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Research Article

A Multiple Attribute Decision Making Method Based on Uncertain Linguistic Heronian Mean

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The Heronian mean is a useful aggregation operator which can capture the interrelationship of the input arguments. In this paper, we develop some Heronian means based on uncertain linguistic variables, such as the generalized uncertain linguistic Heronian mean (GULHM) and uncertain linguistic geometric Heronian mean (ULGHM), and some of their desirable properties are also investigated. Considering the different importance of the input arguments, we define the generalized uncertain linguistic weighted Heronian mean (GULWHM) and uncertain linguistic weighted geometric Heronian mean (ULWGHM). Then, a method of multiple attribute decision making under uncertain linguistic environment is presented based on the GULWHM or the ULWGHM. In the end, an example is given to demonstrate the effectiveness and feasibility of the proposed method.

1. Introduction

Multiple attribute decision making exists here and there, and a multiple attribute decision making problem is to find the most desirable candidate from some feasible alternatives. In real life, decision-makers often provide their preferences on alternatives using linguistic term sets instead of numerical values owing to the fuzziness of human thinking process, and multiple attribute decision making under linguistic environment is a focus in recent years [1–12]. In the process of decision making, the input arguments need to be aggregated by some proper approaches so that the decision makers can select the most desirable alternative. Among these approaches, the operators are widely used. Yager [13] introduced the ordered weighted averaging (OWA) operator, which has only been used in situations in which the input arguments are the exact numerical values. But now, it has been extended to accommodate linguistic environment [2, 14–17], uncertain linguistic environment [18–22], and some other preference representation structures [23, 24]. Uncertain linguistic variable, as a generalization form of linguistic variable, is more powerful in dealing with uncertainty than linguistic variable since it is characterized by a linguistic interval rather than a linguistic value. Since its appearance, the uncertain linguistic variable has received much attention from researchers. Based on the weighted arithmetic averaging (WAA) operator [25] and the ordered weighted averaging (OWA) operator [13], Xu [18] introduced some uncertain linguistic aggregation operators called uncertain linguistic weighted averaging (ULWA) operator, uncertain linguistic ordered weighted averaging (ULOWA) operator, and uncertain linguistic hybrid aggregation (ULHA) operator. The ULWA operator only weights the uncertain linguistic arguments while the ULOWA operator only weights the ordered positions of the uncertain linguistic arguments. The ULHA operator combines the advantages of the ULWA and the ULOWA operator and weights not only the given arguments but also their ordered positions. From a geometric point of view, Xu [20] proposed some uncertain linguistic aggregation operators, such as the uncertain linguistic geometric mean (ULGM), uncertain linguistic weighted geometric mean (ULWGM), and uncertain linguistic ordered weighted geometric (ULOWG) operator. In order to solve the drawbacks of the ULWGM and the ULOWG operator, Wei [21] developed the uncertain linguistic hybrid geometric mean (ULHGM) operator and proposed an approach to multiple attribute group decision making with uncertain linguistic information based on the ULWGM and ULHGM operators. In [22], Park et al. proposed the uncertain linguistic weighted harmonic mean (ULWHM) operator, uncertain linguistic ordered weighted harmonic mean (ULOWHM) operator, and uncertain linguistic hybrid harmonic mean (ULHHM) operator, and an illustrative example about determining the air-conditioning system is also given to demonstrate the effectiveness and feasibility of the proposed method. Motivated by Yager and Filev [26], Xu [27] proposed some induced uncertain linguistic aggregation operators which can aggregate the decision making information in environments of mixing numeric and linguistic variables, such as the induced uncertain linguistic ordered weighted averaging (IULOWA) operator and the induced uncertain linguistic ordered weighted geometric (IULOWG) operator [20]. In [28], Xu generalized the IULOWA and the IULOWG operator and developed some generalized induced uncertain linguistic aggregation operators, including the generalized induced uncertain linguistic ordered weighted averaging (GIULOWA) operator and the generalized induced uncertain linguistic ordered weighted geometric (GIULOWG) operator.

However, the above uncertain linguistic aggregation approaches designed for solving multiple attribute decision making problems only consider the importance of the given arguments but ignore the correlation of them. Up to now, we are only aware of one paper on uncertain linguistic decision making that pays attention to the correlation of the input arguments [29]. In [29], Wei et al. utilized the uncertain linguistic Bonferroni mean (ULBM) operator and the uncertain linguistic geometric Bonferroni mean (ULGBM) operator which are an extension of the Bonferroni mean (BM) [30] to aggregate the uncertain linguistic arguments. The main advantage of the ULBM and ULGBM is that they can reflect the interrelationship of the input uncertain linguistic arguments. Nevertheless, these two means have their own disadvantages. For example, given a set of attributes C_i $(i = 1, 2, \ldots, n)$, the BM can reflect the correlation between any pair of attributes C_i and C_j $(i \neq j)$ but neglect the relationship between the attribute C_i and itself. Moreover, the BM considers the correlation between C_i and C_j $(i \neq j)$ and the correlation between C_j and C_i $(i \neq j)$ simultaneously, which results in potential redundancy. In order to solve these issues, we introduce the Heronian mean (HM) [31], the generalized Heronian mean (GHM_1) [32], and the geometric Heronian mean (GHM_2) [33] and extend them to accommodate uncertain linguistic environment.

To do so, the remainder of this paper is organized as follows. In Section 2, we briefly review some basic concepts, such as the uncertain linguistic variable, HM, GHM_1, and GHM_2. In Section 3, we extend these means to accommodate the situation in which the input arguments are uncertain linguistic variables and develop some uncertain linguistic Heronian means, such as generalized uncertain linguistic Heronian mean (GULHM), generalized uncertain linguistic weighted Heronian mean (GULWHM), uncertain linguistic geometric Heronian mean (ULGHM), and uncertain linguistic weighted geometric Heronian mean (ULWGHM). In Section 4, we propose a method for multiple attribute decision making with uncertain linguistic information based on GULWHM or ULWGHM. In Section 5, an example is given to verify the effectiveness and feasibility of the proposed method. Section 6 ends the paper with some concluding remarks.

2. Uncertain Linguistic Variables and Heronian Mean

2.1. Uncertain Linguistic Variables. Let $S = \{s_i \mid i = 1, 2, \ldots, t\}$ be a linguistic term set with odd cardinality, where s_i represents a possible value for a linguistic variable. For example, a set of seven terms, S, could be defined as follows:

$$\begin{aligned} S = \{ & s_1 = \text{extremely poor}, s_2 = \text{very poor}, s_3 = \text{poor}, \\ & s_4 = \text{medium}, s_5 = \text{good}, s_6 = \text{very good}, \\ & s_7 = \text{extremely good}\}. \end{aligned} \tag{1}$$

It is usually required that there exist the following [7, 17, 21].

(1) The set is ordered as $s_i \geq s_j$ if $i \geq j$.

(2) There is the negation operator $\operatorname{neg}(s_i) = s_j$ such that $i + j = t + 1$.

(3) Max operator $\max(s_i, s_j) = s_i$, if $s_i \geq s_j$.

(4) Min operator $\min(s_i, s_j) = s_i$, if $s_i \leq s_j$.

To preserve all the given information, the discrete term set S should be extended to a continuous term set $\overline{S} = \{s_\alpha \mid s_1 \leq s_\alpha \leq s_q,\ \alpha \in [1, q]\}$, where q is a sufficiently large positive integer; if $s_\alpha \in S$, then we call s_α the original term; otherwise, we call s_α the virtual term [17, 21]. The decision maker, in general, uses the original linguistic terms to evaluate alternatives, and the virtual linguistic terms can only appear in operations.

Definition 1 (see [18–22, 27, 28]). Let $\widetilde{s} = [s_\alpha, s_\beta]$, where $s_\alpha, s_\beta \in \overline{S}$, s_α, and s_β are the lower and the upper limits, respectively, and then we call $\widetilde{s}$ the uncertain linguistic variable. Suppose that $\overline{S}$ is the set of all uncertain linguistic variables.

If $s_\alpha = s_\beta$, then the uncertain linguistic variable $\widetilde{s}$ is reduced to a linguistic value. Consider any three uncertain linguistic variables $\widetilde{s} = [s_\alpha, s_\beta]$, $\widetilde{s}_1 = [s_{\alpha_1}, s_{\beta_1}]$, $\widetilde{s}_2 = [s_{\alpha_2}, s_{\beta_2}]$, and let $\lambda \in [0, 1]$; then their operational laws are defined as follows [18–21, 27, 28]:

(1) $\widetilde{s}_1 \oplus \widetilde{s}_2 = [s_{\alpha_1}, s_{\beta_1}] \oplus [s_{\alpha_2}, s_{\beta_2}] = [s_{\alpha_1} \oplus s_{\alpha_2}, s_{\beta_1} \oplus s_{\beta_2}] = [s_{\alpha_1+\alpha_2}, s_{\beta_1+\beta_2}]$;

(2) $\lambda \widetilde{s} = \lambda [s_\alpha, s_\beta] = [\lambda s_\alpha, \lambda s_\beta] = [s_{\lambda\alpha}, s_{\lambda\beta}]$;

(3) $\widetilde{s}_1 \otimes \widetilde{s}_2 = [s_{\alpha_1}, s_{\beta_1}] \otimes [s_{\alpha_2}, s_{\beta_2}] = [s_{\alpha_1} \otimes s_{\alpha_2},\ s_{\beta_1} \otimes s_{\beta_2}] = [s_{\alpha_1\alpha_2},\ s_{\beta_1\beta_2}]$;

(4) $\widetilde{s}^\lambda = ([s_\alpha, s_\beta])^\lambda = [(s_\alpha)^\lambda, (s_\beta)^\lambda] = [s_{\alpha^\lambda}, s_{\beta^\lambda}]$.

Moreover, the following relationship can be easily proved:

(5) $\tilde{s}_1 \oplus \tilde{s}_2 = \tilde{s}_2 \oplus \tilde{s}_1$;

(6) $\tilde{s}_1 \otimes \tilde{s}_2 = \tilde{s}_2 \otimes \tilde{s}_1$;

(7) $\lambda(\tilde{s}_1 \oplus \tilde{s}_2) = \lambda\tilde{s}_1 \oplus \lambda\tilde{s}_2$;

(8) $(\tilde{s}_1 \otimes \tilde{s}_2)^\lambda = \tilde{s}_1^\lambda \otimes \tilde{s}_2^\lambda$;

(9) $\lambda_1\tilde{s} \oplus \lambda_2\tilde{s} = (\lambda_1 + \lambda_2)\tilde{s}$;

(10) $\tilde{s}^{\lambda_1} \otimes \tilde{s}^{\lambda_2} = \tilde{s}^{\lambda_1+\lambda_2}$.

In order to compare the uncertain linguistic variables, we give the following definition.

Definition 2 (see [34]). Let $\tilde{s}_1 = [s_{\alpha_1}, s_{\beta_1}]$ and $\tilde{s}_2 = [s_{\alpha_2}, s_{\beta_2}]$ be two uncertain linguistic variables, and let $\operatorname{len}(\tilde{s}_1) = \beta_1 - \alpha_1$ and $\operatorname{len}(\tilde{s}_2) = \beta_2 - \alpha_2$; then the degree of possibility of $\tilde{s}_1 \geq \tilde{s}_2$ is defined as

$$p(\tilde{s}_1 \geq \tilde{s}_2) = \frac{\max(0, \beta_1 - \alpha_2) - \max(0, \alpha_1 - \beta_2)}{\operatorname{len}(\tilde{s}_1) + \operatorname{len}(\tilde{s}_2)}. \quad (2)$$

From Definition 2, we can easily get the following results:

(1) $0 \leq p(\tilde{s}_1 \geq \tilde{s}_2) \leq 1$, $0 \leq p(\tilde{s}_2 \geq \tilde{s}_1) \leq 1$;

(2) $p(\tilde{s}_1 \geq \tilde{s}_2) + p(\tilde{s}_2 \geq \tilde{s}_1) = 1$. Especially, $p(\tilde{s}_1 \geq \tilde{s}_1) = p(\tilde{s}_2 \geq \tilde{s}_2) = 1/2$.

2.2. Heronian Mean. Heronian mean (HM), which is one of the aggregation methods, has the desirable characteristic that it can reflect the interrelationship of the input arguments. The definition of HM is as follows.

Definition 3 (see [31]). Let a_i $(i = 1, 2, \ldots, n)$ be a collection of nonnegative numbers. If

$$\operatorname{HM}(a_1, a_2, \ldots, a_n) = \frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} \sqrt{a_i a_j}, \quad (3)$$

then HM is called the Heronian mean (HM).

Based on Definition 3, Yu and Wu [32, 33] proposed the generalized Heronian mean (GHM_1) and the geometric Heronian mean (GHM_2).

Definition 4 (see [32]). Let $p, q \geq 0$ and p, q do not take the value 0 simultaneously. Let a_i $(i = 1, 2, \ldots, n)$ be a collection of nonnegative numbers. If

$$\mathrm{GHM}_1^{p,q}(a_1, a_2, \ldots, a_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} a_i^p a_j^q \right)^{1/(p+q)}, \quad (4)$$

then GHM_1 is called the generalized Heronian mean (GHM_1). If $p = q = 1/2$ especially, then the GHM_1 is reduced to HM.

It is noted that the GHM_1 has the following properties:

(1) $\mathrm{GHM}_1^{p,q}(0, 0, \ldots, 0) = 0$;

(2) $\mathrm{GHM}_1^{p,q}(a, a, \ldots, a) = a$, if $a_i = a$, for all i;

(3) $\mathrm{GHM}_1^{p,q}(a_1, a_2, \ldots, a_n) \geq \mathrm{GHM}_1^{p,q}(b_1, b_2, \ldots, b_n)$, that is, $\mathrm{GHM}_1^{p,q}$ is monotonic, if $a_i \geq b_i$, for all i;

(4) $\min_i\{a_i\} \leq \mathrm{GHM}_1^{p,q}(a_1, a_2, \ldots, a_n) \leq \max_i\{a_i\}$.

Example 5. Let a_1, a_2, a_3 be three nonnegative numbers and $p = q = 2$; then

$$\begin{aligned} &\mathrm{GHM}_1^{p,q}(a_1, a_2, \ldots, a_n) \\ &= \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} a_i^p a_j^q \right)^{1/(p+q)} \\ &= \left(\frac{1}{6} \left(a_1^2 a_1^2 + a_1^2 a_2^2 + a_1^2 a_3^2 + a_2^2 a_2^2 + a_2^2 a_3^2 + a_3^2 a_3^2 \right) \right)^{1/4}. \end{aligned} \quad (5)$$

If we use Bonferroni mean (BM) [30] to aggregate the above three nonnegative numbers, then

$$\begin{aligned} &\mathrm{BM}^{p,q}(a_1, a_2, \ldots, a_n) \\ &= \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_i^p a_j^q \right)^{1/(p+q)} \\ &= \left(\frac{1}{6} \left(a_1^2 a_2^2 + a_1^2 a_3^2 + a_2^2 a_1^2 + a_2^2 a_3^2 + a_3^2 a_1^2 + a_3^2 a_2^2 \right) \right)^{1/4}. \end{aligned} \quad (6)$$

From the above analysis, we can find that the BM computes $a_1^2 a_2^2$, $a_2^2 a_1^2$, $a_1^2 a_3^2$, $a_3^2 a_1^2$, $a_2^2 a_3^2$, and $a_3^2 a_2^2$ separately. However, $a_1^2 a_2^2$ is equal to $a_2^2 a_1^2$, $a_1^2 a_3^2$ is equal to $a_3^2 a_1^2$, and $a_2^2 a_3^2$ is equal to $a_3^2 a_2^2$. Hence, it results in potential redundancy. Moreover, the BM has not paid attention to $a_1^2 a_1^2$, $a_2^2 a_2^2$, and $a_3^2 a_3^2$. Nevertheless, the GHM_1 can solve the two problems effectively.

Definition 6 (see [33]). Let $p, q \geq 0$ and p, q do not take the value 0 simultaneously. Let a_i $(i = 1, 2, \ldots, n)$ be a collection of nonnegative numbers. If

$$\mathrm{GHM}_2^{p,q}(a_1, a_2, \ldots, a_n) = \frac{1}{p+q} \prod_{i=1, j=i}^{n} \left(p a_i + q a_j \right)^{2/n(n+1)}, \quad (7)$$

then GHM_2 is called the geometric Heronian mean (GHM_2).

It is noted that the GHM_2 has the following properties:

(1) $\mathrm{GHM}_2^{p,q}(0, 0, \ldots, 0) = 0$;

(2) $\mathrm{GHM}_2^{p,q}(a, a, \ldots, a) = a$, if $a_i = a$, for all i;

(3) $\mathrm{GHM}_2^{p,q}(a_1, a_2, \ldots, a_n) \geq \mathrm{GHM}_2^{p,q}(b_1, b_2, \ldots, b_n)$ that is, $\mathrm{GHM}_2^{p,q}$ is monotonic, if $a_i \geq b_i$, for all i;

(4) $\min_i\{a_i\} \leq \mathrm{GHM}_2^{p,q}(a_1, a_2, \ldots, a_n) \leq \max_i\{a_i\}$.

Example 7. Let a_1, a_2, a_3 be three nonnegative numbers and $p = q = 1$; then

$$\begin{aligned} &\mathrm{GHM}_2^{p,q}(a_1, a_2, \ldots, a_n) \\ &\quad = \frac{1}{p+q} \prod_{i=1, j=i}^{n} \left(pa_i + qa_j\right)^{2/n(n+1)} \\ &\quad = \frac{1}{2}\left(\left(a_1 + a_1\right)\left(a_1 + a_2\right)\left(a_1 + a_3\right)\right. \\ &\qquad \left. \times \left(a_2 + a_2\right)\left(a_2 + a_3\right)\left(a_3 + a_3\right)\right)^{1/6}. \end{aligned} \tag{8}$$

If we use geometric Bonferroni mean (GBM) proposed by Xia et al. [35] to aggregate the above three nonnegative numbers, then

$$\begin{aligned} &\mathrm{GBM}^{p,q}(a_1, a_2, \ldots, a_n) \\ &\quad = \frac{1}{p+q} \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(pa_i + qa_j\right)^{1/n(n-1)} \\ &\quad = \frac{1}{2}\left(\left(a_1 + a_2\right)\left(a_1 + a_3\right)\left(a_2 + a_1\right)\right. \\ &\qquad \left. \times \left(a_2 + a_3\right)\left(a_3 + a_1\right)\left(a_3 + a_2\right)\right)^{1/6}. \end{aligned} \tag{9}$$

Similar to BM, the GBM also results in potential redundancy. Furthermore, it has not paid attention to $(a_1 + a_1)$, $(a_2 + a_2)$, and $(a_3 + a_3)$. However, the GHM_2 can solve the two problems effectively.

3. Uncertain Linguistic Heronian Means

3.1. The GULHM and the GULWHM. The GHM_1 has the desirable characteristic capturing the interrelationship of the input arguments. However, the arguments suitable to be aggregated by the GHM_1 usually take the forms of nonnegative real numbers. In this section, we will extend the GHM_1 to accommodate the situations in which the input arguments are uncertain linguistic variables. Based on the operational rules on uncertain linguistic variables and Definition 4, we give the generalized uncertain linguistic Heronian mean (GULWHM) in the following.

Definition 8. Let $p, q \geq 0$ and p, q do not take the value 0 simultaneously. Let $\tilde{s}_i = [s_{\alpha_i}, s_{\beta_i}]$ $(i = 1, 2, \ldots, n)$ be a collection of uncertain linguistic variables. If

$$\begin{aligned} &\mathrm{GULHM}^{p,q}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \\ &\quad = \left(\frac{2}{n(n+1)} \bigoplus_{i=1, j=i}^{n} \left(\tilde{s}_i^p \otimes \tilde{s}_j^q\right)\right)^{1/(p+q)} \\ &\quad = \left[\left(\frac{2}{n(n+1)} \bigoplus_{i=1, j=i}^{n} \left(s_{\alpha_i}^p \otimes s_{\alpha_j}^q\right)\right)^{1/(p+q)}, \right. \\ &\qquad \left. \left(\frac{2}{n(n+1)} \bigoplus_{i=1, j=i}^{n} \left(s_{\beta_i}^p \otimes s_{\beta_j}^q\right)\right)^{1/(p+q)}\right], \end{aligned} \tag{10}$$

then the GULHM is called the generalized uncertain linguistic Heronian mean (GULHM). If $p = q = 1/2$; then the GULHM reduces to

$$\begin{aligned} &\mathrm{GULHM}^{1/2,1/2}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \\ &\quad = \frac{2}{n(n+1)} \bigoplus_{i=1, j=i}^{n} \left(\tilde{s}_i^{1/2} \otimes \tilde{s}_j^{1/2}\right) \\ &\quad = \left[\frac{2}{n(n+1)} \bigoplus_{i=1, j=i}^{n} \left(s_{\alpha_i}^{1/2} \otimes s_{\alpha_j}^{1/2}\right), \right. \\ &\qquad \left. \frac{2}{n(n+1)} \bigoplus_{i=1, j=i}^{n} \left(s_{\beta_i}^{1/2} \otimes s_{\beta_j}^{1/2}\right)\right], \end{aligned} \tag{11}$$

which we call the uncertain linguistic Heronian mean (ULHM).

In the following, we investigate the desirable properties of the GULHM.

Theorem 9 (idempotency). *Let $\tilde{s}_i = [s_{\alpha_i}, s_{\beta_i}]$ $(i = 1, 2, \ldots, n)$ be a collection of uncertain linguistic variables. If all $\tilde{s}_i$ are equal, that is, $\tilde{s}_i = \tilde{s} = [s_\alpha, s_\beta]$ for all i, then*

$$GULHM^{p,q}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \tilde{s}. \tag{12}$$

Proof. Consider the following:

$$\begin{aligned} &\mathrm{GULHM}^{p,q}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \\ &\quad = \mathrm{GULHM}^{p,q}(\tilde{s}, \tilde{s}, \ldots, \tilde{s}) \\ &\quad = \left(\frac{2}{n(n+1)} \bigoplus_{i=1, j=i}^{n} \left(\tilde{s}^p \otimes \tilde{s}^q\right)\right)^{1/(p+q)} \\ &\quad = \left[\frac{2}{n(n+1)} \bigoplus_{i=1, j=i}^{n} \left(s_\alpha^p \otimes s_\alpha^q\right), \right. \\ &\qquad \left. \frac{2}{n(n+1)} \bigoplus_{i=1, j=i}^{n} \left(s_\beta^p \otimes s_\beta^q\right)\right]^{1/(p+q)} \\ &\quad = \left[s_\alpha, s_\beta\right]. \end{aligned} \tag{13}$$

□

Theorem 10 (permutation). *Let* $\tilde{s}_i = [s_{\alpha_i}, s_{\beta_i}]$ *and* $\tilde{s}'_i = [s'_{\alpha_i}, s'_{\beta_i}]$ $(i = 1, 2, \ldots, n)$ *be two collections of uncertain linguistic variables; then*

$$\mathrm{GULHM}^{p,q}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \mathrm{GULHM}^{p,q}(\tilde{s}'_1, \tilde{s}'_2, \ldots, \tilde{s}'_n), \tag{14}$$

where $\tilde{s}'_i = [s'_{\alpha_i}, s'_{\beta_i}]$ $(i = 1, 2, \ldots, n)$ *is any permutation of* $\tilde{s}_i = [s_{\alpha_i}, s_{\beta_i}]$ $(i = 1, 2, \ldots, n)$.

Proof. Since $\tilde{s}'_i = [s'_{\alpha_i}, s'_{\beta_i}]$ $(i = 1, 2, \ldots, n)$ is any permutation of $\tilde{s}_i = [s_{\alpha_i}, s_{\beta_i}]$ $(i = 1, 2, \ldots, n)$, then

$$\begin{aligned}
&\mathrm{GULHM}^{p,q}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \\
&\quad = \left(\frac{2}{n(n+1)} \bigoplus_{i=1,j=i}^{n} (\tilde{s}_i^p \otimes \tilde{s}_j^q)\right)^{1/(p+q)} \\
&\quad = \left(\frac{2}{n(n+1)} \bigoplus_{i=1,j=i}^{n} (\tilde{s}_i'^p \otimes \tilde{s}_j'^q)\right)^{1/(p+q)} \\
&\quad = \mathrm{GULHM}^{p,q}(\tilde{s}'_1, \tilde{s}'_2, \ldots, \tilde{s}'_n).
\end{aligned} \tag{15}$$

□

Theorem 11 (monotonicity). *Let* $\tilde{s}_i = [s_{\alpha_i}, s_{\beta_i}]$ *and* $\tilde{s}'_i = [s'_{\alpha_i}, s'_{\beta_i}]$ $(i = 1, 2, \ldots, n)$ *be two collections of uncertain linguistic variables. If* $s_{\alpha_i} \le s'_{\alpha_i}$, $s_{\beta_i} \le s'_{\beta_i}$ *for all i, then*

$$\mathrm{GULHM}^{p,q}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \le \mathrm{GULHM}^{p,q}(\tilde{s}'_1, \tilde{s}'_2, \ldots, \tilde{s}'_n). \tag{16}$$

Proof. Since $s_{\alpha_i} \le s'_{\alpha_i}, s_{\beta_i} \le s'_{\beta_i}$ for all i, then

$$\begin{aligned}
&\left(\frac{2}{n(n+1)} \bigoplus_{i=1,j=i}^{n} (s_{\alpha_i}^p \otimes s_{\alpha_j}^q)\right)^{1/(p+q)} \\
&\quad \le \left(\frac{2}{n(n+1)} \bigoplus_{i=1,j=i}^{n} (s_{\alpha_i}'^p \otimes s_{\alpha_j}'^q)\right)^{1/(p+q)} \\
&\left(\frac{2}{n(n+1)} \bigoplus_{i=1,j=i}^{n} (s_{\beta_i}^p \otimes s_{\beta_j}^q)\right)^{1/(p+q)} \\
&\quad \le \left(\frac{2}{n(n+1)} \bigoplus_{i=1,j=i}^{n} (s_{\beta_i}'^p \otimes s_{\beta_j}'^q)\right)^{1/(p+q)}.
\end{aligned} \tag{17}$$

By Definition 2, we get that

$$\begin{aligned}
&\left[\left(\frac{2}{n(n+1)} \bigoplus_{i=1,j=i}^{n} (s_{\alpha_i}^p \otimes s_{\alpha_j}^q)\right)^{1/(p+q)}, \left(\frac{2}{n(n+1)} \bigoplus_{i=1,j=i}^{n} (s_{\beta_i}^p \otimes s_{\beta_j}^q)\right)^{1/(p+q)}\right] \\
&\quad \le \left[\left(\frac{2}{n(n+1)} \bigoplus_{i=1,j=i}^{n} (s_{\alpha_i}'^p \otimes s_{\alpha_j}'^q)\right)^{1/(p+q)}, \left(\frac{2}{n(n+1)} \bigoplus_{i=1,j=i}^{n} (s_{\beta_i}'^p \otimes s_{\beta_j}'^q)\right)^{1/(p+q)}\right].
\end{aligned} \tag{18}$$

Thus,

$$\begin{aligned}
&\mathrm{GULHM}^{p,q}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \\
&\quad = \left(\frac{2}{n(n+1)} \bigoplus_{i=1,j=i}^{n} (\tilde{s}_i^p \otimes \tilde{s}_j^q)\right)^{1/(p+q)} \\
&\quad \le \left(\frac{2}{n(n+1)} \bigoplus_{i=1,j=i}^{n} (\tilde{s}_i'^p \otimes \tilde{s}_j'^q)\right)^{1/(p+q)} \\
&\quad = \mathrm{GULHM}^{p,q}(\tilde{s}'_1, \tilde{s}'_2, \ldots, \tilde{s}'_n).
\end{aligned} \tag{19}$$

□

Theorem 12 (boundedness). *Let* $\tilde{s}_i = [s_{\alpha_i}, s_{\beta_i}]$ $(i = 1, 2, \ldots, n)$ *be a collection of uncertain linguistic variables, and*

$$\begin{aligned}
\tilde{s}^- &= \min_i \tilde{s}_i = \left[\min_i s_{\alpha_i}, \min_i s_{\beta_i}\right], \\
\tilde{s}^+ &= \max_i \tilde{s}_i = \left[\max_i s_{\alpha_i}, \max_i s_{\beta_i}\right].
\end{aligned} \tag{20}$$

Then,

$$\tilde{s}^- \le \mathrm{GULHM}^{p,q}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \le \tilde{s}^+. \tag{21}$$

Proof. Consider the following:

$$\begin{aligned}
&\mathrm{GULHM}^{p,q}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \\
&= \left(\frac{2}{n(n+1)} \bigoplus_{i=1,j=i}^{n} (\tilde{s}_i^p \otimes \tilde{s}_j^q)\right)^{1/(p+q)} \\
&= \left[\left(\frac{2}{n(n+1)} \bigoplus_{i=1,j=i}^{n} (s_{\alpha_i}^p \otimes s_{\alpha_j}^q)\right)^{1/(p+q)}, \left(\frac{2}{n(n+1)} \bigoplus_{i=1,j=i}^{n} (s_{\beta_i}^p \otimes s_{\beta_j}^q)\right)^{1/(p+q)}\right] \\
&\le \left[\left(\frac{2}{n(n+1)} \bigoplus_{i=1,j=i}^{n} \left(\left\{\max_i s_{\alpha_i}\right\}^p \otimes \left\{\max_i s_{\alpha_i}\right\}^q\right)\right)^{1/(p+q)}, \left(\frac{2}{n(n+1)} \bigoplus_{i=1,j=i}^{n} \left(\left\{\max_i s_{\beta_i}\right\}^p \otimes \left\{\max_i s_{\beta_i}\right\}^q\right)\right)^{1/(p+q)}\right] \\
&= \left[\max_i s_{\alpha_i}, \max_i s_{\beta_i}\right] = \tilde{s}^+.
\end{aligned} \tag{22}$$

Similarly, we can prove

$$\begin{aligned}
&\mathrm{GULHM}^{p,q}(\tilde{s}_1,\tilde{s}_2,\ldots,\tilde{s}_n)\\
&=\left(\frac{2}{n(n+1)}\bigoplus_{i=1,j=i}^{n}\left(\tilde{s}_i^{p}\otimes\tilde{s}_j^{q}\right)\right)^{1/(p+q)}\\
&\geq\left[\min_i s_{\alpha_i},\min_i s_{\beta_i}\right]=\tilde{s}^{-},
\end{aligned}\tag{23}$$

which completes the proof of Theorem 12. □

In most cases, the input arguments have their own importance. Each argument should be assigned a weight. Hence, it is necessary to consider the weighted form of the GULHM. In the following, we define the generalized uncertain linguistic weighted Heronian mean (GULWHM).

Definition 13. Let $p,q\geq 0$ and p,q do not take the value 0 simultaneously. Let $\tilde{s}_i=[s_{\alpha_i},s_{\beta_i}]$ $(i=1,2,\ldots,n)$ be a collection of uncertain linguistic variables. And $w=(w_1,w_2,\ldots,w_n)^T$ is the weight vector of $\tilde{s}_i=[s_{\alpha_i},s_{\beta_i}]$ $(i=1,2,\ldots,n)$, where w_i indicates the importance degree of $\tilde{s}_i$, satisfying $w_i\geq 0$, and $\sum_{i=1}^{n}w_i=1$. If

$$\begin{aligned}
&\mathrm{GULWHM}_w^{p,q}(\tilde{s}_1,\tilde{s}_2,\ldots,\tilde{s}_n)\\
&=\left(\frac{2}{n(n+1)}\bigoplus_{i=1,j=i}^{n}\left((w_i\tilde{s}_i)^{p}\otimes\left(w_j\tilde{s}_j\right)^{q}\right)\right)^{1/(p+q)}\\
&=\left[\left(\frac{2}{n(n+1)}\bigoplus_{i=1,j=i}^{n}\left(\left(w_is_{\alpha_i}\right)^{p}\otimes\left(w_js_{\alpha_j}\right)^{q}\right)\right)^{1/(p+q)},\right.\\
&\qquad\left.\left(\frac{2}{n(n+1)}\bigoplus_{i=1,j=i}^{n}\left(\left(w_is_{\beta_i}\right)^{p}\otimes\left(w_js_{\beta_j}\right)^{q}\right)\right)^{1/(p+q)}\right],
\end{aligned}\tag{24}$$

then GULWHM is called the generalized uncertain linguistic weighted Heronian mean (GULWHM). If $p=q=1/2$; then the GULWHM reduces to

$$\begin{aligned}
&\mathrm{GULWHM}_w^{1/2,1/2}(\tilde{s}_1,\tilde{s}_2,\ldots,\tilde{s}_n)\\
&=\left(\frac{2}{n(n+1)}\bigoplus_{i=1,j=i}^{n}\left((w_i\tilde{s}_i)^{1/2}\otimes\left(w_j\tilde{s}_j\right)^{1/2}\right)\right)\\
&=\left[\left(\frac{2}{n(n+1)}\bigoplus_{i=1,j=i}^{n}\left(\left(w_is_{\alpha_i}\right)^{1/2}\otimes\left(w_js_{\alpha_j}\right)^{1/2}\right)\right),\right.\\
&\qquad\left.\left(\frac{2}{n(n+1)}\bigoplus_{i=1,j=i}^{n}\left(\left(w_is_{\beta_i}\right)^{1/2}\otimes\left(w_js_{\beta_j}\right)^{1/2}\right)\right)\right],
\end{aligned}\tag{25}$$

which we call the uncertain linguistic weighted Heronian mean (ULWHM).

3.2. The ULGHM and the ULWGHM. The geometric Heronian mean (GHM_2) proposed by Yu [33] has the capability to capture the interrelationship among the input arguments. In this section, we will extend the GHM_2 to accommodate the situations in which the input arguments are uncertain linguistic variables. Based on the operational rules on uncertain linguistic variables and Definition 6, we give the uncertain linguistic geometric Heronian mean (ULGHM) as follows.

Definition 14. Let $p,q\geq 0$ and p,q do not take the value 0 simultaneously. Let $\tilde{s}_i=[s_{\alpha_i},s_{\beta_i}]$ $(i=1,2,\ldots,n)$ be a collection of uncertain linguistic variables. If

$$\begin{aligned}
&\mathrm{ULGHM}^{p,q}(\tilde{s}_1,\tilde{s}_2,\ldots,\tilde{s}_n)\\
&=\frac{1}{p+q}\bigotimes_{i=1,j=i}^{n}\left(p\tilde{s}_i\oplus q\tilde{s}_j\right)^{2/n(n+1)}\\
&=\left[\frac{1}{p+q}\bigotimes_{i=1,j=i}^{n}\left(ps_{\alpha_i}\oplus qs_{\alpha_j}\right)^{2/n(n+1)},\right.\\
&\qquad\left.\frac{1}{p+q}\bigotimes_{i=1,j=i}^{n}\left(ps_{\beta_i}\oplus qs_{\beta_j}\right)^{2/n(n+1)}\right],
\end{aligned}\tag{26}$$

then the ULGHM is called the uncertain linguistic geometric Heronian mean (ULGHM). If $p=q$, then the ULGHM reduces to

$$\begin{aligned}
&\mathrm{ULGHM}^{p,q}(\tilde{s}_1,\tilde{s}_2,\ldots,\tilde{s}_n)\\
&=\frac{1}{2}\bigotimes_{i=1,j=i}^{n}\left(\tilde{s}_i\oplus\tilde{s}_j\right)^{2/n(n+1)}\\
&=\frac{1}{2}\left[\bigotimes_{i=1,j=i}^{n}\left(s_{\alpha_i}\oplus s_{\alpha_j}\right)^{2/n(n+1)},\bigotimes_{i=1,j=i}^{n}\left(s_{\beta_i}\oplus s_{\beta_j}\right)^{2/n(n+1)}\right]
\end{aligned}\tag{27}$$

which we call the uncertain linguistic evolution Heronian mean (ULEHM).

In the following, we investigate the desirable properties of the ULGHM, and they can be derived easily.

Theorem 15 (idempotency). *Let $\tilde{s}_i=[s_{\alpha_i},s_{\beta_i}]$ $(i=1,2,\ldots,n)$ be a collection of uncertain linguistic variables. If all $\tilde{s}_i$ are equal, that is, $\tilde{s}_i=\tilde{s}=[s_\alpha,s_\beta]$ for all i, then*

$$\mathit{ULGHM}^{p,q}(\tilde{s}_1,\tilde{s}_2,\ldots,\tilde{s}_n)=\tilde{s}.\tag{28}$$

Theorem 16 (permutation). *Let $\tilde{s}_i=[s_{\alpha_i},s_{\beta_i}]$ and $\tilde{s}_i'=[s_{\alpha_i}',s_{\beta_i}']$ $(i=1,2,\ldots,n)$ be two collections of uncertain linguistic variables; then*

$$\mathit{ULGHM}^{p,q}(\tilde{s}_1,\tilde{s}_2,\ldots,\tilde{s}_n)=\mathit{ULGHM}^{p,q}\left(\tilde{s}_1',\tilde{s}_2',\ldots,\tilde{s}_n'\right),\tag{29}$$

where $\tilde{s}_i'=[s_{\alpha_i}',s_{\beta_i}']$ $(i=1,2,\ldots,n)$ is any permutation of $\tilde{s}_i=[s_{\alpha_i},s_{\beta_i}]$ $(i=1,2,\ldots,n)$.

Theorem 17 (monotonicity). *Let $\tilde{s}_i = [s_{\alpha_i}, s_{\beta_i}]$ and $\tilde{s}_i' = [s_{\alpha_i}', s_{\beta_i}']$ $(i = 1, 2, \ldots, n)$ be two collections of uncertain linguistic variables. If $s_{\alpha_i} \le s_{\alpha_i}'$, $s_{\beta_i} \le s_{\beta_i}'$ for all i, then*

$$ULGHM^{p,q}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \le ULGHM^{p,q}(\tilde{s}_1', \tilde{s}_2', \ldots, \tilde{s}_n'). \quad (30)$$

Theorem 18 (boundedness). *Let $\tilde{s}_i = [s_{\alpha_i}, s_{\beta_i}]$ $(i = 1, 2, \ldots, n)$ be a collection of uncertain linguistic variables, and*

$$\tilde{s}^- = \min_i \tilde{s}_i = \left[\min_i s_{\alpha_i}, \min_i s_{\beta_i}\right],$$
$$\tilde{s}^+ = \max_i \tilde{s}_i = \left[\max_i s_{\alpha_i}, \max_i s_{\beta_i}\right], \quad (31)$$

then

$$\tilde{s}^- \le ULGHM^{p,q}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \le \tilde{s}^+. \quad (32)$$

It is noted that the uncertain linguistic geometric Heronian mean (ULGHM) does not consider the importance of each argument. In the following, we introduce the uncertain linguistic weighted geometric Heronian mean (ULWGHM).

Definition 19. Let $p, q \ge 0$ and p, q do not take the value 0 simultaneously. Let $\tilde{s}_i = [s_{\alpha_i}, s_{\beta_i}]$ $(i = 1, 2, \ldots, n)$ be a collection of uncertain linguistic variables. If

$$\begin{aligned}
&\mathrm{ULWGHM}_w^{p,q}(a_1, a_2, \ldots, a_n) \\
&= \frac{1}{p+q} \bigotimes_{i=1, j=i}^{n} \left(p\tilde{s}_i^{w_i} \oplus q\tilde{s}_j^{w_j}\right)^{2/n(n+1)} \\
&= \left[\frac{1}{p+q} \bigotimes_{i=1, j=i}^{n} \left(ps_{\alpha_i}^{w_i} \oplus qs_{\alpha_j}^{w_j}\right)^{2/n(n+1)}, \right. \\
&\qquad \left. \frac{1}{p+q} \bigotimes_{i=1, j=i}^{n} \left(ps_{\beta_i}^{w_i} \oplus qs_{\beta_j}^{w_j}\right)^{2/n(n+1)}\right],
\end{aligned} \quad (33)$$

then ULWGHM is called the uncertain linguistic weighted geometric Heronian mean (ULWGHM). If $p = q$, then the ULWGHM reduces to

$$\begin{aligned}
&\mathrm{ULWGHM}_w^{p,q}(a_1, a_2, \ldots, a_n) \\
&= \frac{1}{2} \bigotimes_{i=1, j=i}^{n} \left(\tilde{s}_i^{w_i} \oplus \tilde{s}_j^{w_j}\right)^{2/n(n+1)} \\
&= \frac{1}{2} \left[\bigotimes_{i=1, j=i}^{n} \left(s_{\alpha_i}^{w_i} \oplus s_{\alpha_j}^{w_j}\right)^{2/n(n+1)}, \bigotimes_{i=1, j=i}^{n} \left(s_{\beta_i}^{w_i} \oplus s_{\beta_j}^{w_j}\right)^{2/n(n+1)}\right]
\end{aligned} \quad (34)$$

which we call the uncertain linguistic weighted evolution Heronian mean (ULWGHM).

4. A Method for Multiple Attribute Decision Making Based on Heronian Means under Uncertain Linguistic Environment

In this section, we consider a multiple attribute decision making problem with uncertain linguistic information. The generalized uncertain linguistic weighted Heronian mean (GULWHM) or the uncertain linguistic weighted geometric Heronian mean (ULWGHM) proposed in Section 3 will be used to solve the multiple attribute decision making problem.

Let $A = \{A_1, A_2, \ldots, A_m\}$ be the set of alternatives and $C = \{C_1, C_2, \ldots, C_n\}$ the set of attributes, whose weight vector is $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$. The decision makers use the uncertain linguistic variable to provide the linguistic expression under the attribute C_j for the alternative A_i and construct the uncertain linguistic decision matrix $D = (\tilde{d}_{ij})_{m\times n}$. In the following, based on the GULWHM or the ULWGHM, we develop an approach to multiple attribute decision making with uncertain linguistic information.

Step 1. Utilize the GULWHM as

$$\begin{aligned}
\tilde{d}_i &= \left[s_{\alpha_i}, s_{\beta_i}\right] = \mathrm{GULWHM}_w^{p,q}\left(\tilde{d}_{i1}, \tilde{d}_{i2}, \ldots, \tilde{d}_{in}\right) \\
&= \left(\frac{2}{n(n+1)} \bigoplus_{k=1, l=k}^{n} \left(\left(w_k\tilde{d}_{ik}\right)^p \otimes \left(w_l\tilde{d}_{il}\right)^q\right)\right)^{1/(p+q)} \\
&\qquad (i = 1, 2, \ldots, m),
\end{aligned} \quad (35)$$

or the ULWGHM as

$$\begin{aligned}
\tilde{d}_i &= \left[s_{\alpha_i}, s_{\beta_i}\right] = \mathrm{ULWGHM}_w^{p,q}\left(\tilde{d}_{i1}, \tilde{d}_{i2}, \ldots, \tilde{d}_{in}\right) \\
&= \frac{1}{p+q} \bigotimes_{k=1, l=k}^{n} \left(p\tilde{d}_{ik}^{w_k} \oplus q\tilde{d}_{il}^{w_l}\right)^{2/n(n+1)} \quad (i = 1, 2, \ldots, m)
\end{aligned} \quad (36)$$

to get the overall attribute value $\tilde{d}_i$ of the alternative A_i $(i = 1, 2, \ldots, m)$.

Step 2. To rank these overall attribute values $\tilde{d}_i$ $(i = 1, 2, \ldots, m)$, we first compare each $\tilde{d}_i$ with all the $\tilde{d}_j$ $(j = 1, 2, \ldots, m)$ by using (2). Then a complementary matrix $P = (p_{ij})_{m\times m}$ is developed, where

$$\begin{aligned}
p_{ij} = p\left(\tilde{d}_i \ge \tilde{d}_j\right), \quad p_{ij} \ge 0,\ p_{ij} + p_{ji} = 1, \\
p_{ii} = 0.5,\ i, j = 1, 2, \ldots, m.
\end{aligned} \quad (37)$$

Summing all the elements in each line of matrix $P = (p_{ij})_{m\times m}$, we have $p_i = \sum_{j=1}^{m} p_{ij}$, $i = 1, 2, \ldots, m$. Then we rank the overall attribute values $\tilde{d}_i$ in descending order according to the values of p_i $(i = 1, 2, \ldots, m)$.

Step 3. Rank all the alternatives A_i and select the desirable one in accordance with the values of $\tilde{d}_i$ $(i = 1, 2, \ldots, m)$.

Step 4. End.

Table 1: Uncertain linguistic decision matrix D.

	C_1	C_2	C_3	C_4
A_1	$[s_3, s_4]$	$[s_5, s_7]$	$[s_2, s_3]$	$[s_3, s_4]$
A_2	$[s_2, s_3]$	$[s_2, s_3]$	$[s_4, s_6]$	$[s_4, s_5]$
A_3	$[s_4, s_5]$	$[s_4, s_6]$	$[s_5, s_6]$	$[s_6, s_7]$
A_4	$[s_3, s_5]$	$[s_6, s_7]$	$[s_4, s_5]$	$[s_5, s_6]$

5. Example Illustration and Discussion

In this section, an example adapted from [29] is given to illustrate the application of the methods proposed in this paper.

5.1. Example Illustration

Example 20 (see [29]). Suppose an organization plans to implement ERP system. The first step is to form a project team that consists of CIO and two senior representatives from user departments. By collecting all possible information about ERP vendors and systems, project team chooses four potential ERP systems A_i $(i = 1, 2, 3, 4)$ as candidates. The company employs some external professional organizations (or experts) to aid this decision making. The project team selects four attributes to evaluate the alternatives: (1) function and technology C_1, (2) strategic fitness C_2, (3) vendor's ability C_3, and (4) vendor's reputation C_4. Decision makers use the uncertain linguistic variables to evaluate the four possible alternatives A_i $(i = 1, 2, 3, 4)$ under the above four attributes (whose weight vector is $w = (0.2, 0.1, 0.3, 0.4)^T$) and construct the uncertain linguistic decision matrix $D = (\tilde{d}_{ij})_{4\times 4}$ listed in Table 1.

In the following, we use the proposed methods to get the most desirable system.

Step 1. Utilize the GULWHM as

$$\tilde{d}_i = [s_{\alpha_i}, s_{\beta_i}] = \mathrm{GULWHM}_w^{p,q}(\tilde{d}_{i1}, \tilde{d}_{i2}, \dots, \tilde{d}_{in}) = \left(\frac{2}{n(n+1)} \bigoplus_{k=1,\, l=k}^{n} \left((w_k \tilde{d}_{ik})^p \otimes (w_l \tilde{d}_{il})^q\right)\right)^{1/(p+q)} \quad (i = 1, 2, \dots, m) \tag{38}$$

to obtain the overall attribute value $\tilde{d}_i$ for the alternative A_i $(i = 1, 2, 3, 4)$, and let $p = q = 1$. We have

$$\tilde{d}_1 = [s_{0.74}, s_{1.01}], \quad \tilde{d}_2 = [s_{0.89}, s_{1.22}], \quad \tilde{d}_3 = [s_{1.32}, s_{1.59}], \quad \tilde{d}_4 = [s_{1.13}, s_{1.43}]. \tag{39}$$

Step 2. To rank these overall attribute values $\tilde{d}_i$ $(i = 1, \dots, 4)$, we first compare each $\tilde{d}_i$ with all the $\tilde{d}_j$ $(j = 1, \dots, 4)$ by using (2). Then a complementary matrix $P = (p_{ij})_{4\times 4}$ is developed as

$$P = (p_{ij})_{4\times 4} = \begin{bmatrix} 0.500 & 0.200 & 0.000 & 0.000 \\ 0.800 & 0.500 & 0.000 & 0.143 \\ 1.000 & 1.000 & 0.500 & 0.807 \\ 1.000 & 0.857 & 0.193 & 0.500 \end{bmatrix}. \tag{40}$$

Summing all the elements in each line of matrix $P = (p_{ij})_{4\times 4}$, we have

$$p_1 = 0.700, \quad p_2 = 1.443, \quad p_3 = 3.307, \quad p_4 = 2.550. \tag{41}$$

Then we rank the overall attribute values $\tilde{d}_i$ in descending order according to the values of p_i $(i = 1, 2, 3, 4)$ as

$$\tilde{d}_3 \succ \tilde{d}_4 \succ \tilde{d}_2 \succ \tilde{d}_1. \tag{42}$$

Step 3. Rank all the alternatives A_i in accordance with the values of $\tilde{d}_i$ $(i = 1, 2, 3, 4)$ as

$$A_3 \succ A_4 \succ A_2 \succ A_1. \tag{43}$$

Thus, the most desirable system is A_3.

If we use the ULWGHM to solve the above multiple attribute decision making problem and let $p = q$, then the overall attribute values $\tilde{d}_i$ of the alternative A_i $(i = 1, 2, 3, 4)$ can be obtained as follows:

$$\tilde{d}_1 = [s_{1.30}, s_{1.41}], \quad \tilde{d}_2 = [s_{1.35}, s_{1.47}], \quad \tilde{d}_3 = [s_{1.51}, s_{1.59}], \quad \tilde{d}_4 = [s_{1.45}, s_{1.55}]. \tag{44}$$

To rank these overall attribute values $\tilde{d}_i$ $(i = 1, 2, 3, 4)$, we first compare each $\tilde{d}_i$ with all the $\tilde{d}_j$ $(j = 1, 2, 3, 4)$ by using (2). Then a complementary matrix $P = (p_{ij})_{4\times 4}$ is developed as

$$P = (p_{ij})_{4\times 4} = \begin{bmatrix} 0.500 & 0.261 & 0.000 & 0.000 \\ 0.739 & 0.500 & 0.000 & 0.091 \\ 1.000 & 1.000 & 0.500 & 0.778 \\ 1.000 & 0.909 & 0.222 & 0.500 \end{bmatrix}. \tag{45}$$

Summing all the elements in each line of matrix $P = (p_{ij})_{4\times 4}$, we have

$$p_1 = 0.761, \quad p_2 = 1.330, \quad p_3 = 3.278, \quad p_4 = 2.631. \tag{46}$$

Then we rank the overall attribute values $\tilde{d}_i$ in descending order according to the values of p_i $(i = 1, 2, 3, 4)$ as

$$\tilde{d}_3 \succ \tilde{d}_4 \succ \tilde{d}_2 \succ \tilde{d}_1. \tag{47}$$

Rank all the alternatives A_i in accordance with the values of $\tilde{d}_i$ $(i = 1, 2, 3, 4)$ as

$$A_3 \succ A_4 \succ A_2 \succ A_1. \tag{48}$$

Thus, the most desirable system is A_3 and the ranking is the same as obtained by the GULWHM.

Table 2: Overall attribute values by the $\mathrm{GULWHM}_w^{p,p}$ and the rankings of the alternatives.

	A_1	A_2	A_3	A_4	Ranking
$\mathrm{GULWHM}_w^{1/2,1/2}$	$[s_{0.71}, s_{0.98}]$	$[s_{0.76}, s_{1.07}]$	$[s_{1.18}, s_{1.46}]$	$[s_{1.04}, s_{1.34}]$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$\mathrm{GULWHM}_w^{1,1}$	$[s_{0.74}, s_{1.01}]$	$[s_{0.89}, s_{1.22}]$	$[s_{1.32}, s_{1.59}]$	$[s_{1.13}, s_{1.43}]$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$\mathrm{GULWHM}_w^{2,2}$	$[s_{0.80}, s_{1.09}]$	$[s_{1.07}, s_{1.44}]$	$[s_{1.55}, s_{1.84}]$	$[s_{1.30}, s_{1.60}]$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$\mathrm{GULWHM}_w^{5,5}$	$[s_{0.96}, s_{1.28}]$	$[s_{1.30}, s_{1.70}]$	$[s_{1.93}, s_{2.25}]$	$[s_{1.60}, s_{1.79}]$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$\mathrm{GULWHM}_w^{10,10}$	$[s_{1.07}, s_{1.43}]$	$[s_{1.43}, s_{1.82}]$	$[s_{2.14}, s_{2.50}]$	$[s_{1.78}, s_{2.14}]$	$A_3 \succ A_4 \succ A_2 \succ A_1$

Table 3: Overall attribute values by the $\mathrm{GULWHM}_w^{1,q}$ and the rankings of the alternatives.

	A_1	A_2	A_3	A_4	Ranking
$\mathrm{GULWHM}_w^{1,1/2}$	$[s_{0.69}, s_{0.95}]$	$[s_{0.77}, s_{1.08}]$	$[s_{1.17}, s_{1.43}]$	$[s_{1.02}, s_{1.31}]$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$\mathrm{GULWHM}_w^{1,1}$	$[s_{0.74}, s_{1.01}]$	$[s_{0.89}, s_{1.22}]$	$[s_{1.32}, s_{1.59}]$	$[s_{1.13}, s_{1.43}]$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$\mathrm{GULWHM}_w^{1,2}$	$[s_{0.81}, s_{1.10}]$	$[s_{1.04}, s_{1.41}]$	$[s_{1.53}, s_{1.83}]$	$[s_{1.29}, s_{1.60}]$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$\mathrm{GULWHM}_w^{1,5}$	$[s_{0.95}, s_{1.28}]$	$[s_{1.26}, s_{1.65}]$	$[s_{1.87}, s_{2.19}]$	$[s_{1.56}, s_{1.90}]$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$\mathrm{GULWHM}_w^{1,10}$	$[s_{1.05}, s_{1.41}]$	$[s_{1.39}, s_{1.78}]$	$[s_{2.09}, s_{2.44}]$	$[s_{1.74}, s_{2.10}]$	$A_3 \succ A_4 \succ A_2 \succ A_1$

Table 4: Overall attribute values by the $\mathrm{GULWHM}_w^{p,1}$ and the rankings of the alternatives.

	A_1	A_2	A_3	A_4	Ranking
$\mathrm{GULWHM}_w^{1/2,1}$	$[s_{0.76}, s_{1.04}]$	$[s_{0.90}, s_{1.24}]$	$[s_{1.36}, s_{1.64}]$	$[s_{1.17}, s_{1.48}]$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$\mathrm{GULWHM}_w^{1,1}$	$[s_{0.74}, s_{1.01}]$	$[s_{0.89}, s_{1.22}]$	$[s_{1.32}, s_{1.59}]$	$[s_{1.13}, s_{1.43}]$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$\mathrm{GULWHM}_w^{2,1}$	$[s_{0.74}, s_{1.02}]$	$[s_{0.96}, s_{1.31}]$	$[s_{1.38}, s_{1.65}]$	$[s_{1.16}, s_{1.45}]$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$\mathrm{GULWHM}_w^{5,1}$	$[s_{0.84}, s_{1.12}]$	$[s_{1.16}, s_{1.55}]$	$[s_{1.68}, s_{1.96}]$	$[s_{1.39}, s_{1.68}]$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$\mathrm{GULWHM}_w^{10,1}$	$[s_{0.97}, s_{1.30}]$	$[s_{1.31}, s_{1.70}]$	$[s_{1.95}, s_{2.28}]$	$[s_{1.62}, s_{1.95}]$	$A_3 \succ A_4 \succ A_2 \succ A_1$

Table 5: Overall attribute values by the $\mathrm{ULWGHM}_w^{1,q}$ and the rankings of the alternatives.

	A_1	A_2	A_3	A_4	Ranking
$\mathrm{ULWGHM}_w^{1,1/2}$	$[s_{1.28}, s_{1.39}]$	$[s_{1.32}, s_{1.43}]$	$[s_{1.47}, s_{1.57}]$	$[s_{1.41}, s_{1.51}]$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$\mathrm{ULWGHM}_w^{1,1}$	$[s_{1.30}, s_{1.41}]$	$[s_{1.35}, s_{1.47}]$	$[s_{1.51}, s_{1.59}]$	$[s_{1.45}, s_{1.55}]$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$\mathrm{ULWGHM}_w^{1,2}$	$[s_{1.31}, s_{1.43}]$	$[s_{1.39}, s_{1.51}]$	$[s_{1.55}, s_{1.64}]$	$[s_{1.49}, s_{1.58}]$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$\mathrm{ULWGHM}_w^{1,5}$	$[s_{1.33}, s_{1.45}]$	$[s_{1.42}, s_{1.55}]$	$[s_{1.59}, s_{1.68}]$	$[s_{1.52}, s_{1.62}]$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$\mathrm{ULWGHM}_w^{1,10}$	$[s_{1.34}, s_{1.46}]$	$[s_{1.44}, s_{1.57}]$	$[s_{1.61}, s_{1.70}]$	$[s_{1.53}, s_{1.64}]$	$A_3 \succ A_4 \succ A_2 \succ A_1$

5.2. Discussion. If the parameter p or q takes the value of zero, then the GULWHM and ULWGHM cannot capture the interrelationship of the input arguments. Moreover, different overall attribute values $\tilde{d}_i$ of the alternatives A_i $(i = 1, 2, 3, 4)$ can be obtained, and it needs much more calculation effort as the parameters p and q change. Here, we will list some of them. From Table 2, we can find that the overall attribute values obtained by the GULWHM become bigger as the parameters p and q increase simultaneously for the same aggregation arguments. If the parameter p is fixed (without loss of generality, p takes the value 1) and the parameter q increases, the overall attribute values obtained by the GULWHM and shown in Table 3 become bigger for the same aggregation arguments. Similarly, if the parameter q is fixed $(q = 1)$, the aggregated results in Table 4 show that the overall attribute values obtained by the GULWHM for the same aggregation arguments firstly experience a decrease and then become bigger as the parameter p increases. The different parameters play an important part in decision making. The decision makers who take a pessimistic view for prospect can choose the smaller values of the parameters p and q, while the decision makers who take an optimistic view for prospect can choose the bigger values of the parameters p or q.

If we utilize the ULWGHM to aggregate the arguments, some different overall attribute values $\tilde{d}_i$ of the alternatives A_i $(i = 1, 2, 3, 4)$ are listed in Tables 5 and 6. If the parameter p is fixed $(p = 1)$, the overall attribute values obtained by the ULWGHM become bigger as the parameter q increases for the same aggregation arguments. If the parameter q is fixed $(q = 1)$, the overall attribute values obtained by the ULWGHM become smaller as the parameter p increases for the same aggregation arguments. Therefore, the decision makers who take a pessimistic view for prospect can choose the smaller values of the parameter q or the bigger values of the parameter p, while the decision makers who take an optimistic view for prospect can choose the bigger values of the parameter q or the smaller values of the parameter p. From Tables 2 to 6, we can find that the overall attribute values

Table 6: Overall attribute values by the $\text{ULWGHM}_w^{p,1}$ and the rankings of the alternatives.

	A_1	A_2	A_3	A_4	Ranking
$\text{ULWGHM}_w^{1/2,1}$	$[s_{1.31}, s_{1.43}]$	$[s_{1.39}, s_{1.51}]$	$[s_{1.55}, s_{1.64}]$	$[s_{1.49}, s_{1.58}]$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$\text{ULWGHM}_w^{1,1}$	$[s_{1.30}, s_{1.41}]$	$[s_{1.35}, s_{1.47}]$	$[s_{1.51}, s_{1.59}]$	$[s_{1.45}, s_{1.55}]$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$\text{ULWGHM}_w^{2,1}$	$[s_{1.28}, s_{1.39}]$	$[s_{1.32}, s_{1.43}]$	$[s_{1.47}, s_{1.54}]$	$[s_{1.41}, s_{1.51}]$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$\text{ULWGHM}_w^{5,1}$	$[s_{1.26}, s_{1.36}]$	$[s_{1.28}, s_{1.39}]$	$[s_{1.43}, s_{1.50}]$	$[s_{1.38}, s_{1.47}]$	$A_3 \succ A_4 \succ A_2 \succ A_1$
$\text{ULWGHM}_w^{10,1}$	$[s_{1.25}, s_{1.35}]$	$[s_{1.26}, s_{1.37}]$	$[s_{1.40}, s_{1.47}]$	$[s_{1.36}, s_{1.45}]$	$A_3 \succ A_4 \succ A_2 \succ A_1$

of each alternative derived by the GULWHM or ULWGHM depend on the choice of the parameters p and q, but the ranking is kept unchanged.

6. Concluding Remarks

The Heronian mean can reflect the correlation of the aggregated arguments and is usually used to aggregate the information taken the form of numerical numbers. In this paper, we extend the Heronian mean to accommodate the situation where the input arguments are uncertain linguistic variables and develop some uncertain linguistic Heronian means such as the generalized uncertain linguistic Heronian mean (GULHM) and uncertain linguistic geometric Heronian mean (ULGHM). Some desirable properties of these means such as idempotency, permutation, monotonicity, and boundedness are also discussed. Moreover, to aggregate uncertain linguistic variables and embody different importance of the input arguments, we then define the generalized uncertain linguistic weighted Heronian mean (GULWHM) and uncertain linguistic weighted geometric Heronian mean (ULWGHM). The proposed means take the interrelationship of the input arguments into account, and it is a flexible multiple attribute decision making method in that the decision makers can choose different values of the parameters p and q according to their actual needs. To demonstrate the effectiveness and feasibility of the developed uncertain linguistic Heronian means, an example about ERP system is given. In future research, we will continue to study the Heronian mean, and some other types of Heronian mean will also be investigated.

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