Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2014, Article ID 140790, 4 pages
http://dx.doi.org/10.1155/2014/140790

Research Article

On Maps of Period 2 on Prime and Semiprime Rings

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Received 29 March 2014; Accepted 16 August 2014; Published 22 September 2014

Academic Editor: Ram N. Mohapatra

A map f of the ring R into itself is of period 2 if $f^2(x) = x$ for all $x \in R$; involutions are much studied examples. We present some commutativity results for semiprime and prime rings with involution, and we study the existence of derivations and generalized derivations of period 2 on prime and semiprime rings.

1. Introduction

Let R be a ring with center $Z = Z(R)$, and for each $x, y \in R$, let $[x, y]$ denote the commutator $xy - yx$. Note that $[xy, w] = x[y, w] + [x, w]y$ and $[w, xy] = x[w, y] + [w, x]y$ for all $x, y, w \in R$—facts we will use repeatedly.

Let S be a nonempty subset of R. A map $f : R \to R$ is said to be of period 2 on S if $f^2(x) = x$ for all $x \in S$, and S is called an f-subset if $f(S) = S$. If $[x, f(x)] = 0$ for all $x \in S$, then f is said to be *commuting* on S; if $[x, y] = [f(x), f(y)]$ for all $x, y \in S$, then, as in [1], f is called *strong commutativity-preserving* on S.

We assume the reader is familiar with the definitions of *derivation* and *involution*. We define an additive map $F : R \to R$ to be a *right (resp. left) generalized derivation* on R if $F(xy) = F(x)y + xd(y)$ (resp., $F(xy) = d(x)y + xF(y)$) for all $x, y \in R$, where d is a derivation on R, called the associated derivation. If F is both a right generalized derivation and a left generalized derivation with the same associated derivation, we call F a generalized derivation. (Note that this definition is different from that of Hvala in [2]; his generalized derivations are our right generalized derivations.)

Our purpose is to study existence and properties of involutions, derivations, and generalized derivations of period 2 on certain subsets of semiprime and prime rings.

2. Two Commutativity Results for Rings with Involution

There are several known commutativity results for rings with involution (cf. [3, Chapter 3]). We now present a result showing the equivalence of two commutativity conditions on a $*$-ideal of a semiprime ring with involution.

Theorem 1. *Let R be a semiprime ring with involution $*$, and let U be a $*$-ideal of R. Then $*$ is commuting on U if and only if $*$ is strong commutativity-preserving on U.*

Proof. Assume first that $*$ is commuting on U; that is, $[x, x^*] = 0$ for all $x \in U$. By linearizing we get

$$[x, y^*] = [x^*, y] \quad \forall x, y \in U. \tag{1}$$

It follows that $[xy, x^*] = [(xy)^*, x]$ and hence $x[y, x^*] = [y^*, x]x^*$, and by (1) we get

$$x[y^*, x] = [y^*, x]x^* \quad \forall x, y \in U. \tag{2}$$

Since U is a $*$-ideal, (2) yields

$$x[x, y] = [x, y]x^* \quad \forall x, y \in U. \tag{3}$$

Substituting zy for $y, z \in U$, we now get $x[x, zy] = [x, zy]x^*$, so that $xz[x, y] + x[x, z]y = [x, z]yx^* + z[x, y]x^*$. Using

(3) to replace the last term in this equation by $zx[x, y]$ and the second term by $[x, z]x^* y$, we see that $[x, z][x, y] + [x, z][x^*, y] = 0$, so by (1),

$$[x, z]\,[x, y + y^*] = 0 \quad \forall x, y, z \in U. \tag{4}$$

Replacing z by zw yields $[x, z]U[x, y + y^*] = \{0\}$ for all $x, y \in U$, so $[x, y + y^*]U[x, y + y^*] = \{0\}$ for all $x, y \in U$. Since an ideal of a semiprime ring is a semiprime ring, we conclude that

$$[x, y + y^*] = 0 \quad \forall x, y \in U. \tag{5}$$

Now (5) may be rewritten as

$$[x, y] = [y^*, x], \tag{6}$$

so by replacing x by x^* we get $[x^*, y] = [y^*, x^*]$. But by (6) $[x^*, y] = [y, x]$; hence $[y, x] = [y^*, x^*]$ for all $x, y \in U$, so that $*$ is strong commutativity-preserving on U.

For the converse, we assume that $*$ is strong commutativity-preserving on U, which means that $[x, y] = [x^*, y^*]$ for all $x, y \in U$. Substituting xy for y, we get

$$x\,[x, y] = [x, y]\,x^* \quad \forall x, y \in U. \tag{7}$$

This is just equation (3), so we may argue as before that

$$[x, y + y^*] = 0 \quad \forall x, y \in U, \tag{8}$$

and $y + y^* \in Z(U)$ for all $y \in U$. It follows at once that $[y, y^*] = 0$ for all $y \in U$; that is, $*$ is commuting on U. □

The proof just given yields a result for prime rings with involution. Before stating our theorem, we mention that we are using the symbols S and K to denote the sets of symmetric elements and skew elements, respectively, in the ring R with involution $*$.

Theorem 2. *Let R be a prime ring with involution $*$, with $\mathrm{char}(R) \neq 2$. If $*$ is commuting on some nonzero $*$-ideal U, then $S \subseteq Z$.*

Proof. It follows from (5) that if $y \in S \cap U$, then $y \in Z(U)$, and since in a prime ring the center of a nonzero ideal is contained in the center of R, $S \cap U \subseteq Z(R)$. Suppose that $S \cap U \neq \{0\}$, and let $z \in (S \cap U) \setminus \{0\}$. Then for any $s \in S$, $sz \in U \cap S$, so $sz \in Z(R)$. Since z is not a zero divisor, we get $s \in Z(R)$.

To complete the proof, we need only show that $S \cap U \neq \{0\}$. Suppose, on the contrary, that $S \cap U = \{0\}$. Then for any $y \in U$, $y + y^* = 0$; hence $U \subseteq K$. But for any $k \in K$, $k^2 \in S$; therefore $y^2 = 0$ for all $y \in U$, and we have contradicted a well-known result of Levitzki [4, Lemma 1.1]. □

3. On Nonexistence of Derivations of Period 2

If R is an algebra over $GF(p)$ with trivial multiplication, the map given by $d(x) = (p - 1)x$ is a derivation of period 2. We do not know whether there exist less obvious examples.

Clearly any derivation d of period 2 must be a bijection, so there exists no $c \neq 0$ such that $d(c) = 0$. It follows that a ring R with 1 admits no derivation which is of period 2 on R.

There do exist semiprime rings R admitting a derivation which is a bijection, for example, the $\mathbb{R}$—algebra with basis $\{e^{kx} \mid k = 1, 2, \ldots\}$, with d being the usual differentiation. Obviously this example is not of period 2, and we will show that a semiprime ring admits no derivation of period 2 on R.

Theorem 3. *Let R be a semiprime ring and U a nonzero right ideal. Then R admits no derivation d such that d is of period 2 on U.*

Proof. Suppose there exists a derivation d on R such that $d^2(x) = x$ for all $x \in U$. For $x, y \in U$, $xd(y) \in U$ and the condition that $xd(y) = d^2(xd(y))$ yields

$$xd(y) + 2d(x)\,y = 0 \quad \forall x, y \in U. \tag{9}$$

Since $d^2(xy) = xy = d(d(x)y + xd(y))$, we get

$$xy + 2d(x)\,d(y) = 0 \quad \forall x, y \in U; \tag{10}$$

and replacing y by yr in (10), we obtain

$$2d(x)\,yd(r) = 0 \quad \forall x, y \in U,\ r \in R. \tag{11}$$

Substituting rx for r in (11), we get $2d(x)yrd(x) = 0$; hence

$$2d(x)\,yRd(x)\,y = \{0\} \quad \forall x, y \in U. \tag{12}$$

But R is semiprime, so $2d(x)y = 0$ for all $x, y \in U$, and by (9),

$$xd(y) = 0 \quad \forall x, y \in U. \tag{13}$$

Therefore

$$\begin{aligned} d(xd(y)) &= d(x)\,d(y) + xd^2(y) \\ &= d(x)\,d(y) + xy = 0 \quad \forall x, y \in U, \end{aligned} \tag{14}$$

which together with (10) yields $xy = 0$ for all $x, y \in U$. In particular, $x^2 = 0$ for all $x \in U$, contrary to Levitzki's result. □

Corollary 4. *A semiprime ring R admits no derivation of period 2 on R.*

Remark 5. Of course any derivation d of period 2 on R satisfies $d^3 = d$. It is shown in [5] that a noncommutative semiprime ring, though it has no derivations of period 2, may have many nonzero derivations for which $d^3 = d$; for any noncentral idempotent e, the inner derivation d_e is an example.

4. Generalized Derivations of Period 2

Any ring admits right generalized derivations of period 2, namely, the identity map and its negative. Moreover, if R has 1 and $c \in R$ with $c^2 = 1$, then $F(x) = cx$ defines a right generalized derivation of period 2. We show that, in many prime rings, there are no other possibilities.

We will make use of several easy lemmas.

Lemma 6. *Let R be an arbitrary ring. If F is a generalized derivation on R, then* $F(Z) \subseteq Z$.

Proof. Let $x \in R$ and $z \in Z$. Then $F(zx) = F(xz)$, so that $F(z)x + zd(x) = d(x)z + xF(z)$, where d is the associated derivation of F. Since $[z, d(x)] = 0$, the result follows at once. □

Lemma 7. *Let R be a prime ring with* $char(R) \neq 2$, *and let d be a derivation on R. If the right generalized derivation given by* $F(x) = x + d(x)$ *(resp.,* $F(x) = -x + d(x)$*) for all* $x \in R$ *is of period 2 on R, then F is the identity map (resp., the negative of the identity map) on R.*

Proof. Consider the case $F(x) = x + d(x)$ for all $x \in R$. If F is of period 2, we have $x = F^2(x) = F(x + d(x)) = F(x) + F(d(x)) = x + d(x) + d(x) + d^2(x)$; hence

$$2d(x) + d^2(x) = 0 \quad \forall x \in R. \tag{15}$$

Replacing x by xy, we get $2(d(x)y + xd(y)) + d^2(x)y + 2d(x)d(y) + xd^2(y) = 0$; that is, $(2d(x) + d^2(x))y + x(2d(y) + d^2(y)) + 2d(x)d(y) = 0$ for all $x, y \in R$. In view of (15) and the assumption that $char(R) \neq 2$, this equation gives

$$d(x)d(y) = 0 \quad \forall x, y \in R. \tag{16}$$

It is well known and easy to prove that if R is prime and d is a nonzero derivation, then $a \in R$ and $ad(R) = \{0\}$ implies $a = 0$. Thus, from (16) we conclude that $d = 0$ and therefore F is the identity map on R. A similar argument works if $F(x) = -x + d(x)$ for all $x \in R$. □

Lemma 8. *Let R be a prime ring with* $char(R) \neq 2$, *and let F be a right generalized derivation on R with associated derivation d. If F is of period 2 on R, then* $d(Z) = \{0\}$.

Proof. For all $x, y \in R$, $xy = F^2(xy) = F(F(x)y + xd(y)) = F^2(x)y + F(x)d(y) + F(x)d(y) + xd^2(y)$; hence

$$2F(x)d(y) + xd^2(y) = 0 \quad \forall x, y \in R. \tag{17}$$

Replacing x by $F(x)$ in (17) yields

$$2xd(y) + F(x)d^2(y) = 0 \quad \forall x, y \in R. \tag{18}$$

Letting $z \in Z$ and $x \in R$ and replacing x by xz in (18), we get $2xzd(y) + (F(x)z + xd(z))d^2(y) = 0 = z(2xd(y) + F(x)d^2(y)) + d(z)xd^2(y)$, so by (18) we obtain

$$d(z)Rd^2(y) = \{0\} \quad \forall y \in R. \tag{19}$$

If $d(Z) \neq \{0\}$, we conclude that $d^2 = 0$. But since R is prime and $char(R) \neq 2$, it is easy to show that $d \neq 0$ implies $d^2 \neq 0$; hence $d(Z) = \{0\}$ as claimed. □

Theorem 9. *Let R be a (not necessarily commutative) domain with 1, with* $char(R) \neq 2$. *If F is a right generalized derivation on R of period 2, then F is the identity map or its negative.*

Proof. Note that

$$F(x) = F(1x) = F(1)x + d(x) \quad \forall x \in R. \tag{20}$$

Taking $x = 1$ in (17) and (18) and letting $c = F(1)$, we have $2cd(y) + d^2(y) = 0$ and $2d(y) + cd^2(y) = 0$ for all $y \in R$. It follows that $2d(y) + c(-2cd(y)) = 0$; that is, $(2 - 2c^2)d(y) = 0$; hence

$$\left(c^2 - 1\right)d(y) = 0 \quad \forall y \in R. \tag{21}$$

If $d \neq 0$, we have $c = 1$ or $c = -1$, so that $F(x) = x + d(x)$ for all $x \in R$ or $F(x) = -x + d(x)$ for all $x \in R$. But by Lemma 7 this would imply $d = 0$; hence $d = 0$ and

$$F(x) = cx \quad \forall x \in R. \tag{22}$$

Since F is of period 2, $x = c^2x$ and hence $(c^2 - 1)x = 0$ for all $x \in R$. Thus, $c = 1$ or $c = -1$, so by (22), F is the identity map or its negative. □

Corollary 10. *Let R be a commutative integral domain with* $char(R) \neq 2$. *If F is a right generalized derivation of period 2 on R, then F is the identity map or its negative.*

Proof. If R has 1, the result is immediate from Theorem 9. If R does not have 1, define $\widehat{F}$ on the field of fractions K by $\widehat{F}(a/b) = F(a)/b$. Using the fact that $d = 0$ by Lemma 8, we can show that $\widehat{F}$ is well defined and is a right generalized derivation on K. By Theorem 9, $\widehat{F}$ is the identity map or its negative on K and it follows that F is the identity map or its negative on R. □

Theorem 11. *Let R be a prime ring with* $Z \neq \{0\}$ *and with* $char(R) \neq 2$. *If F is a generalized derivation of period 2 on R with associated derivation d, then F is the identity map or its negative.*

Proof. By Lemma 6, $F(Z) \subseteq Z$; hence F restricts to a generalized derivation on Z. Since the center of a prime ring is a commutative domain, it follows from Corollary 10 that $F(z) = z$ for all $z \in Z$ or $F(z) = -z$ for all $z \in Z$. If the former holds, then $F(zx) = F(xz)$, together with the fact that $d(Z) = \{0\}$, gives $z(F(x) - x - d(x)) = 0$ for all $x \in R$ and $z \in Z$. Taking $z \neq 0$ gives $F(x) = x + d(x)$ for all $x \in R$, so by Lemma 7, F is the identity map on R. A similar argument shows that if $F(z) = -z$ for all $z \in Z$, F is the negative of the identity map. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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