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UNIVALENCE CRITERIA FOR A NONLINEAR INTEGRAL OPERATOR

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Abstract. The purpose of this paper is to obtain univalence of a certain nonlinear integral transform of functions belonging to a subclass of analytic functions. We also give several interesting geometric properties of the integral transform.

1. Introduction and preliminaries

Let $\mathscr{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (a_n \geq 0), \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathscr{S}$ be the class of univalent functions in $\mathscr{A}$. Also let $\mathscr{S}^*$, $\mathscr{C}$ and $\mathscr{K}$ denote the familiar classes of functions in $\mathscr{A}$ that are starlike, convex and close-to-convex in $\mathbb{D}$ respectively. For functions f and g, analytic in $\mathbb{D}$, the function f is said to be subordinate to g if there exists a function w analytic in $\mathbb{D}$ with

$$w(0) = 0, \quad |w(z)| < 1 \quad (z \in \mathbb{D}),$$

such that

$$f(z) = g(w(z)) \qquad (z \in \mathbb{D}).$$

We denote this subordination by $f \prec g$ or $f(z) \prec g(z)$. Furthermore, if the function g is univalent in $\mathbb{D}$, then $f(z) \prec g(z) \Longleftrightarrow f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_q$ and $\beta_1, \beta_2, \ldots, \beta_s$ $(q, s \in \mathbb{N} \cup \{0\}, q \leq s+1)$ be complex numbers such that $\beta_k \neq 0, -1, -2, \ldots$ for $k \in \{1, 2, \ldots, s\}$. The generalized hypergeometric function ${}_qF_s$ is given by

$${}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \ldots (\alpha_q)_n}{(\beta_1)_n (\beta_2)_n \ldots (\beta_s)_n} \frac{z^n}{n!}, \qquad (z \in \mathbb{D}),$$

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where $(x)_n$ denotes the Pochhammer symbol defined by

$$(x)_n = x(x+1)(x+2)\cdots(x+n-1) \text{ for } n\in\mathbb{N} \text{ and } (x)_0 = 1.$$

J. Dziok and H. M. Srivastava considered in [5], (see also [6]), a linear operator

$$\mathscr{H}_q^s(\alpha_1,\alpha_2,\ldots,\alpha_q;\beta_1,\beta_2,\ldots,\beta_s)f(z):\mathscr{A}\to\mathscr{A}$$

defined by

$$\mathscr{H}_q^s(\alpha_1,\alpha_2,\ldots,\alpha_q;\beta_1,\beta_2,\ldots,\beta_s)f(z) = [z\ {}_qF_s(\alpha_1,\alpha_2,\ldots,\alpha_q;\beta_1,\beta_2,\ldots,\beta_s;z)] \star f(z) \tag{1.2}$$

where $\star$ denotes the usual Hadamard product (or convolution). For convenience, henceforth we shall denote

$$\mathscr{H}_q^s(\alpha_1,\beta_1) = \mathscr{H}_q^s(\alpha_1,\alpha_2,\ldots,\alpha_q;\beta_1,\beta_2,\ldots,\beta_s).$$

If $f\in\mathscr{A}$, from (1.2) we may easily deduce that

$$\mathscr{H}_q^s(\alpha_1,\beta_1)f(z) = z + \sum_{n=2}^{\infty}\frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}\ldots(\alpha_q)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}\ldots(\beta_s)_{n-1}}\frac{a_n}{(n-1)!}z^n. \tag{1.3}$$

The linear operator $\mathscr{H}_q^s(\alpha_1,\beta_1)$ includes (as its special cases) various other linear operators which were introduced and studied by Hohlov, Carlson and Shaffer and Ruscheweyh. For more details see [1, 10, 13, 14].

In [9], Y. J. Kim and E. P. Merkes considered the nonliner integral transform J_γ defined by

$$J_\gamma[f](z) = \int_0^z \left(\frac{f(t)}{t}\right)^\gamma dt \tag{1.4}$$

for complex numbers γ and functions f in the class $\mathscr{W} = \{f\in\mathscr{A} : f(z)\neq 0, \text{ for all } 0<|z|<1\}$ and showed that $J_\gamma(\mathscr{S}) = \{J_\gamma[f] : f\in\mathscr{S}\}\subset\mathscr{S}$ when $|\gamma|\leq 1/4$. For this result, finding the best constant is still an open problem. Also, V. Singh and P. N. Chichra [12] proved that, for $\gamma\in\mathbb{C}$ with $|\gamma|\leq 1/2$, the inequality $J_\gamma(\mathscr{S}^*)\subset\mathscr{S}$ holds, where $1/2$ is sharp.

By making use of the Dziok-Srivastava operator we now introduce the generalized integral operator $F_\gamma[\alpha_1,\beta_1;z]:\mathscr{A}^n\longrightarrow\mathscr{A}$ as follows:

$$F_\gamma[\alpha_1,\beta_1;z] = \int_0^z \left(\frac{\mathscr{H}_q^s(\alpha_1,\beta_1)f_1(t)}{t}\right)^{\gamma_1}\cdots\left(\frac{\mathscr{H}_q^s(\alpha_1,\beta_1)f_n(t)}{t}\right)^{\gamma_n} dt, \tag{1.5}$$

$$(\gamma_i\in\mathbb{C}, f_i\in\mathscr{A}, i=1,2,\ldots,n).$$

Remark 1.1. It is interesting to note that several well known and new integral operators are the special cases of the operator $F_\gamma[\alpha_1,\beta_1;z]$, here we list a few of them:

(i) When $q=2, s=1, \alpha_1=\beta_1$, and $\alpha_2=1$, then $F_\gamma[\alpha_1,\beta_1;z]$ reduces to

$$F_\gamma(z)=\int_0^z\left(\frac{f_1(t)}{t}\right)^{\gamma_1}\left(\frac{f_2(t)}{t}\right)^{\gamma_2}\cdots\left(\frac{f_n(t)}{t}\right)^{\gamma_n}dt, \tag{1.6}$$

$$(\gamma_i\in\mathbb{C}, f_i\in\mathscr{A}, i=1,2,\dots,n),$$

introduced by D.Breaz and N.Breaz in [3].

(ii) When $q=2, s=1, \alpha_1=\beta_1$, and $\alpha_2=2$, then $F_\gamma[\alpha_1,\beta_1;z]$ reduces to

$$G_\gamma(z)=\int_0^z\left(f_1'(t)\right)^{\gamma_1}\left(f_2'(t)\right)^{\gamma_2}\cdots\left(f_n'(t)\right)^{\gamma_n}dt, \tag{1.7}$$

$$(\gamma_i\in\mathbb{C}, f_i\in\mathscr{A}, i=1,2,\dots,n),$$

recently introduced by D.Breaz and N.Breaz in [3].

(iii) When $q=2, s=1, \alpha_1=\beta_1$, and $\alpha_2=\lambda+1$, then $F_\gamma[\alpha_1,\beta_1;z]$ reduces to

$$I(f_1,f_2,\dots,f_n)(z)=\int_0^z\left(\frac{D^\lambda f_1(t)}{t}\right)^{\gamma_1}\left(\frac{D^\lambda f_2(t)}{t}\right)^{\gamma_2}\cdots\left(\frac{D^\lambda f_n(t)}{t}\right)^{\gamma_n}dt, \tag{1.8}$$

$$(\lambda>-1, \gamma_i\in\mathbb{C}, f_i\in\mathscr{A}, i=1,2,\dots,n),$$

recently introduced by G.I.Oros et al. in [11], where $D^\lambda f$ is the well known Ruscheweyh derivative of f.

Motivated by the works of D. Breaz et al. [4] and Y. C. Kim and H. M. Srivastava [8], in the present paper, we give several interesting conditions for univalence of the nonlinear integral operator $F_\gamma[\alpha_1,\beta_1;z]$. A number of well known and new univalent conditions would follow, upon specializing the parameters involved in $F_\gamma[\alpha_1,\beta_1;z]$.

We now state the following result due to J. Becker [2] which we need to establish our results in the sequel.

Lemma 1. *If $f\in\mathscr{A}$ satisfies the inequality*

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right|\le 1, \qquad \textit{for all} \quad z\in\mathbb{D},$$

then the function f is univalent in $\mathbb{D}$.

2. Main results

Theorem 2.1. *Let $f_i(z)$ be a function in $\mathscr{A}$ such that*

$$\left(\frac{\mathscr{H}_q^s(\alpha_1,\beta_1)f_i(z)}{z}\right)\prec q(z)=\frac{1+Az}{1+Bz},\quad (i=1,2,\dots,n,z\in\mathbb{D})$$

holds for $-1\le B<A\le 1$. If

$$\sum_{i=1}^{n}|\gamma_i|\le\frac{1-AB+\sqrt{(1-A^2)(1-B^2)}}{2(A-B)} \tag{2.1}$$

then the function $F_\gamma[\alpha_1,\beta_1;z]$ given by (1.5) is univalent.

Proof. Let $p(z)=\left(\frac{\mathscr{H}_q^s(\alpha_1,\beta_1)f_i(z)}{z}\right)$. Then, by definition, there exists an analytic function $w:\mathbb{D}\longrightarrow\mathbb{D}$ with $w(0)=0$ such that

$$p(z)=q(w(z))=\frac{1+Aw(z)}{1+Bw(z)}.$$

A simple computation shows that

$$\left|\frac{zp'(z)}{p(z)}\right|=\left|\frac{z\left(\mathscr{H}_q^s(\alpha_1,\beta_1)f_i(z)\right)'}{\mathscr{H}_q^s(\alpha_1,\beta_1)f_i(z)}-1\right|\le\frac{(A-B)|zw'(z)|}{(1-|A||w(z)|)(1-|B||w(z)|)}.$$

By Schwarz-Pick lemma

$$\frac{|w'(z)|}{1-|w(z)|^2}\le\frac{1}{1-|z|^2},\qquad\forall z\in\mathbb{D},$$

and therefore

$$(1-|z|^2)\left|\frac{z\left(\mathscr{H}_q^s(\alpha_1,\beta_1)f_i(z)\right)'}{\mathscr{H}_q^s(\alpha_1,\beta_1)f_i(z)}-1\right|\le\frac{(A-B)(1-|w(z)|^2)}{(1-|A||w(z)|)(1-|B||w(z)|)}$$

for some Schwarz function $w(z)$.

Now,

$$\begin{aligned}(1-|z|^2)\left|\frac{zF_\gamma''[\alpha_1,\beta_1;z]}{F_\gamma'[\alpha_1,\beta_1;z]}\right|&\le(1-|z|^2)\sum_{i=1}^{n}|\gamma_i|\left|\frac{z\left(\mathscr{H}_q^s(\alpha_1,\beta_1)f_i(z)\right)'}{\mathscr{H}_q^s(\alpha_1,\beta_1)f_i(z)}-1\right|\\&\le\frac{(A-B)(1-|w(z)|^2)}{(1-|A||w(z)|)(1-|B||w(z)|)}\sum_{i=1}^{n}|\gamma_i|.\end{aligned}\tag{2.2}$$

But

$$\begin{aligned}\sup_{z\in\mathbb{D}}\frac{(A-B)(1-|w(z)|^2)}{(1-|A||w(z)|)(1-|B||w(z)|)}&=\sup_{0<x<1}\frac{(A-B)(1-x^2)}{(1-|A|x)(1-|B|x)}\\&\le\frac{2(A-B)}{1-AB+\sqrt{(1-A^2)(1-B^2)}}\end{aligned}\tag{2.3}$$

where the supremum is attained by

$$z = x = \frac{A+B}{1+AB+\sqrt{(1-A^2)(1-B^2)}}.$$

Using (2.1) and (2.3) in (2.2) we get,

$$(1-|z|^2)\left|\frac{zF''_\gamma[\alpha_1,\beta_1;z]}{F'_\gamma[\alpha_1,\beta_1;z]}\right| \le 1.$$

Now, by using lemma (1) we conclude that $F_\gamma[\alpha_1,\beta_1;z] \in \mathscr{S}$. □

Taking $q=2, s=1, \alpha_1=\beta_1$, and $\alpha_2=1$ in Theorem (2.1), we have the following

Corollary 2.2. *Let $f_i(z)$ be a function in $\mathscr{A}$ such that*

$$\left(\frac{f_i(z)}{z}\right) \prec q(z) = \frac{1+Az}{1+Bz}, \quad (i=1,2,\dots,n, z \in \mathbb{D})$$

holds for $-1 \le B < A \le 1$. If

$$\sum_{i=1}^{n} |\gamma_i| \le \frac{1-AB+\sqrt{(1-A^2)(1-B^2)}}{2(A-B)} \tag{2.4}$$

then the function $F_\gamma(z)$ defined by (1.6) is univalent in $\mathbb{D}$.

Taking $q=2, s=1, \alpha_1=\beta_1$, and $\alpha_2=2$ in Theorem (2.1), we have the following

Corollary 2.3. *Let $f_i(z)$ be a function in $\mathscr{A}$ such that*

$$f'_i(z) \prec q(z) = \frac{1+Az}{1+Bz}, \quad (i=1,2,\dots,n, z \in \mathbb{D})$$

holds for $-1 \le B < A \le 1$. If

$$\sum_{i=1}^{n} |\gamma_i| \le \frac{1-AB+\sqrt{(1-A^2)(1-B^2)}}{2(A-B)} \tag{2.5}$$

then the function $G_\gamma(z)$ defined by (1.7) is univalent in $\mathbb{D}$.

Taking $q=2, s=1, \alpha_1=\beta_1$, and $\alpha_2=\lambda+1$ in Theorem (2.1), we have the following

Corollary 2.4.*Let $f_i(z)$ be a function in $\mathscr{A}$ such that*

$$\left(\frac{D^\lambda f_i(z)}{z}\right) \prec q(z) = \frac{1+Az}{1+Bz}, \quad (i=1,2,\dots,n, z \in \mathbb{D})$$

holds for $-1 \le B < A \le 1$. If

$$\sum_{i=1}^{n} |\gamma_i| \le \frac{1-AB+\sqrt{(1-A^2)(1-B^2)}}{2(A-B)} \tag{2.6}$$

then the function $I(f_1, f_2, \dots, f_n)(z)$ defined by (1.8) is univalent in $\mathbb{D}$.

Theorem 2.5. *Let $f_i(z)$ be a function in $\mathscr{A}$ such that* $|(\mathscr{H}_q^s(\alpha_1,\beta_1)f_i(z))''| \le 2\lambda$, $z \in \mathbb{D}$, *holds for some constant* $0 < \lambda \le 1$. *If*

$$\sum_{i=1}^{n} |\gamma_i| \le \frac{1+\sqrt{1-\lambda^2}}{2\lambda}, \tag{2.7}$$

then the function $F_\gamma[\alpha_1,\beta_1;z]$ *given by* (1.5) *is univalent.*

Proof. We may write $(\mathscr{H}_q^s(\alpha_1,\beta_1)f_i(z))'' = 2\lambda w(z)$, where $|w| \le 1$. By integration, we have

$$(\mathscr{H}_q^s(\alpha_1,\beta_1)f_i(z))' = 1 + 2\lambda z \int_0^1 w(tz)\,dt$$

and

$$\mathscr{H}_q^s(\alpha_1,\beta_1)f_i(z) = z + 2\lambda z^2 \int_0^1 (1-t)w(tz)\,dt.$$

Since $|\int_0^1 (1-t)w(tz)\,dt| \le 1/2$, we have

$$\begin{aligned}(1-|z|^2)\left|\frac{zF_\gamma''[\alpha_1,\beta_1;z]}{F_\gamma'[\alpha_1,\beta_1;z]}\right| &\le (1-|z|^2)\sum_{i=1}^{n}|\gamma_i|\left|\frac{z\big(\mathscr{H}_q^s(\alpha_1,\beta_1)f_i(z)\big)'}{\mathscr{H}_q^s(\alpha_1,\beta_1)f_i(z)}-1\right| \\ &= (1-|z|^2)\sum_{i=1}^{n}|\gamma_i|\left|\frac{2\lambda z\int_0^1 w(tz)\,dt}{1+2\lambda z\int_0^1(1-t)w(tz)\,dt}\right| \\ &\le \sum_{i=1}^{n}|\gamma_i|\frac{\lambda(1-|z|^2)}{1-\lambda|z|}. \end{aligned} \tag{2.8}$$

But

$$\sup_{z\in\mathbb{D}} \lambda\frac{1-|z|^2}{1-\lambda|z|} = \sup_{0<t<1} \lambda\frac{1-t^2}{1-\lambda t} = 2\frac{1-\sqrt{1-\lambda^2}}{\lambda} = \frac{2\lambda}{1+\sqrt{1-\lambda^2}}. \tag{2.9}$$

where the supremum is attained by

$$z = t = \frac{\lambda}{1+\sqrt{1-\lambda^2}}.$$

Using (2.7) and (2.9) in (2.8) we get,

$$(1-|z|^2)\left|\frac{zF_\gamma''[\alpha_1,\beta_1;z]}{F_\gamma'[\alpha_1,\beta_1;z]}\right| \le 1.$$

Now, by using lemma (1) we conclude that $F_\gamma[\alpha_1,\beta_1;z] \in \mathscr{S}$. □

Letting $q = 2, s = 1, \alpha_1 = \beta_1$, and $\alpha_2 = 1$ in Theorem (2.5), we have the following

Corollary 2.6. *Let $f_i(z)$ be a function in $\mathscr{A}$ such that* $|f_i''(z)| \le 2\lambda$, $z \in \mathbb{D}$, *holds for some constant* $0 < \lambda \le 1$. *If*

$$\sum_{i=1}^{n} |\gamma_i| \le \frac{1+\sqrt{1-\lambda^2}}{2\lambda}, \tag{2.10}$$

then the function $F_\gamma(z)$ *given by* (1.6) *is univalent.*

Letting $n=1, q=2, s=1, \alpha_1=\beta_1$, and $\alpha_2=1$ in Theorem (2.5), we have following

Corollary 2.7. ([7]) *Let $f(z)$ be a function in $\mathscr{A}$ such that $|f''(z)| \le 2\lambda$, $z \in \mathbb{D}$, holds for some constant $0<\lambda\le 1$. If*

$$|\gamma| \le \frac{1+\sqrt{1-\lambda^2}}{2\lambda}, \tag{2.11}$$

then the function $J_\gamma[f](z)$ defined by (1.4) is univalent in $\mathbb{D}$.

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