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Research Article

Solution of Wave Equation in Radial Form by VIM

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An analytic approximation to the solution of wave equation is studied. Wave equation is in radial form with indicated initial and boundary conditions, by variational iteration method it has been used to derive this approximation and some examples are presented to show the simplicity and efficiency of the method.

1. Introduction

Wave equation has attracted much attention and solving these kind of equations has been one of the interesting tasks for mathematicians. Variational iteration method is known as a powerful device for solving functional equations [1–7]. Numerical methods which are commonly used as finite-difference methods and characteristics method need large size of computational works and usually the effect of round-off error causes the loss of accuracy in the results. Analytical methods commonly used for solving wave equation are very restricted and can be used in special cases, so they cannot be used to solve equations resulted by mathematical modeling of numerous realistic scenarios. In this article, the variational iteration method has been applied to solve more general forms of wave equation.

2. He's Variational Iteration Method

The variational iteration method [8–13], which is a modified of general Lagrange multiplier method [14], has been shown to solve effectively, easily, and accurately large class of nonlinear problems with approximations which converge rapidly to accurate solutions. To illustrate the method, consider the following nonlinear equation:

$$Lu(t) + Nu(t) = g(t), \quad (1)$$

where L is a linear operator, N is a nonlinear operator, and $g(t)$ is a known analytic function. According to the

variational iteration method, we can construct the following correction functional:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\xi) (Lu_n(\xi) + Nu_n(\xi) - g(\xi)) d\xi, \quad (2)$$

where λ is general Lagrange multiplier which can be identified via variational theory, $u_0(t)$ is an initial approximation with possible unknowns, and \tilde{u}_n is considered as restricted variation [15] (i.e., $\delta\tilde{u}_n = 0$). Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations $u_{n+1}(t)$ of the solution $u(t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0 . Consequently, the exact solution may be obtained by $u = \lim_{n \rightarrow \infty} u_n$.

3. Numerical Results

To illustrate the method and to show ability of the method, some examples are presented.

Example 1. Let us have one-dimensional wave equation in radial form with initial condition:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r}, \\ u(r_i, 0) &= r. \end{aligned} \quad (3)$$

Its correction functional can be expressed as follows:

$$u_{n+1}(r, t) = u_n(r, t) + \int_0^t (\xi - t) \left(\frac{\partial^2 u_n}{\partial \xi^2} - \frac{\partial^2 \tilde{u}_n}{\partial r^2} - \frac{1}{r} \frac{\partial \tilde{u}_n}{\partial r} \right) d\xi. \quad (4)$$

To make this correct functional stationary, $\delta u_n(r, 0) = 0$,

$$\delta u_{n+1} = \delta u_n + (\lambda_1 (\delta u_n)')_0^t - (\lambda_1' \delta u_n)_0^t + \int_0^t \lambda_1' \delta u_n d\xi = 0. \quad (5)$$

Its stationary conditions can be obtained as follows:

$$\begin{aligned} \delta u_n : 1 - \lambda_1'(t) &= 0, \\ \delta u_n' : \lambda_1(t) &= 0, \\ \delta u_n : \lambda_1'(\xi) &= 0, \end{aligned} \quad (6)$$

from which Lagrange multiplier can be identified as $\lambda_1 = \xi - t$, and the following iteration formula will be obtained:

$$u_{n+1}(r, t) = u_n(r, t) + \int_0^t (\xi - t) \left(\frac{\partial^2 u_n}{\partial \xi^2} - \frac{\partial^2 u_n}{\partial r^2} - \frac{1}{r} \frac{\partial u_n}{\partial r} \right) d\xi. \quad (7)$$

Beginning with $u_0(r, t) = r$, by iteration formula (7), we have

$$\begin{aligned} u_1(r, t) &= r + \frac{1}{2!r} t^2, \\ u_2(r, t) &= r + \frac{1}{2!r} t^2 + \frac{1^2}{4!r^3} t^4, \\ u_3(r, t) &= r + \frac{1}{2!r} t^2 + \frac{1^2}{4!r^3} t^4 + \frac{1^2 \times 3^2}{6! \times r^5} t^6, \\ &\vdots \end{aligned} \quad (8)$$

from which the general term and so the solution will be determined as follows:

$$u(r, t) = r + \sum_{n=1}^{\infty} \frac{1^2 \times 3^2 \times \cdots \times (2n-3)^2}{(2n)!r^{2n-1}} t^{2n}. \quad (9)$$

Example 2. Let us solve two-dimensional wave equation in radial form with the boundary conditions:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2},$$

$$u_r(1, \theta, t) = -1, \quad u_r(10, \theta, t) = 0, \quad u_\theta(r, 1, t) = 1. \quad (10)$$

Its correction functional can be expressed as follows:

$$u_{n+1}(r, \theta, t) = u_n(r, \theta, t) + \int_1^r \lambda_2 \left(\frac{\partial^2 u_n}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial u_n}{\partial \xi} + \frac{1}{\xi^2} \frac{\partial^2 \tilde{u}_n}{\partial \theta^2} - \frac{\partial^2 \tilde{u}_n}{\partial t^2} \right) d\xi. \quad (11)$$

Making the above correct functional stationary, notice that $\delta u_n(1, \theta, t) = 0$:

$$\begin{aligned} \delta u_{n+1} &= \delta u_n + (\lambda_2 (\delta u_n)')_1^r - (\lambda_2' \delta u_n)_1^r \\ &+ \int_1^r \lambda_2'' \delta u_n d\xi + \left(\frac{\lambda_2}{\xi} \delta u_n \right)_1^r \\ &- \int_1^r \frac{\xi \lambda_2' - \lambda_2}{\xi^2} \delta u_n d\xi = 0. \end{aligned} \quad (12)$$

Its stationary conditions can be obtained as follows:

$$\begin{aligned} \delta u_n : \lambda_2'' - \frac{\xi \lambda_2' - \lambda_2}{\xi^2} &= 0, \\ \delta u_n' : \lambda_2(r) &= 0, \\ \delta u_n : 1 - \lambda_2'(r) + \frac{\lambda_2(r)}{r} &= 0, \end{aligned} \quad (13)$$

from which the Lagrange multiplier would be identified as follows:

$$\lambda_2(\xi) = \xi(\ln \xi - \ln r). \quad (14)$$

Substituting (14) into (11) leads to the following iteration formula:

$$\begin{aligned} u_{n+1}(r, \theta, t) &= u_n(r, \theta, t) \\ &+ \int_1^r \xi(\ln \xi - \ln r) \left(\frac{\partial^2 u_n}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial u_n}{\partial \xi} \right. \\ &\quad \left. + \frac{1}{\xi^2} \frac{\partial u_n}{\partial \theta^2} - \frac{\partial^2 u_n}{\partial t^2} \right) d\xi. \end{aligned} \quad (15)$$

Starting with

$$u_0(r, \theta, t) = Ar^2 + B\theta^2 + Ct^2, \quad (16)$$

we have

$$\begin{aligned} u_1(r, \theta, t) &= Ar^2 + B\theta^2 + Ct^2 \\ &+ \int_1^r \xi(\ln \xi - \ln r) \left(4A + \frac{2}{\xi^2} B - 2C \right) d\xi \\ &= A(1 + 2 \ln r) + B(\theta^2 - (\ln r)^2) \\ &+ C \left(t^2 + \frac{r^2}{2} - \frac{1}{2} - \ln r \right). \end{aligned} \quad (17)$$

Imposing the boundary conditions yields to $A = -1/2$, $B = 1$, $C = 1 + \ln 10/99$.

Thus, we have

$$\begin{aligned} u_1(r, \theta, t) &= \frac{1}{198} \left(-100 - \ln 10 + (1 + \ln 10)r^2 - 99(\ln r)^2 \right. \\ &\quad \left. - (200 + 2 \ln 10) \ln r + 99\theta^2 \right. \\ &\quad \left. + 2(1 + \ln 10)t^2 \right), \end{aligned} \quad (18)$$

which is an exact solution.

Example 3. Consider Example 1 with boundary conditions following:

$$u(1, t) = 1, \quad u_r(1, t) = t. \quad (19)$$

Similar to Example 2, the Lagrange multiplier can be identified as $\lambda_3(\xi) = \xi(\ln \xi - \ln r)$ and the following iteration formula will be obtained:

$$u_{n+1}(r, t) = u_n(r, t) + \int_1^r \xi(\ln \xi - \ln r) \left(\frac{\partial^2 u_n}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial u_n}{\partial \xi} - \frac{\partial^2 u_n}{\partial t^2} \right) d\xi. \quad (20)$$

Starting with

$$u_0(r, t) = r + \int_1^r t d\xi = 1 + t(r - 1), \quad (21)$$

by iteration formula (20), we have

$$u_1(r, t) = 1 + t \ln r, \quad (22)$$

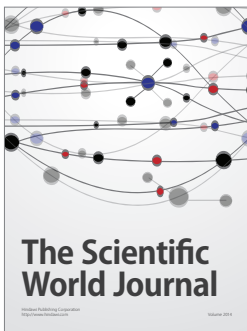
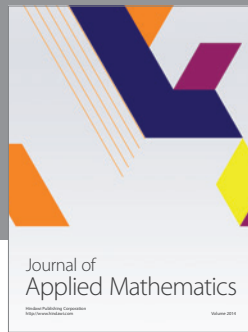
which is an exact solution.

4. Conclusion

In this work, we present an analytical approximation to the solution of wave equation in radial form in different cases. We have achieved this goal by applying variational iteration method. The small size of computations in comparison with the computational size required in numerical methods and the rapid convergence shows that the variational iteration method is reliable and introduces a significant improvement in solving the wave equation over existing methods. The main advantage of the VIM over A.D.M. is that this method provides the solution without a need for calculating Adomian's polynomials [16].

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