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Research Article

The Ideal Convergence of Strongly of Γ^2 in p-Metric Spaces Defined by Modulus

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The aim of this paper is to introduce and study a new concept of the Γ^2 space via ideal convergence defined by modulus and also some topological properties of the resulting sequence spaces were examined.

1. Introduction

Let (x_{mn}) be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^{\infty} x_{mn}$ is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is said to be convergent if and only if the double sequence (S_{mn}) is convergent, where

$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij}, \quad (m, n = 1, 2, 3, \ldots). \tag{1}$$

We denote w^2 as the class of all complex double sequences (x_{mn}). A sequence $x = (x_{mn})$ is said to be double analytic if

$$\sup_{mn} |x_{mn}|^{1/m+n} < \infty. \tag{2}$$

The vector space of all prime sense double analytic sequences is usually denoted by Λ^2. A sequence $x = (x_{mn})$ is called double entire sequence if

$$(|x_{mn}|)^{1/m+n} \longrightarrow 0 \quad \text{as } m, n \longrightarrow \infty. \tag{3}$$

The vector space of all prime sense double entire sequences is usually denoted by Γ^2. The space Λ^2 is a metric space with the metric

$$d(x, y) = \sup_{mn} \left\{ |x_{mn} - y_{mn}|^{1/m+n} : m, n : 1, 2, 3, \ldots \right\}. \tag{4}$$

The space Γ^2 is a metric space with the metric

$$d(x, y) = \sup_{mn} \left\{ (|x_{mn} - y_{mn}|)^{1/m+n} : m, n : 1, 2, 3, \ldots \right\}, \tag{5}$$

for all $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in Γ^2.

Consider a double sequence $x = (x_{ij})$. The (m, n)th section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\delta_{ij}$ for all $m, n \in \mathbb{N}$,

$$\delta_{mn} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots \end{pmatrix}, \tag{6}$$

with 1 in the (m, n)th position and zero otherwise. An FK-space (or a metric space) X is said to have AK property if (δ_{mn}) is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$. We need the following inequality in the sequel of the paper.

Lemma 1. *For $a, b \geq 0$ and $0 < p < 1$, one has*

$$(a + b)^p \leq a^p + b^p. \tag{7}$$

Some initial work on double sequence spaces is found in Bromwich. Later on it was investigated by Moricz [1], Moricz and Rhoades [2], Basarir and Solancan [3], Tripathy [4], Turkmenoglu [5], Subramanian and Misra [6, 7], and many others. Tripathy and Dutta [8] introduced and investigated different types of fuzzy real valued double sequence spaces. Generalizing the concept of ordinary convergence for real sequences Kostyrko et al. introduced the concept of ideal convergence which is a generalization of statistical convergence, by using the ideal I of the subsets of the set of natural numbers.

The notion of different sequence spaces (for single sequences) was introduced by Kizmaz [9] as follows:

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}, \tag{8}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$. Here w, c, c_0, and ℓ_∞ denote the classes of all, convergent, null, and bounded scalar valued single sequences, respectively. The above spaces are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{k\geq 1} |\Delta x_k|. \tag{9}$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\}, \tag{10}$$

where $Z = \Lambda^2$ and Γ^2, respectively. $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$. We further generalized this notion and introduced the following notion. For $m, n \geq 1$,

$$Z\left(\Delta_\gamma^\mu\right) = \left\{x = x_{mn} : \left(\Delta_\gamma^\mu x_{mn}\right) \in Z\right\}, \quad \text{for } Z = \Lambda^2, \Gamma^2. \tag{11}$$

An Orlicz function is a function $f : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing, and convex with $f(0) = 0$, $f(x) > 0$, for $x > 0$ and $f(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function f is replaced by $f(x+y) \leq f(x) + f(y)$, then this function is called modulus function. A modulus function f is said to satisfy Δ^2-condition for all values u, if there exists $K > 0$ such that $f(2u) \leq Kf(u)$, $u \geq 0$.

Remark 2. A modulus function satisfies the inequality $f(\lambda x) \leq \lambda f(x)$ for all λ with $0 < \lambda < 1$.

Lemma 3. *Let f be a modulus function which satisfies Δ^2-condition and let $0 < \delta < 1$. Then for each $t \geq \delta$, one has $f(t) < K\delta^{-1} f(2)$ for some constant $K > 0$.*

Spaces of strongly summable sequences were discussed by Kuttner, Maddox, and others. The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox as an extension of the definition of strongly Cesàro summable sequences. Connor further extended this definition to a definition of strong A-summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong A-summability, strong A-summability with respect to a modulus, and A-statistical convergence. The notion of convergence of double sequences was presented by A. Pringsheim. Also, the four-dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton.

2. Definitions and Preliminaries

Let X be a nonempty set. A nonvoid class $I \subseteq 2^X$ (power set, of X) is called an ideal if I is additive (i.e., $A, B \in I \Rightarrow A \bigcup B \in I$) and hereditary (i.e., $A \in I$ and $B \subseteq A \Rightarrow B \in I$). A nonempty family of sets $F \subseteq 2^X$ is said to be a filter on X if $\phi \notin F$; $A, B \in F \Rightarrow A \bigcap B \in F$ and $A \in F$, $A \subseteq B \Rightarrow B \in F$. For each ideal I there is a filter $F(I)$ given by $F(I) = \{K \subseteq N : N \setminus K \in I\}$. A nontrivial ideal $I \subset 2^X$ is called admissible if and only if $\{\{x\} : x \in X\} \subset I$.

A double sequence space E is said to be solid or normal if $(\alpha_{mn} x_{mn}) \in E$, whenever $(x_{mn}) \in E$ and for all double sequences $\alpha = (\alpha_{mn})$ of scalars with $|\alpha_{mn}| \leq 1$, for all $m, n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and let X be a real vector space of dimension w, where $n \leq w$. A real valued function $d_p(x_1, \ldots, x_n) = \|(d_1(x_1), \ldots, d_n(x_n))\|_p$ on X satisfies the following four conditions:

(i) $\|(d_1(x_1), \ldots, d_n(x_n))\|_p = 0$ if and and only if $d_1(x_1), \ldots, d_n(x_n)$ are linearly dependent,

(ii) $\|(d_1(x_1), \ldots, d_n(x_n))\|_p$ is invariant under permutation,

(iii) $\|(\alpha d_1(x_1), \ldots, d_n(x_n))\|_p = |\alpha| \|(d_1(x_1), \ldots, d_n(x_n))\|_p$, $\alpha \in \mathbb{R}$,

(iv) $d_p((x_1, y_1), (x_2, y_2) \cdots (x_n, y_n)) = (d_X(x_1, x_2, \ldots, x_n)^p + d_Y(y_1, y_2, \ldots, y_n)^p)^{1/p}$ for $1 \leq p < \infty$, or

(v) $d((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)) := \sup\{d_X(x_1, x_2, \ldots, x_n), d_Y(y_1, y_2, \ldots, y_n)\}$, for $x_1, x_2, \ldots, x_n \in X$, $y_1, y_2, \ldots, y_n \in Y$, is called the p product metric of the Cartesian product of n metric spaces which is the p norm of the n-vector of the norms of the n subspaces.

A trivial example of p product metric of n metric spaces is the p norm space $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space which is the p norm:

$$\begin{aligned} &\|(d_1(x_1), \ldots, d_n(x_n))\|_E \\ &\quad = \sup\left(\left|\det\left(d_{mn}(x_{mn})\right)\right|\right) \\ &\quad = \sup\left(\begin{vmatrix} d_{11}(x_{11}) & d_{12}(x_{12}) & \cdots & d_{1n}(x_{1n}) \\ d_{21}(x_{21}) & d_{22}(x_{22}) & \cdots & d_{2n}(x_{1n}) \\ \vdots & & & \\ d_{n1}(x_{n1}) & d_{n2}(x_{n2}) & \cdots & d_{nn}(x_{nn}) \end{vmatrix}\right), \end{aligned} \tag{12}$$

where $x_i = (x_{i1}, \ldots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \ldots, n$.

3. Main Results

In this section we introduce the notion of different types of I-convergent double sequences. This generalizes and unifies different notions of convergence for Γ^2. We will denote the ideal of $2^{N\times N}$ by I_2.

Let I_2 be an ideal of $2^{N\times N}$, f a modulus function, $\eta = (\eta_{mn})$ a double analytic sequence of strictly positive real numbers, and $(X, \|(d_1(x_1),\ldots,d_n(x_n))\|_p)$ a p-product of n metric spaces which is the p norm of the n-vector of the norms of the n subspaces. Further $\Gamma^2(p-X)$ denotes X-valued sequence space. Now, we define the following sequence spaces:

$$\begin{aligned}
&\Gamma_f^{2I_2}\left[\|(d_1(x_1),\ldots,d_n(x_n))\|_p\right]^{\eta}\\
&\quad = x = (x_{mn}) \in \Gamma^2(p-X) : \forall\epsilon > 0,\\
&\Big\{(r,s) \in N\times N :\\
&\quad \frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\Big[f\,\big\|(|x_{mn}|)^{1/m+n},\\
&\qquad d_1(x_1),\ldots,d_n(x_{n-1})\big\|_p\Big]^{\eta_{mn}} \ge \epsilon\Big\} \in I_2,\\
&\qquad \text{for every } d_1(x_1),\ldots,d_n(x_{n-1}) \in X.\\
&\Lambda_f^{2I_2}\left[\|(d_1(x_1),\ldots,d_n(x_n))\|_p\right]^{\eta}\\
&\quad = x = (x_{mn}) \in \Lambda^2(p-X) : \exists K > 0,\\
&\Big\{\Big\{(r,s) \in N\times N :\\
&\quad \frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\Big[f\,\big\|\big(|x_{mn}|^{1/m+n},\\
&\qquad d_1(x_1),\ldots,d_n(x_{n-1})\big)\big\|_p\Big]^{\eta_{mn}}\\
&\quad \ge K\Big\} \in I_2\Big\},\quad \text{for every } d_1(x_1),\ldots,d_n(x_{n-1}) \in X.\\
&\Lambda_f^{2}\left[\|(d_1(x_1),\ldots,d_n(x_n))\|_p\right]^{\eta}\\
&\quad = x = (x_{mn}) \in \Lambda^2(p-X) : \exists K > 0,\\
&\Big\{(r,s) \in N\times N :\\
&\quad \frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\Big[f\,\big\|\big(|x_{mn}|^{1/m+n},\\
&\qquad d_1(x_1),\ldots,d_n(x_{n-1})\big)\big\|_p\Big]^{\eta_{mn}}\\
&\quad \le K\Big\},\quad \text{for every } d_1(x_1),\ldots,d_n(x_{n-1}) \in X.
\end{aligned} \tag{13}$$

If $\eta = \eta_{mn} = 1$ for all $m,n \in \mathbb{N}$ we obtain

$$\begin{aligned}
&\Gamma_f^{2I_2}\left[\|(d_1(x_1),\ldots,d_n(x_n))\|_p\right]^{\eta}\\
&\quad = \Gamma_f^{2I_2}\left[\|(d_1(x_1),\ldots,d_n(x_n))\|_p\right],\\
&\Lambda_f^{2I_2}\left[\|(d_1(x_1),\ldots,d_n(x_n))\|_p\right]^{\eta}\\
&\quad = \Lambda_f^{2I_2}\left[\|(d_1(x_1),\ldots,d_n(x_n))\|_p\right],\\
&\Lambda_f^{2}\left[\|(d_1(x_1),\ldots,d_n(x_n))\|_p\right]^{\eta}\\
&\quad = \Lambda_f^{2}\left[\|(d_1(x_1),\ldots,d_n(x_n))\|_p\right].
\end{aligned} \tag{14}$$

The following well-known inequality will be used in this study: $0 \le \inf_{mn}\eta_{mn} = H_0 \le \eta_{mn} \le \sup_{mn} = H < \infty$, $D = \max(1, 2^{H-1})$; then

$$|x_{mn} + y_{mn}|^{\eta_{mn}} \le D\left\{|x_{mn}|^{\eta_{mn}} + |y_{mn}|^{\eta_{mn}}\right\}, \tag{15}$$

for all $m,n \in \mathbb{N}$ and $x_{mn}, y_{mn} \in \mathbb{C}$. Also $|x_{mn}|^{\eta_{mn}/m+n} \le \max(1, |x_{mn}|^{H/m+n})$ for all $x_{mn} \in \mathbb{C}$.

Theorem 4. *The sets* $\Gamma_f^{2I_2}[\|(d_1(x_1),\ldots,d_n(x_n))\|_p]^{\eta_{mn}}$ *and* $\Lambda_f^{2I_2}[\|(d_1(x_1),\ldots,d_n(x_n))\|_p]^{\eta_{mn}}$ *are linear spaces over the complex field* $\mathbb{C}$

Proof. Now only prove $\Gamma_f^{2I_2}[\|(d_1(x_1),\ldots,d_n(x_n))\|_p]^{\eta_{mn}}$ and the others can be proved similarly. Let $x, y \in \Gamma_f^{2I_2}[\|(d_1(x_1),\ldots,d_n(x_n))\|_p]^{\eta_{mn}}$ and $\alpha, \beta \in \mathbb{C}$. Then

$$\begin{aligned}
&\Big\{(r,s) \in N\times N :\\
&\quad \frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\Big[f\,\big\|(|x_{mn}|)^{1/m+n},\\
&\qquad d_1(x_1),\ldots,d_n(x_{n-1})\big\|_p\Big]^{\eta_{mn}}\\
&\quad \ge \frac{\epsilon}{2}\Big\} \in I_2,\\
&\Big\{(r,s) \in N\times N :\\
&\quad \frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{r}\Big[f\,\big\|(|y_{mn}|)^{1/m+n},\\
&\qquad d_1(x_1),\ldots,d_n(x_{n-1})\big\|_p\Big]^{\eta_{mn}}\\
&\quad \ge \frac{\epsilon}{2}\Big\} \in I_2.
\end{aligned} \tag{16}$$

Since $\|(d_1(x_1),\ldots,d_n(x_n))\|_p$ is a p-product of n metric spaces which is the p norm of the n-vector of the norms of the n subspaces and f is a modulus function, the following inequality holds:

$$\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left(\left\|\frac{|\alpha x_{mn}+\beta y_{mn}|^{1/m+n}}{|\alpha|^{1/m+n}+|\beta|^{1/m+n}}, d_1(x_1),\ldots,d_n(x_{n-1})\right\|_p\right)\right]^{\eta_{mn}}$$
$$\leq \frac{D}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[\frac{|\alpha|^{1/m+n}}{|\alpha|^{1/m+n}+|\beta|^{1/m+n}} \times f\left(\left\|(|x_{mn}|)^{1/m+n}, d_1(x_1),\ldots,d_n(x_{n-1})\right\|_p\right)\right]^{\eta_{mn}}$$
$$+\frac{D}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[\frac{|\beta|^{1/m+n}}{|\alpha|^{1/m+n}+|\beta|^{1/m+n}} \times f\left(\left\|(|y_{mn}|)^{1/m+n}, d_1(x_1),\ldots,d_n(x_{n-1})\right\|_p\right)\right]^{\eta_{mn}}$$
$$\leq \frac{D}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left(\left\|(|x_{mn}|)^{1/m+n}, d_1(x_1),\ldots,d_n(x_{n-1})\right\|_p\right)\right]^{\eta_{mn}}$$
$$+\frac{D}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left(\left\|(|y_{mn}|)^{1/m+n}, d_1(x_1),\ldots,d_n(x_{n-1})\right\|_p\right)\right]^{\eta_{mn}}. \quad (17)$$

From the above inequality we get

$$\left\{(r,s)\in N\times N : \frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left(\left\|\left(\frac{(|\alpha x_{mn}+\beta y_{mn}|)^{1/m+n}}{|\alpha|^{1/m+n}+|\beta|^{1/m+n}}, d_1(x_1),\ldots,d_n(x_{n-1})\right)\right\|_p\right)\right]^{\eta_{mn}} \geq \epsilon\right\} \subset \left\{(r,s)\in N\times N : \frac{D}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left((|x_{mn}|)^{1/m+n}, d_1(x_1),\ldots,d_n(x_{n-1})\right)\right\|_p\right]^{\eta_{mn}} \geq \frac{\epsilon}{2}\right\}\in I_2$$
$$\bigcup\left\{(r,s)\in N\times N : \frac{D}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left(|y_{mn}|^{1/m+n}, d_1(x_1),\ldots,d_n(x_{n-1})\right)\right\|_p\right]^{\eta_{mn}} \geq \frac{\epsilon}{2}\right\}\in I_2. \quad (18)$$

This completes the proof. □

Theorem 5. *$\Gamma_f^{2I_2}[\|(d_1(x_1),\ldots,d_n(x_n))\|_p]^{\eta}$ paranormed space with respect to the paranorm is defined by*

$$g_{rs}(x) = \inf\left\{\left(\sup_{rs}\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left((|x_{mn}|)^{1/m+n}, d_1(x_1),\ldots,d_n(x_{n-1})\right)\right\|_p\right]^{\eta_{mn}}\right)^{1/H} \leq 1\right\}, \quad \textit{for every } d_1(x_1),\ldots,d_n(x_{n-1})\in X. \quad (19)$$

Proof. $g_{rs}(\theta)=0$ and $g_{rs}(-x)=g_{rs}(x)$ are easy to prove, so we omit them. Let us take $x,y \in \Gamma_f^{2I_2}[\|(d_1(x_1),\ldots,d_n(x_n))\|_p]^{\eta_{mn}}$. Let

$$g_{rs}(x) = \inf\left\{\sup_{rs}\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left((|x_{mn}|)^{1/m+n}, d_1(x_1),\ldots,d_n(x_{n-1})\right)\right\|_p\right]^{\eta_{mn}} \leq 1,\ \forall x\in X\right\},$$

$$g_{rs}(y) = \inf\left\{\sup_{rs}\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left((|y_{mn}|)^{1/m+n}, d_1(x_1), \ldots, d_n(x_{n-1})\right)\right\|_p\right]^{\eta_{mn}} \leq 1,\ \forall x \in X\right\}. \tag{20}$$

Then we have

$$\begin{aligned}\sup_{rs}\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}&\left[f\left\|\left((|x_{mn}+y_{mn}|)^{1/m+n}, d_1(x_1), \ldots, d_n(x_{n-1})\right)\right\|_p\right]^{\eta_{mn}} \\ &\leq \sup_{rs}\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left((|x_{mn}|)^{1/m+n}, d_1(x_1), \ldots, d_n(x_{n-1})\right)\right\|_p\right]^{\eta_{mn}} \\ &\quad + \sup_{rs}\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left((|y_{mn}|)^{1/m+n}, d_1(x_1), \ldots, d_n(x_{n-1})\right)\right\|_p\right]^{\eta_{mn}}.\end{aligned} \tag{21}$$

Thus

$$\sup_{rs}\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left((|x_{mn}+y_{mn}|)^{1/m+n}, d_1(x_1), \ldots, d_n(x_{n-1})\right)\right\|_p\right]^{\eta_{mn}} \leq 1, \tag{22}$$

and $g_{rs}(x+y) = g_{rs}(x) + g_{rs}(y)$.

Now, let $\lambda_{mn}^u \to \lambda$, where $\lambda_{mn}^u, \lambda \in \mathbb{C}$ and $g_{rs}(x_{mn}^u - x_{mn}) \to 0$ as $u \to \infty$. We have to prove that $g_{rs}(\lambda_{mn}x_{mn}^u - \lambda x_{mn}) \to 0$ as $u \to \infty$. Let

$$g_{rs}(x^u) = \left\{\sup_{rs}\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left((|x_{mn}^u|)^{1/m+n}, d_1(x_1), \ldots, d_n(x_{n-1})\right)\right\|_p\right]^{\eta_{mn}} \leq 1,\ \forall x \in X\right\},$$

$$g_{rs}(x^u - x) = \left\{\sup_{rs}\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left((|x_{mn}^u - x_{mn}|)^{1/m+n}, d_1(x_1), \ldots, d_n(x_{n-1})\right)\right\|_p\right]^{\eta_{mn}} \leq 1,\right\} \quad \forall x \in X. \tag{23}$$

We observe that

$$\begin{aligned}f&\left(\left\|\left(\frac{(|\lambda_{mn}^u x_{mn}^u - \lambda x_{mn}|)^{1/m+n}}{|\lambda_{mn}^u - \lambda|^{1/m+n} + |\lambda|^{1/m+n}}, d_1(x_1), \ldots, d_n(x_{n-1})\right)\right\|_p\right) \\ &\leq f\left(\left\|\left(\frac{(|\lambda_{mn}^u x_{mn}^u - \lambda x_{mn}^u|)^{1/m+n}}{|\lambda_{mn}^u - \lambda|^{1/m+n} + |\lambda|^{1/m+n}}, d_1(x_1) \ldots, d_n(x_{n-1})\right)\right\|_p\right) \\ &\quad + f\left(\left\|\left(\frac{(|\lambda x_{mn}^u - \lambda x_{mn}|)^{1/m+n}}{|\lambda_{mn}^u - \lambda|^{1/m+n} + |\lambda|^{1/m+n}}, d_1(x_1), \ldots, d_n(x_{n-1})\right)\right\|_p\right) \\ &\leq \frac{|\lambda_{mn}^u - \lambda|}{|\lambda_{mn}^u - \lambda| + |\lambda|} \times f\left(\left\|\left((|x_{mn}^u|)^{1/m+n}, d_1(x_1), \ldots, d_n(x_{n-1})\right)\right\|_p\right) \\ &\quad + \frac{|\lambda|}{|\lambda_{mn}^u - \lambda| + |\lambda|} \times f\left(\left\|\left((|x_{mn}^u - x_{mn}|)^{1/m+n}, d_1(x_1), \ldots, d_n(x_{n-1})\right)\right\|_p\right).\end{aligned} \tag{24}$$

From this inequality, it follows that

$$\left[f\left(\left\|\left(\frac{(|\lambda_{mn}^u x_{mn}^u - \lambda x_{mn}|)^{1/m+n}}{|\lambda_{mn}^u - \lambda|^{1/m+n} + |\lambda|^{1/m+n}}, d_1(x_1), \ldots, d_n(x_{n-1})\right)\right\|_p\right)\right]^{\eta_{mn}} \leq 1, \tag{25}$$

and consequently

$$
\begin{aligned}
& g_{rs}\left(\lambda_{mn}^{u} x_{mn}^{u}-\lambda x_{mn}\right) \\
& \quad \leq\left(\left|\lambda_{mn}^{u}-\lambda\right|\right)^{\eta_{mn}/H} \inf \left\{g_{rs}\left(x_{mn}^{u}\right)\right\} \\
& \qquad +(|\lambda|)^{\eta_{mn}/H} \inf \left\{g_{rs}\left(x_{mn}^{u}-x\right)\right\} \\
& \quad \leq \max \left\{|\lambda|,(|\lambda|)^{\eta_{mn}/H}\right\} g_{rs}\left(x_{mn}^{u}-x_{mn}\right).
\end{aligned} \tag{26}
$$

Hence by our assumption the right-hand side tends to zero as u, m, and $n \rightarrow \infty$. This completes the proof. □

Theorem 6. *(i) If* $0 < \inf_{mn}\eta_{mn} = H_0 \leq \eta_{mn} < 1$, *then* $\Gamma_f^{2I_2}[\|(d_1(x_1),\ldots,d_n(x_n))\|_p]^\eta \subset \Gamma_f^{2I_2}[\|(d_1(x_1),\ldots,d_n(x_n))\|_p]$.

(ii) If $1 \leq \eta_{mn} \leq \sup_{mn}\eta_{mn} = H < \infty$, *then* $\Gamma_f^{2I_2}[\|(d_1(x_1),\ldots,d_n(x_n))\|_p] \subset \Gamma_f^{2I_2}[\|(d_1(x_1),\ldots,d_n(x_n))\|_p]^\eta$.

(iii) If $0 < \eta_{mn} < \mu_{mn} < \infty$ *and* $\{\mu_{mn}/\eta_{mn}\}$ *is analytic, then* $\Gamma_f^{2I_2}[\|(d_1(x_1),\ldots,d_n(x_n))\|_p]^\eta \subset \Gamma_f^{2I_2}[\|(d_1(x_1),\ldots,d_n(x_n))\|_p]^\mu$.

Proof. The proof is easy. Therefore omit it. □

Lemma 7. *If a sequence E is solid, then it is monotone. (See [10, page 53].)*

Theorem 8. $\Gamma_f^{2I_2}[\|(d_1(x_1),\ldots,d_n(x_n))\|_p]^\eta$ *is solid and also monotone.*

Proof. Let $x \in \Gamma_f^{2I_2}[\|(d_1(x_1),\ldots,d_n(x_n))\|_p]^\eta$ and $\alpha = (\alpha_{mn})$ be scalars such that $|\alpha_{mn}|^{1/m+n} \leq 1$ for $m, n \in \mathbb{N}$. Then we have

$$
\begin{aligned}
& \left\{(r,s) \in N \times N :\right. \\
& \quad \frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left(\left(|\alpha_{mn}x_{mn}|\right)^{1/m+n}, d_1(x_1),\right.\right.\right. \\
& \qquad \left.\left.\left.\ldots, d_n(x_{n-1})\right)\right\|_p\right]^{\eta_{mn}} \\
& \left.\leq \epsilon\right\} \subset \left\{\left\{(r,s) \in N \times N :\right.\right. \\
& \quad \frac{T}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left(\left(|x_{mn}|\right)^{1/m+n}, d_1(x_1),\right.\right.\right. \\
& \qquad \left.\left.\left.\left.\ldots, d_n(x_{n-1})\right)\right\|_p\right]^{\eta_{mn}} \leq \epsilon\right\} \\
& \left.\in I_2\right\},
\end{aligned} \tag{27}
$$

where $T = \max_{mn}\{1, |\alpha_{mn}|^{H/m+n}\}$. Hence $\alpha x \in \Gamma_f^{2I_2}[\|(d_1(x_1),\ldots,d_n(x_n))\|_p]^\eta$ with $|\alpha|^{1/m+n} \leq 1$ for all $m, n \in \mathbb{N}$ whenever $x \in \Gamma_f^{2I_2}[\|(d_1(x_1),\ldots,d_n(x_n))\|_p]^\eta$. Also by Lemma 7, it follows that $\Gamma_f^{2I_2}[\|(d_1(x_1),\ldots,d_n(x_n))\|_p]^\eta$ is monotone. This completes the proof. □

Theorem 9. *Let* f, f_1, *and* f_2 *be modulus functions. Then one has*

(i) $\Gamma_{f_1}^{2I_2}[\|(d_1(x_1),\ldots,d_n(x_n))\|_p]^\eta \subset \Gamma_{f\circ f_1}^{2I_2}[\|(d_1(x_1),\ldots,d_n(x_n))\|_p]^\eta$,

(ii) $\Gamma_{f_1}^{2I_2}[\|(d_1(x_1),\ldots,d_n(x_n))\|_p]^\eta \bigcap \Gamma_{f_2}^{2I_2}[\|(d_1(x_1),\ldots,d_n(x_n))\|_p]^\eta \subset \Gamma_{f_1+f_2}^{2I_2}[\|(d_1(x_1),\ldots,d_n(x_n))\|_p]^\eta$.

Proof. (i) Let $\inf_{mn}\eta_{mn} = H_0$. For given $\epsilon > 0$, we first choose $\epsilon_0 > 0$ such that $\max\{\epsilon_0^H, \epsilon_0^{H_0}\} < \epsilon$. Now using the continuity of f, choose $0 < \delta < 1$ such that $0 < t < \delta$ implies $f(t) < \epsilon_0$. Let $x \in \Gamma_{f_1}^{2I_2}[\|(d_1(x_1),\ldots,d_n(x_n))\|_p]^\eta$.

We observe that

$$
\begin{aligned}
A(\delta) = & \left\{(r,s) \in N \times N :\right. \\
& \frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f_1\left\|\left(\left(|x_{mn}|\right)^{1/m+n}, d_1(x_1),\right.\right.\right. \\
& \qquad \left.\left.\left.\left.\ldots, d_n\left(x_{n-1}\right)\right)\right\|_p\right]^{\eta_{mn}} \geq \delta^H\right\} \in I_2.
\end{aligned} \tag{28}
$$

Thus if $(r,s) \notin A(\delta)$ then

$$
\begin{aligned}
& \frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f_1\left\|\left(\left(|x_{mn}|\right)^{1/m+n}, d_1(x_1),\right.\right.\right. \\
& \qquad \left.\left.\left.\ldots, d_n(x_{n-1})\right)\right\|_p\right]^{\eta_{mn}} < \delta^H \\
& \Longrightarrow \sum_{m=1}^{r}\sum_{n=1}^{s}\left[f_1\left\|\left(\left(|x_{mn}|\right)^{1/m+n}, d_1(x_1),\right.\right.\right. \\
& \qquad \left.\left.\left.\ldots, d_n(x_{n-1})\right)\right\|_p\right]^{\eta_{mn}} < rs\delta^H, \\
& \Longrightarrow \left[f_1\left\|\left(\left(|x_{mn}|\right)^{1/m+n}, d_1(x_1),\right.\right.\right. \\
& \qquad \left.\left.\left.\ldots, d_n(x_{n-1})\right)\right\|_p\right]^{\eta_{mn}} < \delta^H, \quad \forall m, n = 1, 2, \ldots. \\
& \Longrightarrow f_1\left(\left\|\left(\left(|x_{mn}|\right)^{1/m+n}, d_1(x_1),\right.\right.\right. \\
& \qquad \left.\left.\left.\ldots, d_n(x_{n-1})\right)\right\|_p\right) < \delta, \quad \forall m, n = 1, 2, \ldots.
\end{aligned} \tag{29}
$$

Hence from above inequality and using continuity of f, we must have

$$f\left(f_1\left(\left\|\left((|x_{mn}|)^{1/m+n}, d_1(x_1), \ldots, d_n(x_{n-1})\right)\right\|_p\right)\right) < \epsilon_0, \quad \forall m, n = 1, 2, \ldots.$$

$$\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left(f_1\left\|\left((|x_{mn}|)^{1/m+n}, d_1(x_1), \ldots, d_n(x_{n-1})\right)\right\|_p\right)\right]^{\eta_{mn}} < rs\max\left\{\epsilon_0^{H}, \epsilon_0^{H_0}\right\} < rs\epsilon$$
$$\Longrightarrow \frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left(f_1\left\|\left((|x_{mn}|)^{1/m+n}, d_1(x_1), \ldots, d_n(x_{n-1})\right)\right\|_p\right)\right]^{\eta_{mn}} < \epsilon. \tag{30}$$

Hence we have

$$\left\{(r,s) \in N \times N : \frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left(f_1\left\|\left((|x_{mn}|)^{1/m+n}, d_1(x_1), \ldots, d_n(x_{n-1})\right)\right\|_p\right)\right]^{\eta_{mn}} \geq \epsilon\right\} \subset A(\delta) \in I_2. \tag{31}$$

(ii) Let $x \in \Gamma_{f_1}^{2I_2}[\|(d_1(x_1), \ldots, d_n(x_n))\|_p]^{\eta} \bigcap \Gamma_{f_2}^{2I_2}[\|(d_1(x_1), \ldots, d_n(x_n))\|_p]^{\eta}$. Then

$$\frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\Big[(f_1+f_2)\times\left(\left\|\left((|x_{mn}|)^{1/m+n}, d_1(x_1), \ldots, d_n(x_{n-1})\right)\right\|_p\right)\Big]^{\eta_{mn}}$$
$$\leq \frac{D}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f_1\left(\left\|\left((|x_{mn}|)^{1/m+n}, d_1(x_1), \ldots, d_n(x_{n-1})\right)\right\|_p\right)\right]^{\eta_{mn}} + \frac{D}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f_2\left(\left\|\left((|x_{mn}|)^{1/m+n}, d_1(x_1), \ldots, d_n(x_{n-1})\right)\right\|_p\right)\right]^{\eta_{mn}}. \tag{32}$$

This completes the proof. □

Theorem 10. *Let the double sequence $\eta = (\eta_{mn})$ be analytic. Then*

$$\Gamma_f^{2I_2}\left[\|(d_1(x_1), \ldots, d_n(x_n))\|_p\right]^{\eta} \subset \Lambda_f^{2I_2}\left[\|(d_1(x_1), \ldots, d_n(x_n))\|_p\right]^{\eta} \tag{33}$$

and the inclusion are strict.

Theorem 11. *The class of sequence $\Lambda_f^{2I_2}[\|(d_1(x_1), \ldots, d_n(x_n))\|_p]^{\eta}$ is sequence algebras.*

Proof. Let $(x_{mn}), (y_{mn}) \in \Lambda_f^{2I_2}[\|(d_1(x_1), \ldots, d_n(x_n))\|_p]^{\eta}$ and $0 < \epsilon < 1$. Then the result follows from the following inclusion relation:

$$\left\{\left\{(r,s) \in N \times N : \frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left(|x_{mn} \otimes y_{mn}|^{1/m+n}, d_1(x_1), \ldots, d_n(x_{n-1})\right)\right\|_p\right]^{\eta_{mn}} < \epsilon\right\} \in I_2\right\} \supseteq \left\{\left\{(r,s) \in N \times N : \frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left(|x_{mn}|^{1/m+n}, d_1(x_1), \ldots, d_n(x_{n-1})\right)\right\|_p\right]^{\eta_{mn}} < \epsilon\right\} \in I_2\right\}$$
$$\bigcap\left\{\left\{(r,s) \in N \times N : \frac{1}{rs}\sum_{m=1}^{r}\sum_{n=1}^{s}\left[f\left\|\left(|y_{mn}|^{1/m+n}, d_1(x_1), \ldots, d_n(x_{n-1})\right)\right\|_p\right]^{\eta_{mn}} < \epsilon\right\} \in I_2\right\}. \tag{34}$$

Similarly we can prove the result for other cases. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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