ z^*(X) &= -iX. \end{aligned} \end{aligned} \tag{118}$$

A straightforward computation shows that $(\mathscr{A}_C, \mathscr{A}_W, z, z^*)$ is "almost" twin Frobenius, in the sense that all properties of a twin Frobenius algebra are satisfied except for the "genus-one condition" which holds up to a minus sign.

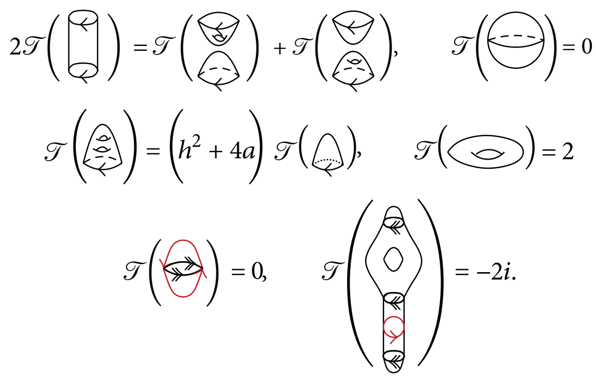
Let $\mathbf{R}$-$\mathbf{Mod}$ be the category of R-modules and module homomorphisms. We denote by $\mathscr{T} : \mathbf{Sing}$-$\mathbf{2Cob} \rightarrow \mathbf{R}$-$\mathbf{Mod}$ the TQFT corresponding to $(\mathscr{A}_C, \mathscr{A}_W, z, z^*)$ (c.f. Section 4), which assigns the ground ring R to the empty 1-manifold and assigns $\mathscr{A}^{\otimes k}$ to a generic object $\mathbf{n} = (n_1, n_2, \ldots, n_k)$ in $\mathbf{Sing}$-$\mathbf{2Cob}$ with $|\mathbf{n}| = k$. The jth factor of $\mathscr{A}^{\otimes k}$ is endowed with the structure $\mathscr{A}_C$ if $n_j = 0 =$ , or with the structure $\mathscr{A}_W$ if $n_j = 1 =$ .

On the generating morphisms of the category $\mathbf{Sing}$-$\mathbf{2Cob}$, the functor $\mathscr{T}$ is defined as follows:



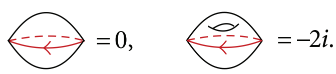
(119)

It is worth noting that the TQFT defined in this example satisfies the local relations for the "dot free" version of the universal $\mathfrak{sl}(2)$ foam cohomology for links (see [15, Section 4]). To be precise, the following identities hold:



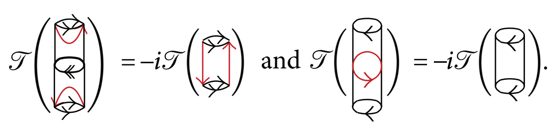
(120)

The last two identities are the "UFO" local relations used in [15] and depicted below:



(121)

Notice that we also have $z \circ z^* = -i\,\mathrm{id}(\ )$ and $z^* \circ z = -i\,\mathrm{id}(\ )$, which are equivalent to



(122)

The latter are versions of the "curtain identities" used in [6, 9, 15].

Even though we have here only an "almost" twin Frobenius algebra, motivated by the above remarks, we believe that this example will play the key role in describing the universal $\mathfrak{sl}(2)$ foam link cohomology using twin TQFTs. We will consider this problem in a subsequent paper.

Example 33. Let k be a field. Consider the commutative Frobenius algebra structure $(C, m_C, \iota_C, \Delta_C, \epsilon_C)$ on the truncated polynomial algebra $C = k[x]/(x^2)$, where

$$\begin{gathered}
m_C(1 \otimes 1) = 1, \\
m_C(1 \otimes x) = m_C(x \otimes 1) = x, \\
m_C(x \otimes x) = 0, \\
\Delta_C(1) = 1 \otimes x + x \otimes 1, \\
\Delta_C(x) = x \otimes x, \\
\epsilon_C(1) = 0, \qquad \epsilon_C(x) = 1.
\end{gathered} \tag{123}$$

Consider also the truncated polynomial algebra $W = k[y]/(y^n)$, $n \geq 2$, which admits a commutative and therefore symmetric Frobenius algebra structure $(W, m_W, \iota_W, \Delta_W, \epsilon_W)$ with $\epsilon_W(y^{n-1}) = 1$ and $\epsilon_W(y^a) = 0$ for all $0 \leq a \leq n-2$. The rules for multiplication m_W are obvious from the definition of W. The comultiplication $\Delta_W : W \to W \otimes_k W$ is dual to the multiplication via the counit map ϵ_W, and it is defined by

$$\Delta_W(y^a) = \sum_{i=0}^{n-1-a} y^{i+a} \otimes y^{n-1-i} \quad \forall 0 \leq a \leq n-1. \tag{124}$$

If char $k = n$, then (C, W, z, z^*) forms a twin Frobenius algebra with $z(1) = 1$, $z(x) = 0$ and $z^*(y^{n-1}) = x$, $z^*(y^a) = 0$, for all $0 \leq a \leq n-2$.

For a field k with char $k \neq n$, the above example does not satisfy the genus-one condition, since $z \circ m_C \circ \Delta_C \circ z^*(y^a) = 0$ for all $0 \leq a \leq n-1$, while $m_W \circ \tau_{W,W} \circ \Delta_W(1) = ny^{n-1}$ and $m_W \circ \tau_{W,W} \circ \Delta_W(y^a) = 0$, for all $1 \leq a \leq n-1$.

Example 34. Consider the polynomial ring $R = \mathbb{Z}[a,h]$, and the truncated polynomial algebras $C = R[x]/(x^2 - hx - a)$ and $W = R[y]/(y^2 - hy - a)$. Both C and W are commutative Frobenius with structure maps

$$\begin{gathered}
\Delta_C(1) = 1 \otimes x + x \otimes 1 - h1 \otimes 1, \\
\Delta_C(x) = x \otimes x + a1 \otimes 1, \\
\epsilon_C(1) = 0, \qquad \epsilon_C(x) = 1, \\
\Delta_W(1) = 1 \otimes y + y \otimes 1 - h1 \otimes 1, \\
\Delta_W(y) = y \otimes y + a1 \otimes 1, \\
\epsilon_W(1) = 0, \qquad \epsilon_W(y) = 1.
\end{gathered} \tag{125}$$

Then (C, W, z, z^*) is twin Frobenius, with $z(1) = 1$, $z(x) = y$ and $z^*(1) = 1$, $z^*(y) = x$.

Example 35. Let R be a commutative ring, and let $\alpha \in R$ such that $\alpha^2 = 1$. Consider the algebras $C = R[x]/(x^2 - hx - a)$ and $W = R[y]/(y^2 - hy - a)$, where $a, h \in R$, with Frobenius algebra structures given by

$$\begin{gathered}
\Delta_C(1) = 1 \otimes x + x \otimes 1 - h1 \otimes 1, \\
\Delta_C(x) = x \otimes x + a1 \otimes 1, \\
\epsilon_C(1) = 0, \qquad \epsilon_C(x) = 1, \\
\Delta_W(1) = \alpha\left(1 \otimes y + y \otimes 1 - h1 \otimes 1\right), \\
\Delta_W(y) = \alpha\left(y \otimes y + a1 \otimes 1\right), \\
\epsilon_W(1) = 0, \qquad \epsilon_W(y) = \alpha.
\end{gathered} \tag{126}$$

We have that (C, W, z, z^*) is twin Frobenius, with $z(1) = 1$, $z(x) = y$ and $z^*(1) = \alpha$, $z^*(y) = \alpha x$.

Other Examples. Examples of knowledgeable Frobenius algebras given by Lauda and Pfeiffer in [18, Examples 3.6–3.8] are also twin Frobenius algebras, with the same restrictions on char k. Example 3.13 from [18] is a twin Frobenius algebra as well, but we need no restriction on char k; actually, this is a particular case of our Example 34 over an arbitrary field k and for $h = 1$.

In all of the above examples, the algebra W is commutative. We provide now a couple of examples where W is non commutative, but symmetric (we remark that these are particular cases of the algebras given in Examples 3.1 and 3.2 in [18]).

Example 36. Let k be a field, $n \in \mathbb{N}$, and let W be the matrix algebra of $n\times n$ matrices over k; that is $W = M_n(k)$. Denote by $\{e_{ij}\}$ the standard basis for W. There is a symmetric Frobenius algebra structure $(W, m_W, \iota_W, \Delta_C, \epsilon_W)$, with

$$\begin{gathered}\Delta_W\left(e_{ij}\right) = \alpha\sum_{k=1}^{n} e_{ik}\otimes e_{kj},\\ \epsilon_W\left(e_{ij}\right) = \alpha^{-1}\delta_{ij},\\ m_W\left(e_{ij}\otimes e_{kl}\right) = \delta_{jk}e_{il},\\ \iota_W(1) = \sum_{i=1}^{n} e_{ii},\end{gathered} \tag{127}$$

where $\alpha = \pm 1 \in k$.

Consider the (trivial) Frobenius algebra structure $(C, m_C, \iota_C, \Delta_C, \epsilon_C)$ on $C := k$, with $\Delta_C(1) = 1\otimes 1$, $\epsilon_C(1) = 1$, $m_C(1\otimes 1) = 1$, and $\iota(1) = 1$. Then (C, W, z, z^*) is a twin Frobenius algebra, with $z(1) = \sum_{i=1}^{n} e_{ii}$ and $z^*(e_{ij}) = \alpha\delta_{ij}$.

Example 37. Let k be a field containing 2. Consider the free associative unital k-algebra $W := \mathbb{H}_k$ of quaternions, generated by I, J, K subject to the relations

$$\begin{gathered}I^2 = J^2 = K^2 = -1, \qquad IJ = -JI = K,\\ JK = -KJ = I, \qquad KI = -IK = J.\end{gathered} \tag{128}$$

Then $(W, m_W, \iota_W, \Delta_C, \epsilon_W)$ is symmetric Frobenius, with

$$\begin{gathered}\Delta_W(1) = \alpha(1\otimes 1 - I\otimes I - J\otimes J - K\otimes K),\\ \Delta_W(I) = \alpha(1\otimes I + I\otimes 1 + J\otimes K - K\otimes J),\\ \Delta_W(J) = \alpha(1\otimes J + J\otimes 1 + K\otimes I - I\otimes K),\\ \Delta_W(K) = \alpha(1\otimes K + K\otimes 1 + I\otimes J - J\otimes I),\\ \epsilon_W(1) = \alpha^{-1},\\ \epsilon_W(I) = \epsilon_W(J) = \epsilon_W(K) = 0,\end{gathered} \tag{129}$$

where $\alpha = \pm 1/2$.

There is a twin Frobenius algebra (C, W, z, z^*), where C is the underlying field k equipped with the trivial Frobenius structure (as in the previous example), and where

$$\begin{gathered}z(1) = 1, \qquad z^*(1) = 4\alpha,\\ z^*(I) = z^*(J) = z^*(K) = 0.\end{gathered} \tag{130}$$

Remark 38. Considering any of these (last two) examples (C_1, W_1, z_1, z_1^*), where $C_1 = k$ with the trivial Frobenius algebra structure, we can tensor it with one of the twin Frobenius algebras (C_2, W_2, z_2, z_2^*) given in the first set of examples (with C_2 commutative). We obtain a twin Frobenius algebra $(C_2, W_1\otimes W_2, z_1\otimes z_2, z_1^*\otimes z_2^*)$ with a bigger and possible noncommutative $W_1\otimes W_2$.

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