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*Research Article*

# Waves Trapped by Submerged Obstacles at High Frequencies

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As is well known, submerged horizontal cylinders can serve as waveguides for surface water waves. For large values of the wavenumber $k$ in the direction of the cylinders, there is only one trapped wave. We construct asymptotics of these trapped modes and their frequencies as $k \to \infty$ in the case of one or two submerged cylinders by means of reducing the initial problem to a system of integral equations on the boundaries and then solving them using a technique suggested by Zhevandrov and Merzon (2003).



## 1. Introduction

It is well known that submerged horizontal cylinders can serve as waveguides for water waves. The first result in this direction was obtained by Ursell [1] for a cylinder of circular cross-section. Later it was discovered that horizontal “bumps” on the bottom (underwater ridges) can also trap waves (see [2, 3]). In [2], Bonnet-Ben Dhia and Joly proved that for large values of the wavenumber $k$ in the direction of the ridge, there is only one trapped mode. Their proof can be straightforwardly carried over to the case of one of several parallel submerged cylinders. They also showed that the distance of the frequency of this mode to the cut-off frequency is exponentially small in $k$ and that the corresponding eigenfunction decays exponentially slowly in the direction orthogonal to the ridge. In our recent paper [4] we have constructed explicitly this trapped mode for large values of $k$ in the case of a ridge and also indicated the formula for the frequency in the case of one submerged cylinder. (We use the opportunity to indicate that the corresponding formulas for the frequencies in [4] lack the numerical factor 2 due to an arithmetical error.)

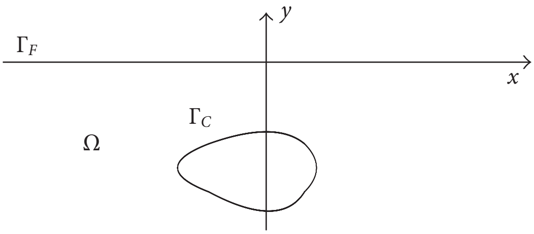


FIGURE 2.1

In the present paper, we give a detailed proof of this result for one cylinder and obtain its generalization to the case of two submerged cylinders. We note that the limit $k \to \infty$ is to some extent analogous to the limit of small height of the underwater ridge: surface water waves decay exponentially with depth $h$ as $\exp(-kh)$, so the influence of an object submerged at a finite distance from the surface is small, just as the influence of a small bump on the bottom. The problem of the ridge of small height was treated in [5], where a close analogy of the problem of water waves and small perturbations of the one-dimensional Schrödinger equation is established. The latter problem was studied by a number of authors (we mention, e.g., [6–8], and, in the context of water waves, [9]). In our case, a technique similar to that of [5] yields the desired result. We note that in contrast to [5], the asymptotics turns out to be exponential, that is, the distance of the trapped wave frequency to the cut-off frequency is exponentially small in $k$. This fact seemingly could have rendered the problem quite complicated from the point of view of asymptotic expansions, but, since in fact we construct an exact convergent expansion, no additional difficulties arise.

## 2. Mathematical formulation and main results

We begin with the problem of one submerged cylinder. We assume that the cylinder cross-section is bounded by the curve $\Gamma_C = \{x = x(t),\ y = y(t),\ t \in [-\pi,\pi]\}$ with smooth $x(t)$ and $y(t)$,

$$x'^2 + y'^2 \neq 0, \tag{2.1}$$

and that $\max y(t) = y(0) < 0$, $y''(0) < 0$, $x'(0) > 0$, where $y$ is the vertical coordinate, $x$ is the horizontal coordinate orthogonal to the direction of the cylinder. The line $\Gamma_F = \{(x,0) : x \in \mathbb{R}\}$ is the free surface. The water layer $\Omega$ is the domain exterior to $\Gamma_C$ and lying below $\Gamma_F$ (see Figure 2.1 for the geometry of the problem).

Looking for the velocity potential in the form $\exp\{i(\omega t - kz)\}\Phi(x,y)$, where $z$ is the horizontal coordinate along the cylinder, $\omega$ is the frequency, we come to the problem

$$\begin{gathered}
\Phi_y = \lambda\Phi, \qquad y = 0,\\
\Phi_{xx} + \Phi_{yy} - k^2\Phi = 0 \quad \text{in } \Omega,\\
\frac{\partial\Phi}{\partial\vec{n}} = 0 \quad \text{on } \Gamma_C,
\end{gathered} \tag{2.2}$$

for the function $\Phi$; here $\lambda = \omega^2/g$. Solutions of this problem from the Sobolev space $H_1(\Omega)$ are called trapped waves and exist only for certain values of $\lambda$ (the eigenparameter) for $k$ fixed.

It is known that the essential spectrum of (2.2) coincides with the interval $[k,\infty)$. There exists only one eigenvalue $\lambda$ below the essential spectrum for large values of $k$ (we do not introduce dimensionless variables for brevity assuming that $1/k$ is small in comparison with some characteristic length, e.g., the diameter of the cylinder). Our goal is to construct an asymptotics of this frequency. The main result in the case of one cylinder consists in the following statement.

Theorem 2.1. *The unique eigenvalue* $\lambda(k)$ *of (2.2) has the form*

$$\lambda(k) = k - \beta^2, \tag{2.3}$$

*where*

$$\beta = k\sqrt{\frac{2\pi}{|y''(0)|}}e^{-2k|y(0)|}x'(0)\left(1+O(k^{-1})\right). \tag{2.4}$$

Sections 3 and 4 are devoted to the proof of this statement and the construction of the corresponding eigenfunction.

## 3. Reduction to a pair of integral equations

As a first step, we reduce (2.2) to a pair of integral equations on $\Gamma_F$ and $\Gamma_C$ for the functions $\varphi = \Phi|_{y=0}$ and $\theta = \Phi|_{\Gamma_C}$. To this end, we apply the Green formula to $\Phi(\xi,\eta)$ and $((-1/2\pi)K_0(kr))$, where $r = \sqrt{(x-\xi)^2+(y-\eta)^2}$ and $K_0$ is the Macdonald function (so that $(-1/2\pi)K_0(kr)$ is the fundamental solution of the operator $\Delta - k^2$).

We have by the Green formula

$$\begin{aligned}
\Phi(\xi,\eta) = {} & \frac{\lambda}{2\pi}\int_{-\infty}^{\infty} K_0\left(k\sqrt{(x-\xi)^2+\eta^2}\right)\varphi(x)dx \\
& + \frac{k\eta}{2\pi}\int_{-\infty}^{\infty} \frac{K_0'\left(k\sqrt{(x-\xi)^2+\eta^2}\right)}{\sqrt{(x-\xi)^2+\eta^2}}\varphi(x)dx \\
& - \frac{k}{2\pi}\int_{-\pi}^{\pi} \frac{K_0'\left(k\sqrt{(x(t)-\xi)^2+(y(t)-\eta)^2}\right)}{\sqrt{(x(t)-\xi)^2+(y(t)-\eta)^2}} \\
& \times \left[y'(t)(x(t)-\xi) - x'(t)(y(t)-\eta)\right]\theta(t)dt, \quad (\xi,\eta)\in\Omega.
\end{aligned} \tag{3.1}$$

Passing in (3.1) to the limits when $\eta \to 0^-$, and $\xi \to x(t)$, $\eta \to y(t)$, we obtain the following integral equations, using the jump formulas for the potentials (see, e.g., [3])

$$\begin{aligned}
\pi\varphi(\xi) &= \lambda\int_{-\infty}^{\infty} K_0(k|x-\xi|)\varphi(x)dx \\
&\quad - k\int_{-\pi}^{\pi} \frac{K_0'\left(k\sqrt{(x(t)-\xi)^2+y(t)^2}\right)}{\sqrt{(x(t)-\xi)^2+y(t)^2}} \\
&\qquad \times \left[y'(t)(x(t)-\xi) - x'(t)y(t)\right]\theta(t)dt, \\
\pi\theta(t) &= \lambda\int_{-\infty}^{\infty} K_0\left(k\sqrt{(x-x(t))^2+y(t)^2}\right)\varphi(x)dx \\
&\quad + ky(t)\int_{-\infty}^{\infty} \frac{K_0'\left(k\sqrt{(x-x(t))^2+y(t)^2}\right)}{\sqrt{(x-x(t))^2+y(t)^2}}\varphi(x)dx \\
&\quad - k\int_{-\pi}^{\pi} \frac{K_0'\left(k\sqrt{(x(t_1)-x(t))^2+(y(t_1)-y(t))^2}\right)}{\sqrt{(x(t_1)-x(t))^2+(y(t_1)-y(t))^2}} \\
&\qquad \times \left[y'(t_1)(x(t_1)-x(t)) - x'(t_1)(y(t_1)-y(t))\right]\theta(t_1)dt_1.
\end{aligned} \tag{3.2}$$

In order to apply the technique of [5] to (3.2), it is necessary to pass to the Fourier transform $\tilde{\varphi}$ of the function $\varphi$,

$$\mathcal{F}_{\xi\to p}[\varphi(\xi)](p) \equiv \tilde{\varphi}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ip\xi}\varphi(\xi)d\xi. \tag{3.3}$$

Using the formulas (see [10])

$$\begin{gathered}
K_0'(z) = -K_1(z), \qquad \mathcal{F}_{\xi\to p}[K_0(k|\xi|)](p) = \frac{\sqrt{\pi/2}}{\sqrt{k^2+p^2}}, \\
\mathcal{F}_{\xi\to p}\left[\frac{K_1\left(k\sqrt{\xi^2+h_0^2}\right)}{\sqrt{\xi^2+h_0^2}}\right](p) = \frac{\sqrt{\pi/2}}{kh_0}e^{-h_0\sqrt{k^2+p^2}}, \\
\mathcal{F}_{\xi\to p}\left[K_0\left(k\sqrt{\xi^2+h_0^2}\right)\right](p) = \frac{\sqrt{\pi/2}}{\sqrt{k^2+p^2}}e^{-h_0\sqrt{k^2+p^2}},
\end{gathered} \tag{3.4}$$

we come to the following system for $\widetilde{\varphi}(p)$, $\theta(t)$:

$$\begin{aligned}\left(1-\frac{\lambda}{\tau(p)}\right)\widetilde{\varphi}(p) &= \int_{-\pi}^{\pi} e^{ipx(t)+y(t)\tau(p)}\left(x'(t)-\frac{ipy'(t)}{\tau(p)}\right)\theta(t)dt,\\ \theta(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ip'x(t)+y(t)\tau(p')}\left(1+\frac{\lambda}{\tau(p')}\right)\widetilde{\varphi}(p')dp'\\ &\quad -\frac{k}{\pi}\int_{-\pi}^{\pi}\frac{K_0'\left(k\sqrt{\varrho(t_1,t)}\right)}{\sqrt{\varrho(t_1,t)}}\sigma(t_1,t)\theta(t_1)dt_1,\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}\tau(p) &:= \sqrt{k^2+p^2},\\ \varrho(t_1,t) &:= \left(x(t_1)-x(t)\right)^2+\left(y(t_1)-y(t)\right)^2,\\ \sigma(t_1,t) &:= y'(t_1)\left(x(t_1)-x(t)\right)-x'(t_1)\left(y(t_1)-y(t)\right).\end{aligned} \tag{3.6}$$

Rewrite system (3.5) as

$$\left(1-\frac{\lambda}{\tau(p)}\right)\widetilde{\varphi}(p) = (\widehat{M}_1\theta)(p), \tag{3.7}$$

$$[(1-\widehat{M}_3)\theta](x) = (\widehat{M}_2\widetilde{\varphi})(x), \tag{3.8}$$

where

$$\begin{aligned}(\widehat{M}_1\theta)(p) &= \int_{-\pi}^{\pi} M_1(p,t)\theta(t)dt,\\ (\widehat{M}_2\widetilde{\varphi})(t) &= \int_{-\infty}^{\infty} M_2(p',t)\widetilde{\varphi}(p')dp',\\ (\widehat{M}_3\theta)(t) &= \int_{-\pi}^{\pi} M_3(t_1,t)\theta(t_1)dt_1,\end{aligned} \tag{3.9}$$

with

$$\begin{aligned}M_1(p,t) &= e^{ipx(t)+y(t)\tau(p)}\left(x'(t)-\frac{ipy'(t)}{\tau(p)}\right),\\ M_2(p',t) &= \frac{1}{2\pi}e^{-ip'x(t)+y(t)\tau(p')}\left(1+\frac{\lambda}{\tau(p')}\right),\\ M_3(t_1,t) &= -\frac{k}{\pi}\frac{K_0'\left(k\sqrt{\varrho(t_1,t)}\right)}{\sqrt{\varrho(t_1,t)}}\sigma(t_1,t).\end{aligned} \tag{3.10}$$

Obviously, a solution of (3.7), (3.8) gives via (3.1) a solution of (2.2).

## 4. Solution of the system of integral equations

Consider (3.8). It is not hard to see, using the asymptotics of $K_1(z)$ for small and large $z$, that the operator $\widehat{M}_3$ in (3.8) is bounded by const$k^{-1/2}$. In fact, the following lemma holds.

Lemma 4.1. *One has*

$$\left|\int_{-\pi}^{\pi} M_3(t_1,t)\theta(t_1)\,dt_1\right| \le Ck^{-1/2}\|\theta\|, \tag{4.1}$$

*where C is a constant and* $\|\theta\| = \sup_{t\in[-\pi,\pi]}|\theta(t)|$.

*Proof.* For a given $\delta > 0$, we divide the interval of integration in two domains, $k|t_1 - t| < \delta$ and $k|t_1 - t| > \delta$. In the first domain, we use the asymptotics $K_0'(z) \sim 1/z$, and in the second, the asymptotics $K_0'(z) \sim \sqrt{\pi/2z}\,e^{-z}$. For $k|t - t_1| < \delta$, we have by (3.10)

$$|M_3(t_1,t)| \le C_1\frac{|\sigma(t_1,t)|}{\varrho(t_1,t)}. \tag{4.2}$$

The numerator here is $O((t_1 - t)^2)$ and

$$\sqrt{\varrho(t_1,t)} \ge c|t_1 - t|, \quad c > 0, \tag{4.3}$$

by (2.1). Hence $M_3(t_1,t)$ is bounded in this domain. For $k|t_1 - t| > \delta$ we have, again by (3.10),

$$|M_3(t_1,t)| \le C_2k^{1/2}e^{-k\sqrt{\varrho(t_1,t)}}\frac{|\sigma(t_1,t)|}{(\varrho(t_1,t))^{3/4}}. \tag{4.4}$$

The last factor is bounded by virtue of (4.3) and by the same inequality we obtain

$$|M_3(t_1,t)| \le C_3k^{1/2}e^{-ck|t_1-t|}. \tag{4.5}$$

Since $e^{-k|t_1-t|} \geq e^{-\delta} = \text{const}$ for $|t_1 - t| < \delta/k$, we see that (4.5) holds for all $t_1$, $t$. Now

$$\left| \int_{-\pi}^{\pi} M_3(t_1,t)\theta(t_1)\,dt_1 \right| \leq \text{const} \int_{-\pi}^{\pi} k^{1/2} e^{-ck|t_1-t|}\,dt_1 \|\theta\| \leq Ck^{-1/2}\|\theta\| \tag{4.6}$$

as claimed. □

Hence, we can invert the operator $(1 - \widehat{M}_3)$ in (3.8) using the Neumann series and obtain

$$\theta(t) = \left[ \left(1 - \widehat{M}_3\right)^{-1} \widehat{M}_2 \widetilde{\varphi} \right](t), \tag{4.7}$$

where $(1 - \widehat{M}_3)^{-1} = \sum_{n=0}^{\infty} \widehat{M}_3^n$. Substituting (4.7) in (3.7), we finally come to

$$\left(1 - \frac{\lambda}{\tau(p)}\right) \widetilde{\varphi}(p) = \left[ \widehat{M}_1 \left(1 - \widehat{M}_3\right)^{-1} \widehat{M}_2 \widetilde{\varphi} \right](p). \tag{4.8}$$

We apply the reasoning of [5] to (4.8). Indeed, we know that $\lambda$ is given by (2.3), where $\beta$ is exponentially small in $k$ [2]. Hence the first factor in the left-hand side of (4.8),

$$L(p) := 1 - \frac{\lambda}{\tau(p)} = 1 - \frac{k - \beta^2}{\sqrt{k^2 + p^2}}, \tag{4.9}$$

is exponentially small in $k$ for $p = 0$. In fact, the roots of $L(p) = 0$ which tend to zero as $k \to \infty$, as it is not hard to see, are simple and given by

$$p = p_{\pm} = \pm \frac{i\sqrt{2}\beta}{\sqrt{\varepsilon}} + O\left(\varepsilon^{1/2}\beta^3\right), \quad \varepsilon = \frac{1}{k}. \tag{4.10}$$

For this reason, the heuristic considerations of [5, Section 2] are applicable to (4.8). Following these arguments, we look for $\widetilde{\varphi}$ in the form $\widetilde{\varphi}(p) = A(p)/L(p)$. As we will see (see formula (4.17) below), $A(p)$ and $M_2(p,t)$ are analytic in a strip containing the real axis, and we can change the contour of integration in the integral

$$\int_{-\infty}^{\infty} M_2(p,t) \frac{A(p)}{L(p)}\,dp \tag{4.11}$$

to the one shown in Figure 4.1 (with a suitable $a > 0$ such that in the disc $|p| < a$ there are no zeros of $L(p)$ apart from $p_{\pm}$).

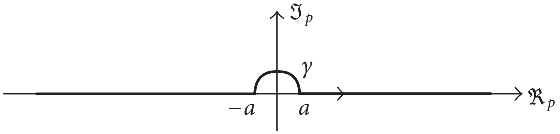


FIGURE 4.1

We have, by the residue theorem,

$$\int_{-\infty}^{\infty} M_2(p,t)\frac{A(p)}{L(p)}dp = \int_{\gamma} M_2(p,t)\frac{A(p)}{L(p)}dp + 2\pi i\frac{M_2(p_+,t)A(p_+)}{(d/dp)L(p)|_{p=p_+}}. \tag{4.12}$$

Thus (4.8) transforms into

$$A(p) = \left[\widehat{M}_1(1-\widehat{M}_3)^{-1}\widehat{M}_4 A\right](p) + \left[\widehat{M}_1(1-\widehat{M}_3)^{-1}f(t)\right]A(p_+), \tag{4.13}$$

where

$$[\widehat{M}_4 A](t) = \int_{\gamma} M_2(p,t)\frac{A(p)}{L(p)}dp, \qquad f(t) = 2\pi i\frac{M_2(p_+,t)}{(d/dp)L(p)|_{p=p_+}}. \tag{4.14}$$

Note that now the operator $\widehat{M}_5 = \widehat{M}_1(1-\widehat{M}_3)^{-1}\widehat{M}_4$ is small in $\varepsilon$ since $|L(p)| \geq \operatorname{const} k^{-2}$ along $\gamma$ and $M_2(p,t)$ is exponentially small. Indeed, on the arc we have up to $O(k^{-\infty})$

$$|L(p)| = \left|1 - \frac{1}{\sqrt{1+p^2/k^2}}\right| = \frac{a^2}{2k^2} + O(k^{-4}), \tag{4.15}$$

and on the part of the contour which lies on the real axis the minimum of $|L(p)|$ is attained at the points $p = \pm a$, hence, the above estimate still holds. Rewriting (4.13) as

$$(1-\widehat{M}_5)A(p) = g(p)A(p_+), \tag{4.16}$$

where $g(p) = \widehat{M}_1(1-\widehat{M}_3)^{-1}f(t)$, we see that $(1-\widehat{M}_5)$ is invertible and

$$A(p) = (1-\widehat{M}_5)^{-1}g(p)A(p_+). \tag{4.17}$$

Let us show that $A(p)$ is indeed analytic in a strip containing the real axis.

LEMMA 4.2. *Let $f(t)$ be continuous in $t \in [-\pi,\pi]$. Then, $g(p) = \widehat{M}_1(1-\widehat{M}_3)^{-1}f(t)$ is analytic in a strip containing $\gamma$, and $|g(p)| \leq Ce^{-h_0\Re\tau/2}\|f\|$, $p \in \gamma$, $h_0 = |y(0)|$.*

*Remark 4.3.* Note that $\Re\tau = k + O(k^{-1})$ for finite $p$ and $\Re\tau = \tau$ for $p$ real.

*Proof.* By Lemma 4.1, $(1-\widehat{M}_3)^{-1}$ is bounded on $C[-\pi,\pi]$. The assertion now follows directly from the explicit formula for $M_1(p,t)$ since $M_1(p,t)$ is analytic in $p$ for $|\Im p|<k$. □

Lemma 4.4. *Let $g(p)$ be analytic in a strip containing $\gamma$ and $\|g\| = \sup_{p\in\gamma}|g(p)| < \infty$. Then, $A(p) = (1-\widehat{M}_5)^{-1}g(p)$ is analytic in $p$ in a strip containing $\gamma$ and $\|A\| \le C\|g\|$.*

*Proof.* We have $(1-\widehat{M}_5)^{-1} = \sum_{n=0}^{\infty}\widehat{M}_5^n$ and

$$|\widehat{M}_5 g| = \left|\widehat{M}_1(1-\widehat{M}_3)^{-1}\widehat{M}_4 g\right| \le Ce^{-h_0\Re\tau - h_0k/2}\|g\| \tag{4.18}$$

by Lemma 4.1, (4.15), and the explicit form of $M_{1,2}$. $\widehat{M}_5 g$ is analytic in the strip since $M_1(p,t)$ is. Iterating (4.18), we see that

$$|\widehat{M}_5^n g| \le C^n e^{-nh_0k/2 - h_0\Re\tau}\|g\|, \tag{4.19}$$

hence, the series $\sum_{n=0}^{\infty}\widehat{M}_5^n g$ converges uniformly for sufficiently large $k$ and its sum is analytic by the Weierstrass theorem. □

Applying now Lemma 4.2 to $g(p) = \widehat{M}_1(1-\widehat{M}_3)^{-1}f(t)$ with $f(t)$ given by (4.14) and then using Lemma 4.4, we see that $A(p)$ given by (4.17) is indeed analytic.

Putting $p = p_+$ in the equality (4.17) and dividing by $A(p_+)$, we obtain an equation for $\beta$:

$$1 = (1-\widehat{M}_5)^{-1}g(p)|_{p=p_+}. \tag{4.20}$$

A standard application of the Laplace method of asymptotic evaluation of integrals to the leading term in (4.20) yields formula (2.4). In fact, from the leading term in (4.20) we have, using (4.10) and multiplying by $\beta$

$$\beta \sim \frac{\sqrt{2}\pi}{\varepsilon^{3/2}}\int_{-\pi}^{\pi} M_1(p_+,t_1)M_2(p_+,t_1)\,dt_1. \tag{4.21}$$

We have $\lambda = k-\beta^2$ and $\tau(p_+) = k(1+O(\varepsilon\beta^2))$, hence $1+(\lambda/\tau(p_+)) = 2(1+O(\varepsilon\beta^2))$. Thus in the leading term we have

$$\beta \sim \frac{\sqrt{2}}{\varepsilon^{3/2}}\int_{-\pi}^{\pi} e^{-2k|y(t_1)|}x'(t_1)\,dt_1. \tag{4.22}$$

Applying the Laplace method to the last integral, we obtain (2.4).

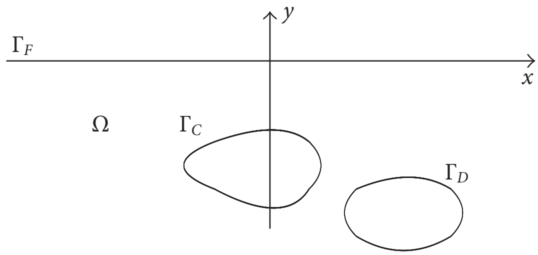


Figure 5.1

## 5. Generalization to the case of two cylinders

The geometry of the problem is as follows: we assume that $\Gamma_C = \{x = x(t),\ y = y(t),\ t \in [-\pi,\pi]\}$ with smooth $x(t)$ and $y(t)$, $x'^2 + y'^2 \neq 0$, and $\max y(t) = y(0) < 0$, $y''(0) < 0$, $x'(0) > 0$; similarly, $\Gamma_D = \{x = u(t),\ y = v(t),\ t \in [-\pi,\pi]\}$ with smooth $u(t)$ and $v(t)$, $x'^2 + y'^2 \neq 0$, and $\max v(t) = v(0) < 0$, $v''(0) < 0$, $u'(0) > 0$, where $y$ is the vertical coordinate, $x$ is the horizontal coordinate orthogonal to the direction of the cylinders, $\Gamma_C$ and $\Gamma_D$ describe curves bounding their cross-sections. We assume that $\Gamma_C$ and $\Gamma_D$ do not intersect, $\sqrt{(x(t)-u(t_1))^2 + (y(t)-v(t_1))^2} \geq d > 0$. $\Gamma_F = \{(x,0) : x \in \mathbb{R}\}$ is the free surface. The water layer $\Omega$ is the domain exterior to $\Gamma_C$ and $\Gamma_D$ and lying below $\Gamma_F$ (see Figure 5.1).

Looking for the velocity potential in the form $\exp\{i(\omega t - kz)\}\Phi(x,y)$, where $z$ is the horizontal coordinate along the cylinders, $\omega$ is the frequency, we come to the problem

$$\begin{gathered} \Phi_y = \lambda\Phi, \quad y = 0, \\ \Phi_{xx} + \Phi_{yy} - k^2\Phi = 0 \quad \text{in } \Omega, \\ \partial\Phi/\partial\vec{n}_C = 0 \quad \text{on } \Gamma_C, \\ \partial\Phi/\partial\vec{n}_D = 0 \quad \text{on } \Gamma_D, \end{gathered} \tag{5.1}$$

for the function $\Phi$; here $\lambda = \omega^2/g$. Just as in Section 2, solutions of this problem from the Sobolev space $H_1(\Omega)$ are called trapped waves and exist only for certain values of $\lambda$ (the eigenparameter) for $k$ fixed. As in the case of one cylinder, the essential spectrum of (5.1) coincides with the interval $[k,\infty)$. There exists only one eigenvalue $\lambda$ below the essential spectrum for large values of $k$. Our goal is to construct an asymptotics of this frequency. Our main result in this case is as follows.

Theorem 5.1. *The unique eigenvalue $\lambda(k)$ of (5.1) has the form*

$$\lambda = k - \beta^2, \tag{5.2}$$

*where*

$$\beta = k\left(\sqrt{\frac{2\pi}{|y''(0)|}}e^{-2k|y(0)|}x'(0)\left(1+O(k^{-1})\right) + \sqrt{\frac{2\pi}{|v''(0)|}}e^{-2k|v(0)|}u'(0)\left(1+O(k^{-1})\right)\right). \tag{5.3}$$

*Remark 5.2.* Of course, if $|y(0)| < |v(0)|$, then the second summand in (5.3) is negligible, and the result in fact is the same as in the case of one cylinder.

In the next section, we perform the steps analogous to Sections 3 and 4 and construct the corresponding eigenfunction.

## 6. Reduction to integral equations and their solution

As in Section 3, we reduce (5.1) to three integral equations on $\Gamma_F$, $\Gamma_C$, and $\Gamma_D$ for the functions $\varphi = \Phi|_{y=0}$, $\theta = \Phi|_{\Gamma_C}$, and $\alpha = \Phi|_{\Gamma_D}$. We have by the Green formula

$$
\begin{aligned}
\Phi(\xi,\eta) = {} & \frac{\lambda}{2\pi}\int_{-\infty}^{\infty} K_0\Big(k\sqrt{(x-\xi)^2+\eta^2}\Big)\varphi(x)dx \\
& + \frac{k\eta}{2\pi}\int_{-\infty}^{\infty} \frac{K_0'\Big(k\sqrt{(x-\xi)^2+\eta^2}\Big)}{\sqrt{(x-\xi)^2+\eta^2}}\varphi(x)dx \\
& - \frac{k}{2\pi}\int_{-\pi}^{\pi} \frac{K_0'\Big(k\sqrt{(x(t)-\xi)^2+(y(t)-\eta)^2}\Big)}{\sqrt{(x(t)-\xi)^2+(y(t)-\eta)^2}} \\
& \qquad \times \big[y'(t)(x(t)-\xi)-x'(t)(y(t)-\eta)\big]\theta(t)dt \\
& - \frac{k}{2\pi}\int_{-\pi}^{\pi} \frac{K_0'\Big(k\sqrt{(u(t)-\xi)^2+(v(t)-\eta)^2}\Big)}{\sqrt{(u(t)-\xi)^2+(v(t)-\eta)^2}} \\
& \qquad \times \big[v'(t)(u(t)-\xi)-u'(t)(v(t)-\eta)\big]\alpha(t)dt, \quad (\xi,\eta)\in\Omega.
\end{aligned}
\tag{6.1}
$$

Passing in (6.1) to the limits when $\eta \to 0^-$, $\xi \to x(t)$, $\eta \to y(t)$, and $\xi \to u(t)$, $\eta \to v(t)$, we obtain the following integral equations:

$$
\begin{aligned}
\pi\varphi(\xi) = {} & \lambda\int_{-\infty}^{\infty} K_0(k|x-\xi|)\varphi(x)dx \\
& - k\int_{-\pi}^{\pi} \frac{K_0'\Big(k\sqrt{(x(t)-\xi)^2+y(t)^2}\Big)}{\sqrt{(x(t)-\xi)^2+y(t)^2}} \\
& \qquad \times \big[y'(t)(x(t)-\xi)-x'(t)y(t)\big]\theta(t)dt \\
& - k\int_{-\pi}^{\pi} \frac{K_0'\Big(k\sqrt{(u(t)-\xi)^2+v(t)^2}\Big)}{\sqrt{(u(t)-\xi)^2+v(t)^2}} \\
& \qquad \times \big[v'(t)(u(t)-\xi)-u'(t)v(t)\big]\alpha(t)dt,
\end{aligned}
$$

$$
\begin{aligned}
\pi\theta(t) = {} & \lambda \int_{-\infty}^{\infty} K_0\left(k\sqrt{(x-x(t))^2+y(t)^2}\right)\varphi(x)dx \\
& + ky(t)\int_{-\infty}^{\infty} \frac{K_0'\left(k\sqrt{(x-x(t))^2+y(t)^2}\right)}{\sqrt{(x-x(t))^2+y(t)^2}}\varphi(x)dx \\
& - k\int_{-\pi}^{\pi} \frac{K_0'\left(k\sqrt{(x(t_1)-x(t))^2+(y(t_1)-y(t))^2}\right)}{\sqrt{(x(t_1)-x(t))^2+(y(t_1)-y(t))^2}} \\
& \qquad \times \left[y'(t_1)(x(t_1)-x(t))-x'(t_1)(y(t_1)-y(t))\right]\theta(t_1)dt_1 \\
& - k\int_{-\pi}^{\pi} \frac{K_0'\left(k\sqrt{(u(t_1)-x(t))^2+(v(t_1)-y(t))^2}\right)}{\sqrt{(u(t_1)-x(t))^2+(v(t_1)-y(t))^2}} \\
& \qquad \times \left[v'(t_1)(u(t_1)-x(t))-u'(t_1)(v(t_1)-y(t))\right]\alpha(t_1)dt_1, \\
\pi\alpha(t) = {} & \lambda \int_{-\infty}^{\infty} K_0\left(k\sqrt{(x-u(t))^2+v(t)^2}\right)\varphi(x)dx \\
& + kv(t)\int_{-\infty}^{\infty} \frac{K_0'\left(k\sqrt{(x-u(t))^2+v(t)^2}\right)}{\sqrt{(x-u(t))^2+v(t)^2}}\varphi(x)dx \\
& - k\int_{-\pi}^{\pi} \frac{K_0'\left(k\sqrt{(x(t_1)-u(t))^2+(y(t_1)-v(t))^2}\right)}{\sqrt{(x(t_1)-u(t))^2+(y(t_1)-v(t))^2}} \\
& \qquad \times \left[y'(t_1)(x(t_1)-u(t))-x'(t_1)(y(t_1)-v(t))\right]\theta(t_1)dt_1 \\
& - k\int_{-\pi}^{\pi} \frac{K_0'\left(k\sqrt{(u(t_1)-u(t))^2+(v(t_1)-v(t))^2}\right)}{\sqrt{(u(t_1)-u(t))^2+(v(t_1)-v(t))^2}} \\
& \qquad \times \left[v'(t_1)(u(t_1)-u(t))-u'(t_1)(v(t_1)-v(t))\right]\alpha(t_1)dt_1.
\end{aligned} \tag{6.2}
$$

As in Section 3, passing to the Fourier transform $\widetilde{\varphi}$ of the function $\varphi$ and using (3.4), we come to the following system for $\widetilde{\varphi}(p)$, $\theta(t)$, and $\alpha(t)$:

$$
\begin{aligned}
\left(1-\frac{\lambda}{\tau(p)}\right)\widetilde{\varphi}(p) = {} & \int_{-\pi}^{\pi} e^{ipx(t)+y(t)\tau(p)}\left(x'(t)-\frac{ipy'(t)}{\tau(p)}\right)\theta(t)dt \\
& + \int_{-\pi}^{\pi} e^{ipu(t)+v(t)\tau(p)}\left(u'(t)-\frac{ipv'(t)}{\tau(p)}\right)\alpha(t)dt,
\end{aligned} \tag{6.3}
$$

$$\begin{aligned}\theta(t) = {} & \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ip'x(t)+y(t)\tau(p')}\left(1+\frac{\lambda}{\tau(p')}\right)\widetilde{\varphi}(p')\,dp' \\ & -\frac{k}{\pi}\int_{-\pi}^{\pi}\frac{K_0'\left(k\sqrt{\varrho(t_1,t)}\right)}{\sqrt{\varrho(t_1,t)}}\sigma_1(t_1,t)\theta(t_1)\,dt_1 \\ & -\frac{k}{\pi}\int_{-\pi}^{\pi}\frac{K_0'\left(k\sqrt{\vartheta(t,t_1)}\right)}{\sqrt{\vartheta(t,t_1)}}\sigma_3(t_1,t)\alpha(t_1)\,dt_1,\end{aligned} \tag{6.4}$$

$$\begin{aligned}\alpha(t) = {} & \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ip'u(t)+v(t)\tau(p')}\left(1+\frac{\lambda}{\tau(p')}\right)\widetilde{\varphi}(p')\,dp' \\ & -\frac{k}{\pi}\int_{-\pi}^{\pi}\frac{K_0'\left(k\sqrt{\vartheta(t_1,t)}\right)}{\sqrt{\vartheta(t_1,t)}}\sigma_4(t_1,t)\theta(t_1)\,dt_1 \\ & -\frac{k}{\pi}\int_{-\pi}^{\pi}\frac{K_0'\left(k\sqrt{\varpi(t_1,t)}\right)}{\sqrt{\varpi(t_1,t)}}\sigma_2(t_1,t)\alpha(t_1)\,dt_1,\end{aligned} \tag{6.5}$$

where

$$\begin{aligned}\tau(p) &:= \sqrt{k^2+p^2}, \\ \varrho(t_1,t) &:= (x(t_1)-x(t))^2+(y(t_1)-y(t))^2, \\ \varpi(t_1,t) &:= (u(t_1)-u(t))^2+(v(t_1)-v(t))^2, \\ \vartheta(t_1,t) &:= (x(t_1)-u(t))^2+(y(t_1)-v(t))^2, \\ \sigma_1(t_1,t) &:= y'(t_1)(x(t_1)-x(t))-x'(t_1)(y(t_1)-y(t)), \\ \sigma_2(t_1,t) &:= v'(t_1)(u(t_1)-u(t))-u'(t_1)(v(t_1)-v(t)), \\ \sigma_3(t_1,t) &:= v'(t_1)(u(t_1)-x(t))-u'(t_1)(v(t_1)-y(t)), \\ \sigma_4(t_1,t) &:= y'(t_1)(x(t_1)-u(t))-x'(t_1)(y(t_1)-v(t)).\end{aligned} \tag{6.6}$$

Rewrite system (6.3)–(6.5) as

$$\left(1-\frac{\lambda}{\tau(p)}\right)\widetilde{\varphi}(p) = (\widehat{M}_1\theta)(p)+(\widehat{M}_2\alpha)(p), \tag{6.7}$$

$$[(1-\widehat{M}_5)\theta](t) = (\widehat{M}_3\widetilde{\varphi})(t)+(\widehat{M}_4\alpha)(t), \tag{6.8}$$

$$[(1-\widehat{M}_8)\alpha](t) = (\widehat{M}_6\widetilde{\varphi})(t)+(\widehat{M}_7\theta)(t), \tag{6.9}$$

where

$$
\begin{aligned}
(\widehat{M}_1\theta)(p) &= \int_{-\pi}^{\pi} M_1(p,t)\theta(t)dt,\\
(\widehat{M}_2\alpha)(p) &= \int_{-\pi}^{\pi} M_2(p,t)\alpha(t)dt,\\
(\widehat{M}_j\widetilde{\varphi})(t) &= \int_{-\infty}^{\infty} M_j(p',t)\widetilde{\varphi}(p')dp', \quad j=3,6,\\
(\widehat{M}_j\alpha)(t) &= \int_{-\pi}^{\pi} M_j(t_1,t)\alpha(t_1)dt_1, \quad j=4,8,\\
(\widehat{M}_j\theta)(t) &= \int_{-\pi}^{\pi} M_j(t_1,t)\theta(t_1)dt_1, \quad j=5,7,
\end{aligned}
\tag{6.10}
$$

with

$$
\begin{aligned}
M_1(p,t) &= e^{ipx(t)+y(t)\tau(p)}\left(x'(t)-\frac{ipy'(t)}{\tau(p)}\right),\\
M_2(p,t) &= e^{ipu(t)+v(t)\tau(p)}\left(u'(t)-\frac{ipv'(t)}{\tau(p)}\right),\\
M_3(p',t) &= \frac{1}{2\pi}e^{-ip'x(t)+y(t)\tau(p')}\left(1+\frac{\lambda}{\tau(p')}\right),\\
M_4(t_1,t) &= -\frac{k}{\pi}\frac{K_0'\left(k\sqrt{\vartheta(t,t_1)}\right)}{\sqrt{\vartheta(t,t_1)}}\sigma_3(t_1,t),\\
M_5(t_1,t) &= -\frac{k}{\pi}\frac{K_0'\left(k\sqrt{\varrho(t_1,t)}\right)}{\sqrt{\varrho(t_1,t)}}\sigma_1(t_1,t),\\
M_6(p',t) &= \frac{1}{2\pi}e^{-ip'u(t)+v(t)\tau(p')}\left(1+\frac{\lambda}{\tau(p')}\right),\\
M_7(t_1,t) &= -\frac{k}{\pi}\frac{K_0'\left(k\sqrt{\vartheta(t_1,t)}\right)}{\sqrt{\vartheta(t_1,t)}}\sigma_4(t_1,t),\\
M_8(t_1,t) &= -\frac{k}{\pi}\frac{K_0'\left(k\sqrt{\varpi(t_1,t)}\right)}{\sqrt{\varpi(t_1,t)}}\sigma_2(t_1,t).
\end{aligned}
\tag{6.11}
$$

Consider equations (6.8), (6.9). Repeating the arguments of Lemma 4.1, we obtain $\|\widehat{M}_{5,8}\| \le \operatorname{const} k^{-1/2}$. Hence the operators $(1-\widehat{M}_5)$, $(1-\widehat{M}_8)$ are invertible. Moreover, $\widehat{M}_{4,7}$ are exponentially small since $\sqrt{\vartheta(t_1,t)} \ge d > 0$. Solving (6.8), (6.9) for $\theta$ and $\alpha$, we obtain

$$\begin{aligned}\theta(t) &= \Big\{\Big[1-(1-\widehat{M}_5)^{-1}\widehat{M}_4(1-\widehat{M}_8)^{-1}\widehat{M}_7\Big]^{-1}\\ &\qquad\times(1-\widehat{M}_5)^{-1}\Big[\widehat{M}_3+\widehat{M}_4(1-\widehat{M}_8)^{-1}\widehat{M}_6\Big]\widetilde{\varphi}\Big\}(t),\\ \alpha(t) &= \Big\{\Big[1-(1-\widehat{M}_8)^{-1}\widehat{M}_7(1-\widehat{M}_5)^{-1}\widehat{M}_4\Big]^{-1}\\ &\qquad\times(1-\widehat{M}_8)^{-1}\Big[\widehat{M}_6+\widehat{M}_7(1-\widehat{M}_5)^{-1}\widehat{M}_3\Big]\widetilde{\varphi}\Big\}(t),\end{aligned} \tag{6.12}$$

where $(1-\widehat{M}_j)^{-1}=\sum_{n=0}^{\infty}\widehat{M}_j^n$. Substituting (6.12) in (6.3), we finally come to

$$\Big(1-\frac{\lambda}{\tau(p)}\Big)\widetilde{\varphi}(p)=[\widehat{M}_{11}\widetilde{\varphi}](p), \tag{6.13}$$

where

$$\begin{aligned}\widehat{M}_{11} &= \widehat{M}_9\widehat{M}_3+\widehat{M}_{10}\widehat{M}_6,\\ \widehat{M}_9 &= \widehat{M}_1\Big[1-(1-\widehat{M}_5)^{-1}\widehat{M}_4(1-\widehat{M}_8)^{-1}\widehat{M}_7\Big]^{-1}(1-\widehat{M}_5)^{-1}\\ &\quad+\widehat{M}_2\Big[1-(1-\widehat{M}_8)^{-1}\widehat{M}_7(1-\widehat{M}_5)^{-1}\widehat{M}_4\Big]^{-1}(1-\widehat{M}_8)^{-1}\widehat{M}_7(1-\widehat{M}_5)^{-1},\\ \widehat{M}_{10} &= \widehat{M}_1\Big[1-(1-\widehat{M}_5)^{-1}\widehat{M}_4(1-\widehat{M}_8)^{-1}\widehat{M}_7\Big]^{-1}(1-\widehat{M}_5)^{-1}\widehat{M}_4(1-\widehat{M}_8)^{-1}\\ &\quad+\widehat{M}_2\Big[1-(1-\widehat{M}_8)^{-1}\widehat{M}_7(1-\widehat{M}_5)^{-1}\widehat{M}_4\Big]^{-1}(1-\widehat{M}_8)^{-1}.\end{aligned} \tag{6.14}$$

We repeat the procedure of Section 4. We look for $\widetilde{\varphi}$ in the form $\widetilde{\varphi}(p)=A(p)/L(p)$. Assuming that $A(p)$ is analytic and using the fact that $M_j(p,t)$, $j=3,6$, are analytic in a strip containing the real axis, we can change the contour of integration in the integrals

$$\int_{-\infty}^{\infty}M_j(p,t)\frac{A(p)}{L(p)}dp,\qquad j=3,6, \tag{6.15}$$

to the one shown in Figure 4.1. We have, by the residue theorem,

$$\int_{-\infty}^{\infty}M_j(p,t)\frac{A(p)}{L(p)}dp=\int_{\gamma}M_j(p,t)\frac{A(p)}{L(p)}dp+2\pi i\frac{M_j(p_+,t)A(p_+)}{(d/dp)L(p)|_{p=p_+}},\quad j=3,6. \tag{6.16}$$

Thus (6.13) transforms into

$$A(p)=\widehat{M}_{11}^{\gamma}A(p)+g(p)A(p_+), \tag{6.17}$$

where

$$\widehat{M}_{11}^{\gamma} = \widehat{M}_9\widehat{M}_3^{\gamma} + \widehat{M}_{10}\widehat{M}_6^{\gamma},$$

$$[\widehat{M}_j^{\gamma}A](t) = \int_{\gamma} M_j(p',t)\frac{A(p')}{L(p')}dp', \quad f_j(t) = 2\pi i\frac{M_j(p_+,t)}{(d/dp)L(p)|_{p=p_+}}, \quad j = 3,6,$$

$$g(p) = \widehat{M}_9 f_3(t) + \widehat{M}_{10} f_6(t). \tag{6.18}$$

The operator $\widehat{M}_{11}^{\gamma}$ is small in $\varepsilon$, $\varepsilon = 1/k$, just as operator $\widehat{M}_5$ in Section 4.

Rewriting (6.17) as

$$[(1-\widehat{M}_{11}^{\gamma})A](p) = g(p)A(p_+), \tag{6.19}$$

we see that $(1-\widehat{M}_{11}^{\gamma})$ is invertible and

$$A(p) = [(1-\widehat{M}_{11}^{\gamma})^{-1}g](p)A(p_+). \tag{6.20}$$

Putting $p = p_+$ in the last equality and dividing by $A(p_+)$, we obtain an equation for $\beta$:

$$1 = [(1-\widehat{M}_{11})^{-1}g](p)|_{p=p_+}. \tag{6.21}$$

A standard application of the Laplace method of asymptotic evaluation of integrals to the leading term in (6.21) yields formula (5.3). In fact, from the leading term in (6.21),

$$\beta \sim \frac{\sqrt{2}\pi}{\varepsilon^{3/2}}\int_{-\pi}^{\pi}(M_1(p_+,t_1)M_3(p_+,t_1) + M_2(p_+,t_1)M_6(p_+,t_1))dt_1, \tag{6.22}$$

with $M_j(p_+,t_1)$, $j = 1,2,3,6$, defined in (6.11). As in Section 4, we have

$$\beta \sim \frac{\sqrt{2}}{\varepsilon^{3/2}}\int_{-\pi}^{\pi}(e^{-2k|y(t_1)|}x'(t_1) + e^{-2k|v(t_1)|}u'(t_1))dt_1. \tag{6.23}$$

Applying the Laplace method to the last integral, we obtain (5.3).

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