

Hindawi Publishing Corporation
 Abstract and Applied Analysis
 Volume 2009, Article ID 417473, 14 pages
 doi:10.1155/2009/417473

Research Article

Solution and Stability of a Mixed Type Cubic and Quartic Functional Equation in Quasi-Banach Spaces

M. Eshaghi Gordji,¹ S. Zolfaghari,¹ J. M. Rassias,² and M. B. Savadkouhi¹

¹ Department of Mathematics, Semnan University, P. O. Box 35195-363, Semnan, Iran

² Section of Mathematics and Informatics, Pedagogical Department, National and Capodistrian University of Athens, 4, Agamemnonos Street, Aghia Paraskevi, 15342 Athens, Greece

Correspondence should be addressed to M. Eshaghi Gordji, madjid.eshaghi@gmail.com

Received 24 January 2009; Accepted 6 August 2009

Recommended by Elena Litsyn

We obtain the general solution and the generalized Ulam-Hyers stability of the mixed type cubic and quartic functional equation $f(x+2y)+f(x-2y)=4(f(x+y)+f(x-y))-24f(y)-6f(x)+3f(2y)$ in quasi-Banach spaces.

Copyright © 2009 M. Eshaghi Gordji et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

We recall some basic facts concerning quasiBanach space. A quasinorm is a real-valued function on X satisfying the following.

- (1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
- (2) $\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.
- (3) There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a quasinormed space if $\|\cdot\|$ is a quasinorm on X . A quasiBanach space is a complete quasinormed space. A quasinorm $\|\cdot\|$ is called a p -norm ($0 < p \leq 1$) if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p \quad (1.1)$$

for all $x, y \in X$. In this case, a quasiBanach space is called a p -Banach space. Given a p -norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on X . By the Aoki-Rolewicz theorem [1] (see also [2]), each quasinorm is equivalent to some p -norm. Since it is much easier to work with p -norms, henceforth we restrict our attention mainly to p -norms. The stability problem of functional equations originated from a question of Ulam [3] in 1940, concerning the stability of group homomorphisms. Let (G_1, \cdot) be a group and let $(G_2, *)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \rightarrow G_2$ satisfies the inequality $d(h(x \cdot y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \rightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In the other words, Under what condition does there exists a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [4] gave the first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \rightarrow E'$ be a mapping between Banach spaces such that

$$\|f(x + y) - f(x) - f(y)\| \leq \delta \quad (1.2)$$

for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \rightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \delta \quad (1.3)$$

for all $x \in E$. Moreover if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is linear. Rassias [5] succeeded in extending the result of Hyers' Theorem by weakening the condition for the Cauchy difference controlled by $(\|x\|^p + \|y\|^p)$, $p \in [0, 1)$ to be unbounded. This condition has been assumed further till now, through the complete Hyers direct method, in order to prove linearity for generalized Hyers-Ulam stability problem forms. A number of mathematicians were attracted to the pertinent stability results of Rassias [6], and stimulated to investigate the stability problems of functional equations. The stability phenomenon that was introduced and proved by Rassias is called Hyers-Ulam-Rassias stability. And then the stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [4, 5, 7-18]).

The following cubic functional equation, which is the oldest cubic functional equation, was introduced by the third author of this paper, Rassias [6] (in 2001):

$$f(x + 2y) + 3f(x) = 3f(x + y) + f(x - y) + 6f(y). \quad (1.4)$$

Jun and Kim [19] introduced the following cubic functional equation:

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x), \quad (1.5)$$

and they established the general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.5). The function $f(x) = x^3$ satisfies the functional equation (1.5), which is thus called a cubic functional equation. Every solution of the cubic functional equation is said to be a cubic function. Jun and Kim proved that a function f between real

vector spaces X and Y is a solution of (1.5) if and only if there exists a unique function $C : X \times X \times X \rightarrow Y$ such that $f(x) = C(x, x, x)$ for all $x \in X$, and C is symmetric for each fixed one variable and is additive for fixed two variables (see also [20]).

The quartic functional equation (1.6) was introduced by Rassias [21] (in 2000) and then (in 2005) was employed by Park and Bae [22] and others, such that:

$$f(x + 2y) + f(x - 2y) = 4[f(x + y) + f(x - y)] + 24f(y) - 6f(x). \quad (1.6)$$

In fact they proved that a function f between real vector spaces X and Y is a solution of (1.6) if and only if there exists a unique symmetric multiadditive function $Q : X \times X \times X \times X \rightarrow Y$ such that $f(x) = Q(x, x, x, x)$ for all x (see also [21–29]). It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.6), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function. In this paper we deal with the following functional equation:

$$f(x + 2y) + f(x - 2y) = 4(f(x + y) + f(x - y)) - 24f(y) - 6f(x) + 3f(2y) \quad (1.7)$$

in quasiBanach spaces. It is easy to see that the function $f(x) = ax^3 + bx^4$ is a solution of the functional equation (1.7). In the present paper we investigate the general solution of functional equation (1.7) when f is a mapping between vector spaces, and we establish the generalized Hyers-Ulam-Rassias stability of the functional equation (1.7) whenever f is a mapping between two quasiBanach spaces. We only mention here the papers [30, 31] concerning the stability of the mixed type functional equations.

2. General Solution

Throughout this section, X and Y will be real vector spaces. Before proceeding to the proof of Theorem 2.3 which is the main result in this section, we shall need the following two lemmas.

Lemma 2.1. *If an even function $f : X \rightarrow Y$ satisfies (1.7), then f is quartic.*

Proof. Putting $x = y = 0$ in (1.7), we get $f(0) = 0$. Setting $x = 0$ in (1.7), by evenness of f we obtain

$$f(2y) = 16f(y) \quad (2.1)$$

for all $y \in X$. Hence (1.7) can be written as

$$f(x + 2y) + f(x - 2y) = 4(f(x + y) + f(x - y)) + 24f(y) - 6f(x). \quad (2.2)$$

This means that f is quartic function, which completes the proof of the lemma. \square

Lemma 2.2. *If an odd function $f : X \rightarrow Y$ satisfies (1.7), then f is a cubic function.*

Proof. Setting $x = y = 0$ in (1.7) gives $f(0) = 0$. Putting $x = 0$ in (1.7), then by oddness of f , we have

$$f(2y) = 8f(y). \quad (2.3)$$

Hence (1.7) can be written as

$$f(x + 2y) + f(x - 2y) = 4f(x + y) + 4f(x - y) - 6f(x). \quad (2.4)$$

Replacing x by $x + y$ in (2.4), we obtain

$$f(x + 3y) + f(x - y) = 4f(x + 2y) - 6f(x + y) + 4f(x). \quad (2.5)$$

Substituting $-y$ for y in (2.5) gives

$$f(x - 3y) + f(x + y) = 4f(x - 2y) - 6f(x - y) + 4f(x). \quad (2.6)$$

If we subtract (2.5) from (2.6), we obtain

$$f(x + 3y) - f(x - 3y) = 4f(x + 2y) - 4f(x - 2y) - 5f(x + y) + 5f(x - y). \quad (2.7)$$

Let us interchange x and y in (2.7). Then we see that

$$f(3x + y) + f(3x - y) = 4f(2x + y) + 4f(2x - y) - 5f(x + y) - 5f(x - y). \quad (2.8)$$

With the substitution $y := x + y$ in (2.4), we have

$$f(3x + 2y) - f(x + 2y) = 4f(2x + y) - 4f(y) - 6f(x). \quad (2.9)$$

From the substitution $y := -y$ in (2.9) it follows that

$$f(3x - 2y) - f(x - 2y) = 4f(2x - y) + 4f(y) - 6f(x). \quad (2.10)$$

If we add (2.9) to (2.10), we have

$$f(3x + 2y) + f(3x - 2y) = 4f(2x + y) + 4f(2x - y) + f(x + 2y) + f(x - 2y) - 12f(x). \quad (2.11)$$

Replacing x by $2x$ in (2.7) and using (2.3), we obtain

$$f(2x + 3y) - f(2x - 3y) = 32f(x + y) - 32f(x - y) - 5f(2x + y) + 5f(2x - y). \quad (2.12)$$

Interchanging x with y in (2.12) gives the equation

$$f(3x + 2y) + f(3x - 2y) = 32f(x + y) + 32f(x - y) - 5f(x + 2y) - 5f(x - 2y). \quad (2.13)$$

If we compare (2.11) and (2.13) and employ (2.4), we conclude that

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x). \quad (2.14)$$

This means that f is cubic function. This completes the proof of Lemma. \square

Theorem 2.3. *A function $f : X \rightarrow Y$ satisfies (1.7) for all $x, y \in X$ if and only if there exists a unique function $C : X \times X \times X \rightarrow Y$ and a unique symmetric multiadditive function $Q : X \times X \times X \times X \rightarrow Y$ such that $f(x) = C(x, x, x) + Q(x, x, x, x)$ for all $x \in X$, and that C is symmetric for each fixed one variable and is additive for fixed two variables.*

Proof. Let f satisfy (1.7). We decompose f into the even part and odd part by setting

$$f_e(x) = \frac{1}{2}(f(x) + f(-x)), \quad f_o(x) = \frac{1}{2}(f(x) - f(-x)) \quad (2.15)$$

for all $x \in X$. By (1.7), we have

$$\begin{aligned} f_e(x + 2y) + f_e(x - 2y) &= \frac{1}{2}[f(x + 2y) + f(-x - 2y) + f(x - 2y) + f(-x + 2y)] \\ &= 4(f_e(x + y) + f_e(x - y)) - 24f_e(y) - 6f_e(x) + 3f_e(2y) \end{aligned} \quad (2.16)$$

for all $x, y \in X$. This means that f_e satisfies in (1.7). Similarly we can show that f_o satisfies (1.7). By Lemmas 2.1 and 2.2, f_e and f_o are quartic and cubic, respectively. Thus there exists a unique function $C : X \times X \times X \rightarrow Y$ and a unique symmetric multiadditive function $Q : X \times X \times X \times X \rightarrow Y$ such that $f_e(x) = Q(x, x, x, x)$ and that $f_o(x) = C(x, x, x)$ for all $x \in X$, and C is symmetric for each fixed one variable and is additive for fixed two variables. Thus $f(x) = C(x, x, x) + Q(x, x, x, x)$ for all $x \in X$. The proof of the converse is trivial. \square

3. Stability

Throughout this section, X and Y will be a uniquely two-divisible abelian group and a quasiBanach spaces respectively, and p will be a fixed real number in $[0, 1]$. We need the following lemma in the main theorems. Now before taking up the main subject, given $f : X \rightarrow Y$, we define the difference operator $D_f : X \times X \rightarrow Y$ by

$$D_f(x, y) = f(x + 2y) + f(x - 2y) - 4[f(x + y) + f(x - y)] - 3f(2y) + 24f(y) + 6f(x) \quad (3.1)$$

for all $x, y \in X$. We consider the following functional inequality:

$$\|D_f(x, y)\| \leq \phi(x, y) \quad (3.2)$$

for an upper bound $\phi : X \times X \rightarrow [0, \infty)$.

Lemma 3.1. *Let x_1, x_2, \dots, x_n be nonnegative real numbers. Then*

$$\left(\sum_{i=1}^n x_i\right)^p \leq \sum_{i=1}^n x_i^p. \quad (3.3)$$

Theorem 3.2. *Let $l \in \{1, -1\}$ be fixed and let $\varphi : X \times X \rightarrow \mathbb{R}^+$ be a function such that*

$$\lim_{n \rightarrow \infty} 16^{ln} \varphi\left(\frac{x}{2^{ln}}, \frac{y}{2^{ln}}\right) = 0 \quad (3.4)$$

for all $x, y \in X$ and

$$\sum_{i=1}^{\infty} 16^{ilp} \varphi^p\left(0, \frac{y}{2^{li}}\right) < \infty \quad (3.5)$$

for all $y \in X$. Suppose that an even function $f : X \rightarrow Y$ with $f(0) = 0$ satisfies the inequality

$$\|D_f(x, y)\|_Y \leq \varphi(x, y), \quad (3.6)$$

for all $x, y \in X$. Then the limit

$$Q(x) := \lim_{n \rightarrow \infty} 16^{ln} f\left(\frac{x}{2^{ln}}\right) \quad (3.7)$$

exists for all $x \in X$ and $Q : X \rightarrow Y$ is a unique quartic function satisfying

$$\|f(x) - Q(x)\|_Y \leq \frac{K}{16} [\tilde{\varphi}_e(x)]^{1/p}, \quad (3.8)$$

where

$$\tilde{\varphi}_e(x) := \sum_{i=|l+1|/2}^{\infty} 16^{ilp} \varphi^p\left(0, \frac{x}{2^i}\right) \quad (3.9)$$

for all $x \in X$.

Proof. Let $l = 1$. By putting $x = 0$ in (3.6), we get

$$\|f(2y) - 16f(y)\|_Y \leq \varphi(0, y) \quad (3.10)$$

for all $y \in X$. Replacing y by x in (3.10) yields

$$\|f(2x) - 16f(x)\|_Y \leq \varphi(0, x) \quad (3.11)$$

for all $x \in X$. Let $\varphi(x) = \varphi(0, x)$ for all $x \in X$, then by (3.11), we get

$$\|f(2x) - 16f(x)\|_Y \leq \varphi(x) \quad (3.12)$$

for all $x \in X$. Interchanging x with $x/2^{n+1}$ in (3.12), and multiplying by 16^n it follows that

$$\left\| 16^{n+1}f\left(\frac{x}{2^{n+1}}\right) - 16^n f\left(\frac{x}{2^n}\right) \right\|_Y \leq K16^n \varphi\left(\frac{x}{2^{n+1}}\right) \quad (3.13)$$

for all $x \in X$ and all nonnegative integers n . Since Y is p -Banach space, then by (3.13) we have

$$\begin{aligned} \left\| 16^{n+1}f\left(\frac{x}{2^{n+1}}\right) - 16^m f\left(\frac{x}{2^m}\right) \right\|_Y^p &\leq \sum_{i=m}^n \left\| 16^{i+1}f\left(\frac{x}{2^{i+1}}\right) - 16^i f\left(\frac{x}{2^i}\right) \right\|_Y^p \\ &\leq K^p \sum_{i=m}^n 16^{ip} \varphi^p\left(\frac{x}{2^{i+1}}\right) \end{aligned} \quad (3.14)$$

for all nonnegative integers n and m with $n \geq m$ and all $x \in X$. Since $\varphi^p(x) = \varphi^p(0, x)$ for all $x \in X$. Therefore by (3.5) we have

$$\sum_{i=1}^{\infty} 16^{ip} \varphi^p\left(\frac{x}{2^i}\right) < \infty \quad (3.15)$$

for all $x \in X$. Therefore we conclude from (3.14) and (3.15) that the sequence $\{16^n f(x/2^n)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, it follows that the sequence $\{16^n f(x/2^n)\}$ converges for all $x \in X$. We define the mapping $Q : X \rightarrow Y$ by (3.7) for all $x \in X$. Letting $m = 0$ and passing the limit $n \rightarrow \infty$ in (3.14), we get

$$\|f(x) - Q(x)\|_Y^p \leq K^p \sum_{i=0}^{\infty} 16^{ip} \varphi^p\left(\frac{x}{2^{i+1}}\right) = \frac{K^p}{16^p} \sum_{i=1}^{\infty} 16^{ip} \varphi^p\left(\frac{x}{2^i}\right) \quad (3.16)$$

for all $x \in X$. Therefore (3.8) follows from (3.9) and (3.16). Now we show that Q is quartic. It follows from (3.4), (3.6) and (3.7)

$$\|D_Q(x, y)\|_Y = \lim_{n \rightarrow \infty} 16^n \left\| D_f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\|_Y \leq \lim_{n \rightarrow \infty} 16^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \quad (3.17)$$

for all $x, y \in X$. Therefore the mapping $Q : X \rightarrow Y$ satisfies (1.7). Since $Q(0) = 0$, then by Lemma 2.1 we get that the mapping $Q : X \rightarrow Y$ is quartic. To prove the uniqueness of Q , let $T : X \rightarrow Y$ be another quartic mapping satisfies (3.8). Since

$$\lim_{n \rightarrow \infty} 16^{np} \sum_{i=1}^{\infty} 16^{ip} \varphi^p \left(\frac{x}{2^{i+n}}, \frac{y}{2^{i+n}} \right) = \lim_{n \rightarrow \infty} \sum_{i=n+1}^{\infty} 16^{ip} \varphi^p \left(\frac{x}{2^i}, \frac{y}{2^i} \right) = 0 \quad (3.18)$$

for all $y \in X$ and all $x \in \{0\}$, then

$$\lim_{n \rightarrow \infty} 16^{np} \tilde{\varphi}_e \left(\frac{x}{2^n} \right) = 0 \quad (3.19)$$

for all $x \in X$. It follows from (3.8), (3.19)

$$\|Q(x) - T(x)\|_Y^p = \lim_{n \rightarrow \infty} 16^{np} \left\| f \left(\frac{x}{2^n} \right) - T \left(\frac{x}{2^n} \right) \right\|_Y^p \leq \frac{K^p}{16^p} \lim_{n \rightarrow \infty} 16^{np} \tilde{\varphi}_e \left(\frac{x}{2^n} \right) = 0 \quad (3.20)$$

for all $x \in X$. Hence $Q = T$. For $l = -1$, we obtain

$$\left\| \frac{f(2^n x)}{16^n} - f(x) \right\|_Y^p \leq \frac{K^p}{16^p} \sum_{i=0}^{\infty} \frac{\varphi^p(0, 2^i x)}{16^{ip}}, \quad (3.21)$$

from which one can prove the result by a similar technique. \square

Corollary 3.3. Let θ, r, s, u, v be nonnegative real numbers such that $s \neq 4 \neq u + v$. Suppose that an even function $f : X \rightarrow Y$ with $f(0) = 0$ satisfies the inequality

$$\|D_f(x, y)\|_Y \leq \theta (\|x\|_X^u \|y\|_X^v + \|x\|_X^r + \|y\|_X^s) \quad (3.22)$$

for all $x, y \in X$. Then there exists a unique quartic function $Q : X \rightarrow Y$ satisfying

$$\|f(x) - Q(x)\|_Y \leq K\theta \left\{ \frac{1}{|16^p - 2^{sp}|} \right\}^{1/p} \|x\|_X^s \quad (3.23)$$

for all $x \in X$.

Proof. It follows from Theorem 3.2 that $\varphi(x, y) := \theta (\|x\|_X^u \|y\|_X^v + \|x\|_X^r + \|y\|_X^s)$ for all $x, y \in X$. \square

Theorem 3.4. Let $l \in \{1, -1\}$ be fixed and let $\varphi : X \times X \rightarrow [0, \infty)$ be a function such that

$$\lim_{n \rightarrow \infty} 8^{ln} \varphi \left(\frac{x}{2^{ln}}, \frac{y}{2^{ln}} \right) = 0 \quad (3.24)$$

for all $x, y \in X$ and

$$\sum_{i=1}^{\infty} 8^{ip} \varphi^p \left(0, \frac{y}{2^i} \right) < \infty \quad (3.25)$$

for all $y \in X$. Suppose that an odd function $f : X \rightarrow Y$ satisfies the inequality

$$\|D_f(x, y)\|_Y \leq \varphi(x, y), \quad (3.26)$$

for all $x, y \in X$. Then the limit

$$C(x) := \lim_{n \rightarrow \infty} 8^{ln} f \left(\frac{x}{2^{ln}} \right) \quad (3.27)$$

exists for all $x \in X$ and $C : X \rightarrow Y$ is a unique cubic function satisfying

$$\|f(x) - C(x)\|_Y \leq \frac{K}{24} [\tilde{\phi}_o(x)]^{1/p} \quad (3.28)$$

for all $x \in X$, where

$$\tilde{\phi}_o(x) := \sum_{i=|l+1|/2}^{\infty} 8^{ip} \varphi^p \left(0, \frac{x}{2^i} \right). \quad (3.29)$$

Proof. Let $l = 1$. Setting $x = 0$ in (3.26), we get

$$\|3f(2y) - 24f(y)\|_Y \leq \varphi(0, y) \quad (3.30)$$

for all $y \in X$. If we replace y in (3.30) by x and divide both sides of (3.30) by 3, we get

$$\|f(2x) - 8f(x)\|_Y \leq \frac{1}{3} \varphi(0, x) \quad (3.31)$$

for all $x \in X$. Let $\phi(x) = (1/3)\varphi(0, x)$ for all $x \in X$, then by (3.31), we get

$$\|f(2x) - 8f(x)\|_Y \leq \phi(x) \quad (3.32)$$

for all $x \in X$. Multiply (3.32) by 8^n and replace x by $x/2^{n+1}$, we obtain that

$$\left\| 8^{n+1} f \left(\frac{x}{2^{n+1}} \right) - 8^n f \left(\frac{x}{2^n} \right) \right\|_Y \leq K 8^n \phi \left(\frac{x}{2^{n+1}} \right) \quad (3.33)$$

for all $x \in X$ and all nonnegative integers n . Since Y is a p -Banach space, (3.33) follows that

$$\begin{aligned} \left\| 8^{n+1} f\left(\frac{x}{2^{n+1}}\right) - 8^n f\left(\frac{x}{2^n}\right) \right\|_Y^p &\leq \sum_{i=m}^n \left\| 8^{i+1} f\left(\frac{x}{2^{i+1}}\right) - 8^i f\left(\frac{x}{2^i}\right) \right\|_Y^p \\ &\leq K^p \sum_{i=m}^n 8^{ip} \phi^p\left(\frac{x}{2^{i+1}}\right) \end{aligned} \quad (3.34)$$

for all nonnegative integers n and m with $n \geq m$ and all $x \in X$. Since $\phi^p(x) = (1/3^p)\varphi^p(0, x)$ for all $x \in X$. Therefore it follows from (3.25) that

$$\sum_{i=1}^{\infty} 8^{ip} \phi^p\left(\frac{x}{2^i}\right) < \infty \quad (3.35)$$

for all $x \in X$, therefore we conclude from (3.34) and (3.35) that the sequence $\{8^n f(x/2^n)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{8^n f(x/2^n)\}$ converges for all $x \in X$. So one can define the mapping $C : X \rightarrow Y$ by (3.27) for all $x \in X$. Letting $m = 0$ and passing the limit $n \rightarrow \infty$ in (3.34), we get

$$\|f(x) - C(x)\|_Y^p \leq K^p \sum_{i=0}^{\infty} 8^{ip} \phi^p\left(\frac{x}{2^{i+1}}\right) = \frac{K^p}{8^p} \sum_{i=1}^{\infty} 8^{ip} \phi^p\left(\frac{x}{2^i}\right) \quad (3.36)$$

for all $x \in X$. Therefore (3.28) follows from (3.29) and (3.36). Now we show that C is cubic. It follows from (3.24), (3.26) and (3.27)

$$\|D_C(x, y)\|_Y = \lim_{n \rightarrow \infty} 8^n \left\| D_f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\|_Y \leq \lim_{n \rightarrow \infty} 8^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \quad (3.37)$$

for all $x, y \in X$. Therefore the mapping $C : X \rightarrow Y$ satisfies (1.7). Since f is an odd function, then (3.27) implies that the mapping odd. Therefore by Lemma 2.2 we get that the mapping $C : X \rightarrow Y$ is cubic. The rest of proof is similar to the proof of Theorem 3.2. \square

Corollary 3.5. *Let θ be a nonnegative real number and r, s be real numbers such that $s \neq 3 \neq u + v$. Suppose that an odd function $f : X \rightarrow Y$ satisfies the inequality*

$$\|D_f(x, y)\|_Y \leq \theta (\|x\|_X^u \|y\|_X^v + \|x\|_X^r + \|y\|_X^s) \quad (3.38)$$

for all $x, y \in X$. Then there exists a unique cubic function $C : X \rightarrow Y$ satisfying

$$\|f(x) - C(x)\|_Y \leq \frac{K\theta}{3} \left\{ \frac{1}{|8^p - 2^{sp}|} \right\}^{1/p} \|x\|_X^s \quad (3.39)$$

for all $x \in X$.

Proof. It follows from (3.38) and Theorem 3.4 that $\varphi(x, y) := \theta (\|x\|_X^u \|y\|_X^v + \|x\|_X^r + \|y\|_X^s)$ for all $x, y \in X$. \square

Theorem 3.6. Let $l \in \{1, -1\}$ be fixed and let $\varphi : X \times X \rightarrow [0, \infty)$ be a function which satisfies

$$\lim_{n \rightarrow \infty} \left\{ \left(\frac{|l|+1}{2} \right) 16^{ln} \varphi \left(\frac{x}{2^{ln}}, \frac{y}{2^{ln}} \right) + \left(\frac{|l|-1}{2} \right) 8^{ln} \varphi \left(\frac{x}{2^{ln}}, \frac{y}{2^{ln}} \right) \right\} = 0 \quad (3.40)$$

for all $x, y \in X$ and

$$\sum_{i=|l+1|/2}^{\infty} \left\{ \left(\frac{|l|+1}{2} \right) 16^{ip} \varphi^p \left(0, \frac{y}{2^i} \right) + \left(\frac{|l|-1}{2} \right) 8^{ip} \varphi^p \left(0, \frac{y}{2^i} \right) \right\} < \infty \quad (3.41)$$

for all $y \in X$. Suppose that a function $f : X \rightarrow Y$ with $f(0) = 0$ satisfies the inequality

$$\|D_f(x, y)\|_Y \leq \varphi(x, y) \quad (3.42)$$

for all $x, y \in X$. Then there exists a unique quartic function $Q : X \rightarrow Y$ and a unique cubic function $C : X \rightarrow Y$ satisfying (1.7) and

$$\|f(x) - Q(x) - C(x)\|_Y \leq \frac{K^3}{32} [\tilde{\psi}_e(x) + \tilde{\psi}_e(-x)]^{1/p} + \frac{K^3}{48} [\tilde{\phi}_o(x) + \tilde{\phi}_o(-x)]^{1/p} \quad (3.43)$$

for all $x \in X$, where $\tilde{\psi}_e(x)$ and $\tilde{\phi}_o(x)$ that have been defined in (3.9) and (3.29), respectively.

Proof. Let $f_e(x) = (1/2)(f(x) + f(-x))$ for all $x \in X$. Then $f_e(0) = 0, f_e(-x) = f_e(x)$, and $\|D_{f_e}(x, y)\| \leq (K/2)[\varphi(x, y) + \varphi(-x, -y)]$ for all $x, y \in X$. Let

$$\Phi(x, y) = \frac{K}{2} [\varphi(x, y) + \varphi(-x, -y)] \quad (3.44)$$

for all $x, y \in X$. So

$$\lim_{n \rightarrow \infty} 16^{ln} \Phi \left(\frac{x}{2^{ln}}, \frac{y}{2^{ln}} \right) = 0 \quad (3.45)$$

for all $x, y \in X$. Since

$$\Phi^p(x, y) \leq \frac{K^p}{2^p} [\varphi^p(x, y) + \varphi^p(-x, -y)] \quad (3.46)$$

for all $x, y \in X$, then

$$\sum_{i=1}^{\infty} 16^{ip} \Phi^p \left(\frac{x}{2^{i1}}, \frac{y}{2^{i1}} \right) < \infty \quad (3.47)$$

for all $y \in X$ and all $x \in \{0\}$. Hence, in view of Theorem 3.2, there exists a unique quartic function $Q : X \rightarrow Y$ satisfying

$$\|f_e(x) - Q(x)\|_Y \leq \frac{K}{16} [\tilde{\Psi}_e(x)]^{1/p} \quad (3.48)$$

for all $x \in X$, where

$$\tilde{\Psi}_e(x) := \sum_{i=1}^{\infty} 16^{ip} \Phi^p \left(0, \frac{x}{2^i} \right). \quad (3.49)$$

We have

$$\tilde{\Psi}_e(x) \leq \frac{K^p}{2^p} [\tilde{\psi}_e(x) + \tilde{\psi}_e(-x)] \quad (3.50)$$

for all $x \in X$. Therefore it follows from (3.48) that,

$$\|f_e(x) - Q(x)\|_Y \leq \frac{K^2}{32} [\tilde{\psi}_e(x) + \tilde{\psi}_e(-x)]^{1/p} \quad (3.51)$$

for all $x \in X$. Let $f_o(x) = (1/2)(f(x) - f(-x))$ for all $x \in X$. Then $f_o(0) = 0$, $f_o(-x) = -f_o(x)$, and $\|D_{f_o}(x, y)\| \leq \Phi(x, y)$ for all $x, y \in X$. From Theorem 3.4, it follows that there exists a unique cubic function $C : X \rightarrow Y$ satisfying

$$\|f_o(x) - C(x)\|_Y \leq \frac{K}{24} [\tilde{\Phi}_o(x)]^{1/p} \quad (3.52)$$

for all $x \in X$, where

$$\tilde{\Phi}_o(x) := \sum_{i=1}^{\infty} 8^{ip} \Phi^p \left(0, \frac{x}{2^i} \right). \quad (3.53)$$

Since

$$\tilde{\Phi}_o(x) \leq \frac{K^p}{2^p} [\tilde{\phi}_o(x) + \tilde{\phi}_o(-x)] \quad (3.54)$$

for all $x \in X$, it follows from (3.52) that,

$$\|f_o(x) - C(x)\|_Y \leq \frac{K^2}{48} [\tilde{\phi}_o(x) + \tilde{\phi}_o(-x)]^{1/p} \quad (3.55)$$

for all $x \in X$. Hence (3.43) follows from (3.51) and (3.55). \square

Corollary 3.7. Let θ, r, s be nonnegative real numbers such that $u + v, s \in (4, \infty) \cup (-\infty, 3)$. Suppose that a function $f : X \rightarrow Y$ with $f(0) = 0$ satisfies the inequality

$$\|D_f(x, y)\|_Y \leq \theta(\|x\|_X^u \|y\|_X^v + \|x\|_X^r + \|y\|_X^s) \quad (3.56)$$

for all $x, y \in X$. Then there exists a unique quartic function $Q : X \rightarrow Y$ and a unique cubic function $C : X \rightarrow Y$ satisfying (1.7) and

$$\|f(x) - Q(x) - C(x)\|_Y \leq \frac{K^3\theta}{3} \left\{ 3 \left[\frac{1}{|16^p - 2^{sp}|} \right]^{1/p} + \left[\frac{1}{|8^p - 2^{sp}|} \right]^{1/p} \right\} \|x\|_X^s \quad (3.57)$$

for all $x \in X$.

Proof. It follows from Theorem 3.6 that

$$\varphi(x, y) := \theta(\|x\|_X^u \|y\|_X^v + \|x\|_X^r + \|y\|_X^s) \quad (3.58)$$

for all $x, y \in X$. □

Acknowledgment

The second and fourth authors would like to thank the office of gifted students at Semnan University for its financial support.

References

- [1] S. Rolewicz, *Metric Linear Spaces*, Polish Scientific, Warsaw, Poland, 2nd edition, 1984.
- [2] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis. Vol. 1*, vol. 48 of *American Mathematical Society Colloquium Publications*, American Mathematical Society, Providence, RI, USA, 2000.
- [3] S. M. Ulam, *Problems in Modern Mathematics*, chapter 6, John Wiley & Sons, New York, NY, USA, 1940.
- [4] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.
- [5] Th. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.
- [6] J. M. Rassias, "Solution of the Ulam stability problem for cubic mappings," *Glasnik Matematički*, vol. 36(56), no. 1, pp. 63–72, 2001.
- [7] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.
- [8] P. W. Cholewa, "Remarks on the stability of functional equations," *Aequationes Mathematicae*, vol. 27, no. 1-2, pp. 76–86, 1984.
- [9] Z. Gajda, "On stability of additive mappings," *International Journal of Mathematics and Mathematical Sciences*, vol. 14, no. 3, pp. 431–434, 1991.
- [10] P. Găvruta, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.
- [11] A. Grabiec, "The generalized Hyers-Ulam stability of a class of functional equations," *Publicationes Mathematicae Debrecen*, vol. 48, no. 3-4, pp. 217–235, 1996.
- [12] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, vol. 34 of *Progress in Nonlinear Differential Equations and Their Applications*, Birkhäuser, Boston, Mass, USA, 1998.

- [13] J. M. Rassias, "On a new approximation of approximately linear mappings by linear mappings," *Discussiones Mathematicae*, vol. 7, pp. 193–196, 1985.
- [14] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Bulletin des Sciences Mathématiques*, vol. 108, no. 4, pp. 445–446, 1984.
- [15] J. M. Rassias, "Solution of a problem of Ulam," *Journal of Approximation Theory*, vol. 57, no. 3, pp. 268–273, 1989.
- [16] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Journal of Functional Analysis*, vol. 46, no. 1, pp. 126–130, 1982.
- [17] Th. M. Rassias, Ed., *Functional Equations and Inequalities*, vol. 518 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.
- [18] Th. M. Rassias, "On the stability of functional equations and a problem of Ulam," *Acta Applicandae Mathematicae*, vol. 62, no. 1, pp. 23–130, 2000.
- [19] K.-W. Jun and H.-M. Kim, "The generalized Hyers-Ulam-Rassias stability of a cubic functional equation," *Journal of Mathematical Analysis and Applications*, vol. 274, no. 2, pp. 267–278, 2002.
- [20] P. Găvruta and L. Cădariu, "General stability of the cubic functional equation," *Buletinul Științific al Universității Politehnica din Timișoara. Seria Matematică-Fizică*, vol. 47(61), no. 1, pp. 59–70, 2002.
- [21] J. M. Rassias, "Solution of the Ulam stability problem for quartic mappings," *The Journal of the Indian Mathematical Society*, vol. 67, no. 1–4, pp. 169–178, 2000.
- [22] W.-G. Park and J.-H. Bae, "On a bi-quadratic functional equation and its stability," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 62, no. 4, pp. 643–654, 2005.
- [23] J. K. Chung and P. K. Sahoo, "On the general solution of a quartic functional equation," *Bulletin of the Korean Mathematical Society*, vol. 40, no. 4, pp. 565–576, 2003.
- [24] L. Cădariu, "Fixed points in generalized metric space and the stability of a quartic functional equation," *Buletinul Științific al Universității Politehnica din Timișoara. Seria Matematică-Fizică*, vol. 50(64), no. 2, pp. 25–34, 2005.
- [25] S. H. Lee, S. M. Im, and I. S. Hwang, "Quartic functional equations," *Journal of Mathematical Analysis and Applications*, vol. 307, no. 2, pp. 387–394, 2005.
- [26] A. Najati, "On the stability of a quartic functional equation," *Journal of Mathematical Analysis and Applications*, vol. 340, no. 1, pp. 569–574, 2008.
- [27] C.-G. Park, "On the stability of the orthogonally quartic functional equation," *Bulletin of the Iranian Mathematical Society*, vol. 31, no. 1, pp. 63–70, 2005.
- [28] K. Ravi and M. Arunkumar, "Hyers-Ulam-Rassias stability of a quartic functional equation," *International Journal of Pure and Applied Mathematics*, vol. 34, no. 2, pp. 247–260, 2007.
- [29] E. Thandapani, K. Ravi, and M. Arunkumar, "On the solution of the generalized quartic functional equation," *Far East Journal of Applied Mathematics*, vol. 24, no. 3, pp. 297–312, 2006.
- [30] A. Gilányi, "On Hyers-Ulam stability of monomial functional equations," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 68, pp. 321–328, 1998.
- [31] A. Gilányi, "Hyers-Ulam stability of monomial functional equations on a general domain," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 96, no. 19, pp. 10588–10590, 1999.



Hindawi

Submit your manuscripts at
<http://www.hindawi.com>

