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Research Article

Convolution Properties for Certain Classes of Analytic Functions Defined by q-Derivative Operator

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We investigate convolution properties and coefficients estimates for two classes of analytic functions involving the q-derivative operator defined in the open unit disc. Some of our results improve previously known results.

1. Introduction

Simply, h-calculus or q-calculus is ordinary classical calculus without the notion of limits. Here h ostensibly stands for Planck's constant, while q stands for quantum. Recently, the area of q-calculus has attracted the serious attention of researchers. This great interest is due to its application in various branches of mathematics and physics. The application of q-calculus was initiated by Jackson [1, 2]. He was the first to develop q-integral and q-derivative in a systematic way. Later, geometrical interpretation of q-analysis has been recognized through studies on quantum groups. It also suggests a relation between integrable systems and q-analysis. Aral and Gupta [3–5] defined and studied the q-analogue of Baskakov Durrmeyer operator which is based on q-analogue of beta function. Another important q-generalization of complex operators is q-Picard and q-Gauss-Weierstrass singular integral operators discussed in [6–8]. Mohammed and Darus [9] studied approximation and geometric properties of these q-operators in some subclasses of analytic functions in compact disk. These q-operators are defined by using convolution of normalized analytic functions and q-hypergeometric functions, where several interesting results are obtained (see also [10, 11]). A comprehensive study on applications of q-calculus in operator theory may be found in [12].

Let $\mathscr{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathscr{S}(\alpha)$ and $\mathscr{K}(\alpha)$ $(0 \le \alpha < 1)$ denote the subclasses of $\mathscr{A}$ that consists, respectively, of starlike of order α and convex of order α in $\mathbb{U}$ (see [13]). If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$, written $f(z) \prec g(z)$ if there exists a Schwarz function ω, which (by definition) is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(\omega(z))$, $z \in \mathbb{U}$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence (see [14–16]):

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0), \quad f(\mathbb{U}) \subset g(\mathbb{U}). \tag{2}$$

For functions f given by (1) and g given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \tag{3}$$

the Hadamard product or convolution of f and g is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \quad (4)$$

Let $\mathcal{S}[A,B]$ and $\mathcal{K}[A,B]$ denote the subclasses of the class $\mathcal{A}$ for $-1 \le B < A \le 1$ which are defined by (see [17–22])

$$\mathcal{S}[A,B] = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz},\ z \in \mathbb{U} \right\},$$
$$\mathcal{K}[A,B] = \left\{ f \in \mathcal{A} : \frac{(zf'(z))'}{f'(z)} \prec \frac{1+Az}{1+Bz},\ \ z \in \mathbb{U} \right\}. \quad (5)$$

We note that

$$\mathcal{S}[1-2\alpha,-1] = \mathcal{S}(\alpha), \qquad \mathcal{K}[1-2\alpha,-1] = \mathcal{K}(\alpha) \quad (0 \le \alpha < 1). \quad (6)$$

For function $f \in \mathcal{A}$ given by (1) and $0 < q < 1$, the q-derivative of a function f is defined by (see [1])

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} \quad (z \ne 0), \quad (7)$$

and $D_q f(0) = f'(0)$. From (7), we deduce that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, \quad z \ne 0, \quad (8)$$

where

$$[k]_q = \frac{1-q^k}{1-q}. \quad (9)$$

As $q \to 1$, $[k]_q \to k$. For a function $h(z) = z^k$, we observe that

$$D_q h(z) = D_q(z^k) = \frac{1-q^k}{1-q} z^{k-1} = [k]_q z^{k-1},$$
$$\lim_{q \to 1} D_q h(z) = \lim_{q \to 1} [k]_q z^{k-1} = k z^{k-1} = h'(z), \quad (10)$$

where h' is the ordinary derivative.

Making use of the q-derivative $D_q f(z)$, we introduce the subclasses $\mathcal{S}_q[A,B]$ and $\mathcal{K}_q[A,B]$ of $\mathcal{A}$ for $0 < q < 1$ and $-1 \le B < A \le 1$ as follows:

$$\mathcal{S}_q[A,B] = \left\{ f \in \mathcal{A} : \frac{zD_q f(z)}{f(z)} \prec \frac{1+Az}{1+Bz},\ \ z \in \mathbb{U} \right\},$$
$$\mathcal{K}_q[A,B] = \left\{ f \in \mathcal{A} : \frac{D_q(zD_q f(z))}{D_q f(z)} \prec \frac{1+Az}{1+Bz},\ \ z \in \mathbb{U} \right\}. \quad (11)$$

We note that

(i) $\mathcal{S}_q[1-2\alpha,-1] = \mathcal{S}_q(\alpha)$ $(0 \le \alpha < 1)$

$$\mathcal{S}_q(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zD_q f(z)}{f(z)} > \alpha,\ z \in \mathbb{U} \right\}; \quad (12)$$

(ii) $\mathcal{K}_q[1-2\alpha,-1] = \mathcal{K}_q(\alpha)$ $(0 \le \alpha < 1)$

$$\mathcal{K}_q(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{D_q(zD_q f(z))}{D_q f(z)} > \alpha,\ z \in \mathbb{U} \right\}; \quad (13)$$

(iii) $\mathcal{S}_q[(1-2\alpha)\beta,-\beta] = \mathcal{S}_q(\alpha,\beta)$ $(0 \le \alpha < 1,\ 0 < \beta \le 1)$

$$\mathcal{S}_q(\alpha,\beta) = \left\{ f \in \mathcal{A} : \left| \frac{(zD_q f(z)/f(z)) - 1}{(zD_q f(z)/f(z)) + 1 - 2\alpha} \right| < \beta,\ z \in \mathbb{U} \right\}, \quad (14)$$

(iv) $\mathcal{K}_q[(1-2\alpha)\beta,-\beta] = \mathcal{K}_q(\alpha,\beta)$ $(0 \le \alpha < 1,\ 0 < \beta \le 1)$

$$\mathcal{K}_q(\alpha,\beta) = \left\{ f \in \mathcal{A} : \left| \frac{(D_q(zD_q f(z))/D_q f(z)) - 1}{(D_q(zD_q f(z))/D_q f(z)) + 1 - 2\alpha} \right| < \beta,\ z \in \mathbb{U} \right\}, \quad (15)$$

(v)

$$\lim_{q \to 1} \mathcal{S}_q[A,B] = \left\{ f \in \mathcal{A} : \lim_{q \to 1} \frac{zD_q f(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \right\} = \mathcal{S}[A,B],$$
$$\lim_{q \to 1} \mathcal{K}_q[A,B] = \left\{ f \in \mathcal{A} : \lim_{q \to 1} \frac{D_q(zD_q f(z))}{D_q f(z)} \prec \frac{1+Az}{1+Bz} \right\} = \mathcal{K}[A,B]. \quad (16)$$

From (11), we have

$$f \in \mathcal{K}_q[A,B] \Longleftrightarrow zD_q f \in \mathcal{S}_q[A,B]. \quad (17)$$

In this paper, we investigate convolution properties, the necessary and sufficient condition and coefficient estimates for the classes $\mathcal{S}_q[A,B]$ and $\mathcal{K}_q[A,B]$ associated with the q-derivative $D_q f(z)$. The motivation of this paper is to improve and generalize previously known results.

2. Convolution Properties

Unless otherwise mentioned, we assume throughout this section that $\theta \in [0, 2\pi)$, $0 < q < 1$ and $-1 \le B < A \le 1$.

Theorem 1. *The function f defined by* (1) *is in the class $\mathcal{S}_q[A, B]$ if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z - Lqz^2}{(1-z)(1-qz)}\right] \ne 0 \quad (z \in \mathbb{U}) \tag{18}$$

for all $L = L_\theta = (e^{-i\theta} + A)/(A - B)$ and also $L = 1$.

Proof. First suppose f defined by (1) is in the class $\mathcal{S}_q[A, B]$; we have

$$\frac{zD_q f(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}. \tag{19}$$

Since the function from the left-hand side of the subordination is analytic in $\mathbb{U}$, it follows $f(z) \ne 0, z \in \mathbb{U}^* = \mathbb{U} \setminus \{0\}$; that is, $(1/z)f(z) \ne 0$, $z \in \mathbb{U}$, and this is equivalent to the fact that (18) holds for $L = 1$. From (19) according to the subordination of two analytic functions we say that there exists a function $w(z)$ analytic in $\mathbb{U}$ with $w(0) = 0$, $|w(z)| < 1$ such that

$$\frac{zD_q f(z)}{f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)} \quad (z \in \mathbb{U}) \tag{20}$$

which is equivalent to

$$\frac{zD_q f(z)}{f(z)} \ne \frac{1 + Ae^{i\theta}}{1 + Be^{i\theta}} \quad (z \in \mathbb{U};\ 0 \le \theta < 2\pi), \tag{21}$$

or

$$\frac{1}{z}\left[\left(1 + Be^{i\theta}\right) zD_q f(z) - \left(1 + Ae^{i\theta}\right) f(z)\right] \ne 0 \quad (z \in \mathbb{U};\ 0 \le \theta < 2\pi). \tag{22}$$

Since

$$f(z) * \frac{z}{1-z} = f(z), \qquad f(z) * \frac{z}{(1-z)(1-qz)} = zD_q f(z). \tag{23}$$

Now from (23), we may write (22) as

$$\frac{1}{z}\left[f(z) * \left(\frac{\left(1 + Be^{i\theta}\right) z}{(1-z)(1-qz)} - \frac{\left(1 + Ae^{i\theta}\right) z}{1-z}\right)\right] = \frac{(B - A)e^{i\theta}}{z} \times \left[f(z) * \frac{z - \left(\left(e^{-i\theta} + A\right)/(A - B)\right) qz^2}{(1-z)(1-qz)}\right] \ne 0 \quad (z \in \mathbb{U};\ 0 \le \theta < 2\pi), \tag{24}$$

which leads to (18), which proves the necessary part of Theorem 1.

Reversely, because assumption (18) holds for $L = 1$, it follows that $(1/z)f(z) \ne 0$ for all $z \in \mathbb{U}$; hence, the function $\varphi(z) = zD_q f(z)/f(z)$ is analytic in $\mathbb{U}$ (i.e., it is regular at $z_0 = 0$, with $\varphi(0) = 0$). Since it was shown in the first part of the proof that assumption (18) is equivalent to (21), we obtain that

$$\frac{zD_q f(z)}{f(z)} \ne \frac{1 + Ae^{i\theta}}{1 + Be^{i\theta}} \quad (z \in \mathbb{U};\ 0 \le \theta < 2\pi), \tag{25}$$

and if we denote

$$\psi(z) = \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}), \tag{26}$$

relation (25) shows that $\varphi(\mathbb{U}) \cap \psi(\mathbb{U}) = \emptyset$. Thus, the simply connected domain $\varphi(\mathbb{U})$ is included in a connected component of $\mathbb{C} \setminus \psi(\partial\mathbb{U})$. From here, using the fact that $\varphi(0) = \psi(0)$ together with the univalence of the function ψ, it follows that $\varphi(z) \prec \psi(z)$, which represents in fact subordination (19); that is, $f \in \mathcal{S}_q[A, B]$. This completes the proof of Theorem 1. □

Taking $q \to 1^-$ in Theorem 1, we obtain the following result which improves the convolution result of Aouf and Seoudy [23, Theorem 1] and also the result of Silverman and Silvia [21, Theorem 7].

Corollary 2. *The function f defined by* (1) *is in the class $\mathcal{S}[A, B]$ if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z - Lz^2}{(1-z)^2}\right] \ne 0 \quad (z \in \mathbb{U}) \tag{27}$$

for all $L = L_\theta = (e^{-i\theta} + A)/(A - B)$ and also $L = 1$.

Putting $A = 1 - 2\alpha$ $(0 \le \alpha < 1)$ and $B = -1$ in Theorem 1, we obtain the following corollary.

Corollary 3. *The function f defined by* (1) *is in the class $\mathcal{S}_q(\alpha)$ $(0 \le \alpha < 1)$ if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z - Mqz^2}{(1-z)(1-qz)}\right] \ne 0 \quad (z \in \mathbb{U}) \tag{28}$$

for all $M = M_\theta = (e^{-i\theta} + 1 - 2\alpha)/2(1 - \alpha)$, $0 \le \alpha < 1$, and also $M = 1$.

Taking $q \to 1^-$ in Corollary 3, we obtain the following result which improves the convolution result of Silverman et al. [22, Theorems 1].

Corollary 4. *The function f defined by* (1) *is in the class $\mathcal{S}(\alpha)$ $(0 \le \alpha < 1)$ if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z - Mz^2}{(1-z)^2}\right] \ne 0 \quad (z \in \mathbb{U}) \tag{29}$$

for all $M = M_\theta = (e^{-i\theta} + 1 - 2\alpha)/2(1 - \alpha)$, $0 \le \alpha < 1$, and also $M = 1$.

Theorem 5. *The function f defined by* (1) *is in the class $\mathcal{K}_q[A, B]$ if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z + [1 - (q+1)L]qz^2}{(1-z)(1-qz)(1-q^2z)}\right] \neq 0 \quad (z \in \mathbb{U}) \tag{30}$$

for all $L = L_\theta = (e^{-i\theta} + A)/(A - B)$ and also $L = 1$.

Proof. Set

$$g(z) = \frac{z - Lqz^2}{(1-z)(1-qz)}, \tag{31}$$

and we note that

$$zD_q g(z) = \frac{z + [1 - (q+1)L]qz^2}{(1-z)(1-qz)(1-q^2z)}. \tag{32}$$

From the identity $zD_qf(z) * g(z) = f(z) * zD_qg(z)$ $(f, g \in \mathcal{A})$ and the fact that

$$f \in \mathcal{K}_q[A, B] \Longleftrightarrow zD_qf(z) \in \mathcal{S}_q[A, B] \tag{33}$$

the result follows from Theorem 1. □

Taking $q \to 1^-$ in Theorem 1, we obtain the following result which improves the result of Aouf and Seoudy [23, Theorem 2].

Corollary 6. *The function f defined by* (1) *is in the class $\mathcal{K}[A, B]$ if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z + [1 - 2L]z^2}{(1-z)^3}\right] \neq 0 \quad (z \in \mathbb{U}) \tag{34}$$

for all $L = L_\theta = (e^{-i\theta} + A)/(A - B)$ and also $L = 1$.

Putting $A = 1 - 2\alpha$ $(0 \le \alpha < 1)$ and $B = -1$ in Theorem 5, we obtain the following corollary.

Corollary 7. *The function f defined by* (1) *is in the class $\mathcal{K}_q(\alpha)$ $(0 \le \alpha < 1)$ if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z + [1 - (q+1)L]qz^2}{(1-z)(1-qz)(1-q^2z)}\right] \neq 0 \quad (z \in \mathbb{U}) \tag{35}$$

for all $M = M_\theta = (e^{-i\theta} + 1 - 2\alpha)/2(1 - \alpha)$, $0 \le \alpha < 1$, and also $L = 1$.

Taking $q \to 1^-$ in Corollary 7, we obtain the following result which improves the convolution result of Silverman et al. [22, Theorem 2].

Corollary 8. *The function f defined by* (1) *is in the class $\mathcal{K}(\alpha)$ $(0 \le \alpha < 1)$ if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z + [1 - 2L]qz^2}{(1-z)^3}\right] \neq 0 \quad (z \in \mathbb{U}) \tag{36}$$

for all $M = M_\theta = (e^{-i\theta} + 1 - 2\alpha)/2(1 - \alpha)$, $0 \le \alpha < 1$, and also $L = 1$.

Theorem 9. *A necessary and sufficient condition for the function f defined by* (1) *to be in the class $\mathcal{S}_q[A, B]$ is that*

$$1 - \sum_{k=2}^{\infty} \frac{[k]_q\left(e^{-i\theta} + B\right) - e^{-i\theta} - A}{A - B} a_k z^{k-1} \neq 0 \quad (z \in \mathbb{U}). \tag{37}$$

Proof. From Theorem 1, we find that $f \in \mathcal{S}_q[A, B]$ if and only if

$$\frac{1}{z}\left[f(z) * \frac{z - Lqz^2}{(1-z)(1-qz)}\right] \neq 0 \quad (z \in \mathbb{U}) \tag{38}$$

for all $L = L_\theta = (e^{-i\theta} + A)/(A - B)$ and also for $L = 1$. The left-hand side of (38) can be written as

$$\begin{aligned} &\frac{1}{z}\left[f(z) * \left(\frac{z}{(1-z)(1-qz)} - \frac{Lqz^2}{(1-z)(1-qz)}\right)\right] \\ &= \frac{1}{z}\left\{zD_qf(z) - L\left[zD_qf(z) - f(z)\right]\right\} \\ &= 1 - \sum_{k=2}^{\infty}\left([k]_q(L-1) - L\right)a_k z^{k-1}. \end{aligned} \tag{39}$$

Thus, the proof of The Theorem 9 is completed. □

Taking $q \to 1^-$ in Theorem 9, we obtain the following result.

Corollary 10. *A necessary and sufficient condition for the function f defined by* (1) *to be in the class $\mathcal{S}[A, B]$ is that*

$$1 - \sum_{k=2}^{\infty} \frac{k\left(e^{-i\theta} + B\right) - e^{-i\theta} - A}{A - B} a_k z^{k-1} \neq 0 \quad (z \in \mathbb{U}). \tag{40}$$

Putting $A = 1 - 2\alpha$ $(0 \le \alpha < 1)$ and $B = -1$ in Theorem 9, we obtain the following corollary.

Corollary 11. *A necessary and sufficient condition for the function f defined by* (1) *to be in the class $\mathcal{S}_q(\alpha)$ is that*

$$1 - \sum_{k=2}^{\infty} \frac{[k]_q\left(e^{-i\theta} - 1\right) - e^{-i\theta} - 1 + 2\alpha}{2(1-\alpha)} a_k z^{k-1} \neq 0 \quad (z \in \mathbb{U}). \tag{41}$$

Taking $q \to 1^-$ in Corollary 11, we obtain the following corollary which improves the result of Ahuja [17, Corollary 1 when $n = 0$].

Corollary 12. *A necessary and sufficient condition for the function f defined by* (1) *to be in the class $\mathcal{S}(\alpha)$ is that*

$$1 - \sum_{k=2}^{\infty} \frac{k\left(e^{-i\theta} - 1\right) - e^{-i\theta} - 1 + 2\alpha}{2(1-\alpha)} a_k z^{k-1} \neq 0 \quad (z \in \mathbb{U}). \tag{42}$$

Theorem 13. *A necessary and sufficient condition for the function $f(z)$ defined by* (1) *to be in the class $\mathcal{K}_q[A,B]$ is that*

$$1-\sum_{k=2}^{\infty}[k]_q\frac{[k]_q\left(e^{-i\theta}+B\right)-e^{-i\theta}-A}{A-B}a_kz^{k-1}\neq 0\quad (z\in\mathbb{U}). \tag{43}$$

Proof. From Theorem 5, we find that $f\in\mathcal{K}_q[A,B]$ if and only if

$$\frac{1}{z}\left\{f(z)*\frac{z+[1-(q+1)L]qz^2}{(1-z)(1-qz)(1-q^2z)}\right\}\neq 0\quad (z\in\mathbb{U}), \tag{44}$$

for all $L=L_\theta=(e^{-i\theta}+A)/(A-B)$ and also for $L=1$. The left-hand side of (44) may be written as

$$\begin{aligned}&\frac{1}{z}\left\{f(z)*\left(\frac{z}{(1-z)(1-qz)(1-q^2z)}\right.\right.\\&\qquad\left.\left.+\frac{[1-(q+1)L]qz^2}{(1-z)(1-qz)(1-q^2z)}\right)\right\}\\&=\frac{1}{z}\left\{qz^2D_q\left(D_qf(z)\right)+zD_qf(z)\right.\\&\qquad\left.-L\left[qz^2D_q\left(D_qf(z)\right)\right]\right\}\\&=1-\sum_{k=2}^{\infty}[k]_q\frac{[k-1]_qqe^{-i\theta}-A+[k]_qB}{A-B}a_kz^{k-1},\end{aligned} \tag{45}$$

and this proves Theorem 13. □

Taking $q\to 1^-$ in Theorem 13, we obtain the following result.

Corollary 14. *A necessary and sufficient condition for the function $f(z)$ defined by* (1) *to be in the class $\mathcal{K}[A,B]$ is that*

$$1-\sum_{k=2}^{\infty}k\frac{k\left(e^{-i\theta}+B\right)-e^{-i\theta}-A}{A-B}a_kz^{k-1}\neq 0\quad (z\in\mathbb{U}). \tag{46}$$

Putting $A=1-2\alpha$ $(0\leq\alpha<1)$ and $B=-1$ in Theorem 13, we obtain the following corollary.

Corollary 15. *A necessary and sufficient condition for the function f defined by* (1) *to be in the class $\mathcal{K}_q(\alpha)$ $(0\leq\alpha<1)$ is that*

$$1-\sum_{k=2}^{\infty}[k]_q\frac{[k]_q\left(e^{-i\theta}-1\right)-e^{-i\theta}-1+2\alpha}{2(1-\alpha)}a_kz^{k-1}\neq 0\quad (z\in\mathbb{U}). \tag{47}$$

Taking $q\to 1^-$ in Corollary 15, we obtain the following corollary which improves the result of Ahuja [17, Corollary 1 when $n=1$].

Corollary 16. *A necessary and sufficient condition for the function f defined by* (1) *to be in the class $\mathcal{K}(\alpha)$ $(0\leq\alpha<1)$ is that*

$$1-\sum_{k=2}^{\infty}k\frac{k\left(e^{-i\theta}-1\right)-e^{-i\theta}-1+2\alpha}{2(1-\alpha)}a_kz^{k-1}\neq 0\quad (z\in\mathbb{U}). \tag{48}$$

3. Coefficient Estimates

As an application of Theorems 9 and 13, we next determine coefficient estimate and inclusion property for a function of form (1) to be in the classes $\mathcal{S}_q[A,B]$ and $\mathcal{K}_q[A,B]$.

Theorem 17. *If the function f defined by* (1) *satisfies the following inequality:*

$$\sum_{k=2}^{\infty}\left\{[k]_q(1-B)-1+A\right\}|a_k|\leq A-B, \tag{49}$$

then $f\in\mathcal{S}_q[A,B]$.

Proof. Since

$$\begin{aligned}&\left|1-\sum_{k=2}^{\infty}\frac{[k]_q\left(e^{-i\theta}+B\right)-e^{-i\theta}-A}{A-B}a_kz^{k-1}\right|\\&>1-\sum_{k=2}^{\infty}\left|\frac{[k]_q\left(e^{-i\theta}+B\right)-e^{-i\theta}-A}{A-B}\right||a_k|\\&=1-\sum_{k=2}^{\infty}\frac{\left|[k]_q\left(e^{-i\theta}+B\right)-e^{-i\theta}-A\right|}{A-B}|a_k|\\&>1-\sum_{k=2}^{\infty}\frac{[k]_q(1-B)-1+A}{A-B}|a_k|>0\end{aligned} \tag{50}$$

the result follows from Theorem 9. □

Taking $q\to 1^-$ in Theorem 17, we obtain the result of Ahuja [17, Theorem 3 when $n=0$].

Corollary 18. *If the function f defined by* (1) *satisfies the following inequality:*

$$\sum_{k=2}^{\infty}[k(1-B)-1+A]|a_k|\leq A-B, \tag{51}$$

then $f\in\mathcal{S}[A,B]$.

Putting $A=1-2\alpha$ $(0\leq\alpha<1)$ and $B=-1$ in Theorem 21, we obtain the following corollary.

Corollary 19. *If the function f defined by* (1) *satisfies the following inequality:*

$$\sum_{k=2}^{\infty}\left([k]_q-\alpha\right)|a_k|\leq 1-\alpha, \tag{52}$$

then $f\in\mathcal{S}_q(\alpha)$.

Taking $q \to 1^-$ in Corollary 19, we obtain the following corollary obtained by Silverman [24].

Corollary 20. *If the function f defined by* (1) *satisfies the following inequality:*

$$\sum_{k=2}^{\infty}(k-\alpha)\left|a_k\right| \leq 1-\alpha, \tag{53}$$

then $f \in \mathcal{S}(\alpha)$.

Similarly, we can prove the following theorem.

Theorem 21. *If the function f defined by* (1) *satisfies the following inequality:*

$$\sum_{k=2}^{\infty}[k]_q\left\{[k]_q(1-B)-1+A\right\}\left|a_k\right| \leq A-B, \tag{54}$$

then $f \in \mathcal{K}_q[A,B]$.

Taking $q \to 1^-$ in Theorem 21, we obtain the result of Ahuja [17, Theorem 3 when $n=1$].

Corollary 22. *If the function f defined by* (1) *satisfies the following inequality:*

$$\sum_{k=2}^{\infty}k\left[k(1-B)-1+A\right]\left|a_k\right| \leq A-B, \tag{55}$$

then $f \in \mathcal{K}[A,B]$.

Putting $A = 1-2\alpha$ $(0 \leq \alpha < 1)$ and $B=-1$ in Theorem 21, we obtain the following corollary.

Corollary 23. *The function f defined by* (1) *belongs to the class $\mathcal{K}_q(\alpha)$ $(0 \leq \alpha < 1)$ if*

$$\sum_{k=2}^{\infty}[k]_q\left([k]_q-\alpha\right)\left|a_k\right| \leq 1-\alpha. \tag{56}$$

Taking $q \to 1^-$ in Corollary 23, we obtain the following corollary obtained by Silverman [24].

Corollary 24. *The function f defined by* (1) *belongs to the class $\mathcal{K}(\alpha)$ $(0 \leq \alpha < 1)$ if*

$$\sum_{k=2}^{\infty}k(k-\alpha)\left|a_k\right| \leq 1-\alpha. \tag{57}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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